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KHOVANOV HOMOLOGY OF SYMMETRIC LINKS

WOJCIECH POLITARCZYK

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HOMOLOGIE KHOVANOVA SPLOTÓW SYMETRYCZNYCH

WOJCIECH POLITARCZYK

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To my Wife, my Parents and my Sister

ABSTRACT

This thesis presents a construction of a variant of the Khovanov homology for periodic links, i.e, links with certain kind of symmetry. This version takes into account symmetries of links. We use elements of homological algebra, like derived functors and spectral sequences, and integral representation theory of finite cyclic groups to construct and describe properties of the equivariant Khovanov homology. Further, we develop a spectral sequence for computing the equivariant Khovanov homology. We use this spectral sequence to compute the rational equivariant Khovanov homology of torus links $T(n, 2)$.

Apart from that, we also study properties of the equivariant analogues of the Jones polynomial. We show that they satisfy certain version of the skein relation and use it to generalize a result of J.H. Przytycki, which is a criterion for periodicity of a link in terms of its Jones polynomial. Additionally, we develop a state sum formula for the equivariant analogues of the Jones polynomial, which enables us to reprove the classical congruence of K. Murasugi.

STRESZCZENIE

Rozprawa ta prezentuje konstrukcję wariantu homologii Khovanova dla tzw. splotów periodycznych, czyli splotów posiadających pewną symetrię. Ta wersja homologii Khovanova uwzględnia symetrie splotów. Przy pomocy metod algebry homologicznej, takich jak funktory pochodne i ciągi spektralne, oraz teorii całkowitoliczbowych reprezentacji grup cyklicznych podajemy konstrukcję i opisujemy podstawowe własności ekwiwariantnych homologii Khovanova. Dodatkowo, konstruujemy ciąg spektralny, który pozwala wyliczać ekwiwariantne homologie Khovanova. Ciąg ten jest adaptacją motkowego ciągu dokładnego. W dalszej części wyliczany wymierne ekwiwariantne homologie Khovanova splotów torusowych $T(n, 2)$.

Oprócz tego, rozważamy ekwiwariantne odpowiedniki wielomianu Jonesa. Pokazujemy, że spełniają one odpowiednik relacji motkowej dla klasycznego wielomianu Jonesa i używamy tej własności do wzmocnienia kryterium periodyczności splotu podanego przez J.H. Przytyckiego. Dodatkowo, wyprowadzamy sumę statystyczną dla ekwiwariantnych odpowiedników wielomianu Jonesa. Konsekwencją tego faktu jest klasyczna kongruencja podana przez K. Murasugiego.

Algebra is the offer made by the devil
to the mathematician. The devil says:
"I will give you this powerful machine,
it will answer any question you like.
All you need to do is give me your soul:
give up geometry and you will have
this marvelous machine."
Sir Michael Atiyah [1]

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Last but not least, I would like to thank my family, especially my wife for her love, support and for being my distraction, my parents and my sister for their support and encouragement.

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1 INTRODUCTION

One of the main themes in topology is the study of symmetries of certain objects like topological spaces or manifolds. In knot theory one is particularly interested in symmetries of knots, that is symmetries of the 3-sphere that preserve the given knot. One such particular example is provided by involutions. Such an involution can preserve or reverse the orientation of the knot and the ambient space, hence we can distinguish four kinds of involutive symmetries of knots: strong invertibility, strong $+$-amphicheirality, strong $-$-amphicheirality and involutions that preserve the orientation of the 3-sphere and the knot. Appart from that, there are many more possible symmetries, which can be, for example, derived from the symmetries of S^3.

In this thesis we study knots which possess certain kind of symmetry of finite order, which is derived from the semi-free action of the cyclic group on the 3-sphere i.e. we are interested in the diffeomorphisms $f\colon (S^3, K) \to (S^3, K)$ of finite order, where K is a knot. Due to the resolution of the Smith Conjecture in [14], the existence of such symmetry can be rephrased in the following way. Let ρ_n be the rotation of $\mathbb{R}^3$ by the $\frac{2\pi}{n}$ angle about the OZ axis. We are interested in knots $K \subset \mathbb{R}^3$, which are disjoint from the OZ axis and invariant under ρ_n. A knot K is n-periodic if it admits such rotational symmetry. n-periodic links are defined analogously.

The importance of periodic links stems from the fact, that according to [21], a 3-manifold M admits an action of the cyclic group of prime order p with the fixed point set being an unknot if, and only if, it can be obtained as a surgery on a p-periodic link. Additionally, M admits a free action of the cyclic group $\mathbb{Z}/p$ if, and only if, it can be obtained as a surgery on a link of the form a p-periodic link L together with the fixed point axis F. Hence, cyclic symmetries of 3-manifolds are determined by the symmetries of their Kirby diagrams.

Another possible application of periodic links is, according to J.H. Przytycki [17], to give a unified theory of skein modules for branched and unbranched coverings. Skein module of a 3-manifold M is a certain algebraic objects associated to M, which serves as a generalization of a certain polynomial link invariant, like the Jones polynomial, for links in M. For more details on skein modules refer to [19].

There are many techniques at hand to study periodic knots. The first significant results were obtained by Trotter in [31], where the author studies actions of the cyclic group on the fundamental group of the complement of the knot, to derive all possible periods of torus links. Murasugi studied periodic links with the aid of the Alexander polynomial in [15], obtaining very strong criterion for detecting periodicity. In [7] authors give partial answers to the converse of the Murasugi's theorem i.e. they consider the question whether a Laurent polynomial, which satisfies the congruence of Murasugi, is the Alexander polynomial of a periodic link.

Several authors studied Jones polynomial of periodic links. The first result in this direction was obtained by Murasugi in [16]. Besides that, [17, 28, 36] give other criteria for detecting periodicity of knots in terms of their Jones polynomial. Several other authors [20, 4, 17, 30, 29, 37] studied SU_n-quantum polynomials and the HOMFLYPT polynomial of periodic links. A summary of these results can be found in [18].

Khovanov in [10] made a breakthrough in knot theory, by constructing certain homology theory of links, called the Khovanov homology, which categorifies the Jones polynomial, i.e., the Jones polynomial can be recovered from the Khovanov homology as an appropriately defined Euler characteristic. Hence, it is natural to ask whether this homology theory can be utilized to study periodic links. The first such trial was made in [5]. However, the author works only with $\mathbb{Z}/2$ coefficients due to certain technical problem with signs, which appeares along the way. Nevertheless, the author obtains an invariant of a periodic link and shows, via transfer argument, that his invariant is isomorphic to the submodule of fixed points of the action on the Khovanov homology.

The purpose of this thesis is to study the equivariant Khovanov homology of periodic links, which considerably generalizes the one constructed in [5]. We give a construction of the equivariant Khovanov homology with integral coeffcients and study its properties such as its relation to the classical Khovanov homology and additional torsion. Next we construct a spectral sequence converging to the equivariant Khovanov homology and use it to compute the 2-equivariant Khovanov homology of torus links. Further we define equivariant analogues of the Jones polynomial and study properties of these polynomials. We prove that they satisfy certain variant of the skein relation and use to to derive certain periodicity criterion, which generalizes the ones given in [17, 28]. We conclude this thesis with a derivation of the state sum formula for the equivariant Jones polynomials, which is applied to recover the congruence from [16].

More specifically, we proceed as follows. In chapter 3 we show that if D is an n-periodic diagram of an n-periodic link L, the Khovanov complex $CKh(D)$ becomes a cochain complex of graded $\mathbb{Z}[\mathbb{Z}/n]$-modules. This enables us to study the Khovanov homology of periodic links with arbitrary coefficients. Next, with the aid of integral representation theory of cyclic groups, we construct the equivariant Khovanov homology – denoted by $Kh^{*,*,*}_{\mathbb{Z}/n}(D)$ – a triply graded homology theory, where the third grading is supported only for $d \mid n$. Further we show that this is indeed an invariant of periodic links, utilizing machinery from [2].

Theorem 3.2.3. Equivariant Khovanov homology groups are invariants of periodic links, that is, they are invariant under equivariant Reidemeister moves.
Next, we show the relation of the equivariant Khovanov homology to the classical Khovanov homology.

Theorem 3.2.4. Let $p_1, \ldots, p_s$ be the collection of all prime divisors of n. Define the ring $R_n = \mathbb{Z}\left[\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_s}\right]$. There exists a natural map

$$\bigoplus_{d|r} \mathrm{Kh}^{*,*,d}_{\mathbb{Z}/n}(L) \to \mathrm{Kh}(L)$$

which, when tensored with R_n, becomes an isomorphism.

Hence, the equivariant Khovanov homology, after collapsing the third grading, encodes the same information as the classical Khovanov homology, modulo torsion of order dividing n.

Further, we analyze the structure of the invariant for trivial links. We have to distinguish two cases. The first case considers the periodic trivial link, whose components are preserved under the action of the cyclic group and the second case considers the periodic trivial link, which posseses components, which are freely permuted by the action of $\mathbb{Z}/n$. In both cases the homology is expressible in terms of the group cohomology of the cyclic group with coefficients in the cyclotomic rings $\mathbb{Z}[\xi_d]$, for $d \mid n$. The result is stated only for the first case provided that the symmetry is of order p^n, for a prime p.

Proposition 3.2.5. Let T_f be an f-component trivial link. The equivariant Khovanov homology of T_f is expressible in terms of the group cohomology of the cyclic group $\mathbb{Z}/p^n$ in the following way.

$$\mathrm{Kh}^{*,*,p^s}_{\mathbb{Z}/p^n}(T_f) = \bigoplus_{i=0}^{f} H^*\left(\mathbb{Z}/p^n, \mathbb{Z}[\xi_{p^s}]\right)^{\binom{f}{i}}\{2i-f\}.$$

Proposition 3.2.11 gives the corresponding result for the second case.

Our next goal is to use additional algebraic structure of the equivariant Khovanov homology to extract some information about the additional torsion. This additional algebraic structure manifest itself in the fact that for any $0 \leqslant s \leqslant n$, $\mathrm{Kh}^{*,*,p^s}_{\mathbb{Z}/p^n}(D)$ is a graded module over the graded ring $\mathbb{E}\mathrm{xt}^*_{\mathbb{Z}[\mathbb{Z}/p^n]}\left(\mathbb{Z}[\xi_{p^s}], \mathbb{Z}[\xi_{p^s}]\right)$. This ring is isomorphic to certain quotient of the polynomial ring $\mathbb{Z}[T_s]$. Analysis of this structure yields the periodicity result, which can be thought of as an analogue of the periodicity of the cohomology groups of the cyclic groups. Below, $n_+(D)$ denotes the number of positive crossings of the link diagram D.

Corollary 3.2.13. Let T_s denote the cohomology class in the ext ring

$$T_s \in \mathrm{Ext}^2_{\mathbb{Z}[\mathbb{Z}/p^n]}\left(\mathbb{Z}[\xi_{p^s}], \mathbb{Z}[\xi_{p^s}]\right)$$

from proposition 2.2.24. Multiplication by T_s

$$- \cup T_s \colon \mathrm{Kh}^{i,*,p^s}_{\mathbb{Z}/p^n}(D) \to \mathrm{Kh}^{i+2,*,p^s}_{\mathbb{Z}/p^n}(D)$$

is an epimorphism for $i = n_+(D)$ and isomorphism for $i > n_+(D)$.

As a consequence we can obtain some information about the additional torsion appearing in the equivariant Khovanov homology.

Corollary 3.2.16. For $i > n_+(D)$, $\mathrm{Kh}^{i,*,1}_{\mathbb{Z}/p^n}(D)$ is annihilated by p^n, and for $1 \leqslant s \leqslant n$, $\mathrm{Kh}^{i,*,p^s}_{\mathbb{Z}/p^n}(D)$ is annihilated by p^{n-s+1}.

Chapter 3 is concluded with some remarks on the structure of the rational equivariant Khovanov homology. These considerations are sufficient to compute the rational equivariant Khovanov homology of torus links $T(n,2)$ and, if $\gcd(n,3) = 1$, for torus knots $T(n,3)$, with respect to the $\mathbb{Z}/d$-symmetry, provided that $d \mid n$ is odd and greater than 2. In all of these cases we have

$$\mathrm{Kh}^{*,*,1}_{\mathbb{Z}/d}(D;\mathbb{Q}) = \mathrm{Kh}^{*,*}(D;\mathbb{Q}),$$
$$\mathrm{Kh}^{*,*,k}_{\mathbb{Z}/d}(D;\mathbb{Q}) = 0, \quad k > 1, \quad k \mid d.$$

Theorem 4.1.11, which is the main result of Chapter 4, yields a spectral sequence converging to the equivariant Khovanov homology of a periodic link. Since the long exact sequence of Khovanov homology, coming from two resolutions of a single crossing of D, cannot be adapted to the equivariant setting, the spectral sequence is supposed to fill in this gap and provide a computational tool. Instead of resolving a single crossing, we resolve crossings from a single orbit. We take all possible resolutions of these crossings and assemble this data into a spectral sequence. This spectral sequence is later used to compute the rational 2-equivariant Khovanov homology of torus links, i.e. the equivariant Khovanov homology with respect to the $\mathbb{Z}/2$-symmetry. It turns out, that something analogous happen as in the case of symmetries of order $d > 2$. Namely, almost always the only non-trivial part is $\mathrm{Kh}^{*,*,1}_{\mathbb{Z}/2}$, with an exception of torus links $T(2n,2)$ for which

$$\mathrm{Kh}^{i,j,2}_{\mathbb{Z}/2}(T(2n,2);\mathbb{Q}) = \begin{cases} \mathbb{Q}, & i = 2n, \quad j = 6n, \\ 0, & \text{otherwise.} \end{cases}$$

In Chapter 5 we take one step back and analyze analogues of the Jones polynomial, which can be derived from the equivariant Khovanov homology. To be more precise, we define the equivariant Jones polynomial in the following way. Choose an n-periodic diagram D and $d \mid n$.

$$J_{n,d}(D) = \sum_{i,j} (-1)^i q^j \dim_{\mathbb{Q}[\xi_d]} \mathrm{Kh}^{i,j,d}_{\mathbb{Z}/n}(D) \in \mathbb{Z}\left[q, q^{-1}\right].$$

However, it turns out that it is better to consider the difference Jones polynomials defined as follows. Let p be an odd prime and let D be a p^n-periodic diagram.

Definition 5.1.2. Suppose that D is a p^n-periodic link diagram. Define the difference Jones polynomials

$$DJ_{n,s}(D) = J_{p^n,p^s}(D) - J_{p^n,p^{s+1}}(D)$$

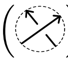
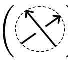
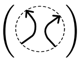
for $0 \leqslant s \leqslant n$.

The first indication, that the difference polynomials have better properties is the following corollary.

Corollary 5.1.3. The following equality holds.

$$J(D) = \sum_{s=0}^{n-1} p^s \, DJ_{n,s}(D) + p^n \, J_{p^n,p^n}(D).$$

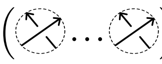
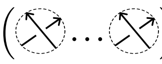
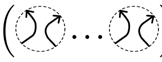
Besides that, it turns out, that the difference polynomials and the classical Jones polynomial have similar properties. For example, the Jones polynomials satisfies the following skein relation.

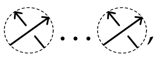
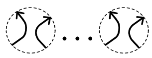
$q^{-2} J\left(\right.$  $\left.\right) - q^2 J\left(\right.$  $\left.\right) = (q^{-1} - q) J\left(\right.$  $\left.\right).$

The main theorem of Chapter 5 shows that a similar result holds for the difference Jones polynomials.

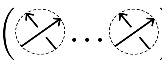
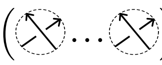
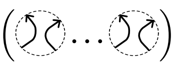
Theorem 5.1.5. If p is an odd prime, then the difference Jones polynomials have the following properties

1. DJ_0 satisfies the following version of the skein relation

   $q^{-2p^n} DJ_{n,0}\left(\right.$  $\left.\right) - q^{2p^n} DJ_{n,0}\left(\right.$  $\left.\right) =$
   $= \left(q^{-p^n} - q^{p^n}\right) DJ_{n,0}\left(\right.$  $\left.\right),$

   where ,  and  denote the orbit of positive, negative and orientation preserving resolutions of crossing, respectively.

2. For any $0 \leqslant s < n$, DJ_s satisfies the following congruences

   $q^{-2p^n} DJ_{n,n-s}\left(\right.$  $\left.\right) - q^{2p^n} DJ_{n,n-s}\left(\right.$  $\left.\right) \equiv$
   $\equiv \left(q^{-p^n} - q^{p^n}\right) DJ_{n,s}\left(\right.$  $\left.\right) \pmod{q^{p^s} - q^{-p^s}}.$

The above theorem has several interesting corollaries. The congruences from [17, 28] follows from this theorem immediately. Furthermore, we can use some other properties of the difference polynomials to strengthen this result.

Theorem 5.1.8. Suppose that L is p^n-periodic link and for all i, j we have $\dim_{\mathbb{Q}} Kh^{i,j}(L;\mathbb{Q}) < \varphi(p^s)$, then the following congruence holds

$$J(L)(q) \equiv J(L)(q^{-1}) \pmod{\mathcal{I}_{p^n,s}},$$

where $\mathcal{J}_{p^n,s}$ is an ideal generated by the following monomials

$$q^{p^n} - q^{-p^n}, p\left(q^{p^{n-1}} - q^{-p^{n-1}}\right), \dots, p^{s-1}\left(q^{p^{n-s+1}} - q^{-p^{n-s+1}}\right).$$

Example 5.1.9 shows that the above theorem is indeed stronger than the one from [17, 28].

We conclude Chapter 5 with considerations regarding state sums for the difference polynomials. We prove the following analogue of [9, Prop. 2.2].

Theorem 5.2.1. Let D be a p^n-periodic diagram of a link. Then for any $0 \leqslant m \leqslant n$ we have the following equality

$$DJ_{n,n-m}(D) = (-1)^{n_-} q^{n_+ - 2n_-} \sum_{m \leqslant \nu \leqslant n} \sum_{\substack{s \in \mathcal{S}(D) \\ \operatorname{Iso}(s) = \mathbb{Z}/p^\nu}} (-q)^{r(s)} DJ_{p^\nu, p^{\nu-s}}(s).$$

For a Kauffman state s we write $r(s) = r$ if $s \in \mathcal{S}_r(D)$, compare Definition 3.1.6.

We show that the Murasugi criterion from [16] follows from the above state sum expansion.

For the sake of the reader we devote Chapter 2 to survey all the necessary material crucial in the remainder part of this material. This exposition is very concise, hence, we refer the interested reader to the more detailed exposition to [6], for material from representation theory, to [13, 26, 27, 35] for homological algebra and to [2, 10, 32] for the Khovanov homology.

2 PRELIMINARIES

2.1 REPRESENTATION THEORY

Before we start, we will briefly recall some notions from representation theory that are essential in the remainder part of this thesis. The exposition of the material in this section is based on [6].

2.1.1 *Rational representation theory*

Let M be a finite-dimensional $\mathbb{Q}[\mathbb{Z}/n]$-module, for some $n > 1$.

Definition 2.1.1. Define the *character* of M to be the function

$$\chi_M \colon \mathbb{Z}/n \to \mathbb{Q}$$
$$\chi_M(g) = \operatorname{tr} \rho(g),$$

where

$$\rho \colon \mathbb{Z}/n \to \operatorname{Aut}(M)$$

is the representation which determines the module structure of M.

Proposition 2.1.2. Let M_1, M_2 be two $\mathbb{Q}[\mathbb{Z}/n]$-modules and let χ_{M_1} and χ_{M_2} be their characters.

1. If $M_1 \cong M_2$ as $\mathbb{Q}[\mathbb{Z}/n]$-modules, then $\chi_{M_1} = \chi_{M_2}$.
2. $\chi_{M_1 \oplus M_2} = \chi_{M_1} + \chi_{M_2}$.
3. $\chi_{M_1 \otimes_{\mathbb{Q}} M_2} = \chi_{M_1} \cdot \chi_{M_2}$.

Example 2.1.3. Consider the group algebra $\mathbb{Q}[\mathbb{Z}/n]$. It is isomorphic to the following quotient of the polynomial algebra

$$\mathbb{Q}[\mathbb{Z}/n] \cong \mathbb{Q}[t]/(t^n - 1).$$

However, the polynomial $t^n - 1$ can be further decomposed over $\mathbb{Q}$

$$t^n - 1 = \prod_{d|n} \Phi_d(t),$$

where

$$\Phi_d(t) = \prod_{\substack{1 \leqslant k \leqslant d \\ \gcd(k,d)=1}} (t - \xi_d^k),$$

and ξ_d is the primitive root of unity of order d

$$\xi_d = \exp\left(\frac{2\pi i}{d}\right).$$

The above implies that $\mathbb{Q}[\mathbb{Z}/n]$ admits the following decomposition

$$\mathbb{Q}[\mathbb{Z}/n] = \bigoplus_{d|n} \mathbb{Q}[\xi_d],$$

where

$$\mathbb{Q}[\xi_d] = \mathbb{Q}[t]/(\Phi_d(t))$$

is the d-th cyclotomic field. Denote by $\chi_{d,n}$ the character of the $\mathbb{Q}[\mathbb{Z}/n]$-module $\mathbb{Q}[\xi_d]$.

The above decomposition exemplifies the so called Wedderburn decomposition of semi-simple artinian algebras.

Theorem 2.1.4. The group algebra $\mathbb{Q}[\mathbb{Z}/n]$ is a semi-simple artinian algebra, hence every finitely-generated $\mathbb{Q}[\mathbb{Z}/n]$-module decomposes into a direct sum of irreducible modules. Every irreducible $\mathbb{Q}[\mathbb{Z}/n]$-module is isomorphic to $\mathbb{Q}[\xi_d]$ for some $d \mid n$.

Proposition 2.1.5 (Schur's Lemma)**.** If M and N are two finite-dimensional and irreducible $\mathbb{Q}[\mathbb{Z}/n]$-modules which are not isomorphic, then

$$\mathrm{Hom}_{\mathbb{Q}[\mathbb{Z}/n]}(M, N) = 0.$$

Example 2.1.6. Consider the cyclic group $\mathbb{Z}/p^n$, where p is a prime. Let $0 \leqslant s < n$, $0 \leqslant j \leqslant p^{n-s} - 1$ and $0 \leqslant m \leqslant p^s - 1$. The characters of $\mathbb{Z}/p^n$ are given by the following formulas.

$$\chi_{1,p^n}(t^j) = 1,$$

$$\chi_{p^{n-s},p^n}(t^{j+m\cdot p^{n-s}}) = \begin{cases} \phi(p^{n-s}), & j = 0, \\ -p^{n-s-1}, & j \mid p^{n-s-1}, \\ 0, & \text{otherwise}, \end{cases}$$

where $0 \leqslant j \leqslant p^{n-s} - 1$.

Definition 2.1.7. Let $e_1, \ldots, e_k$ be a set of central idempotents in a semi-simple $\mathbb{Q}$-algebra A. We say that $e_1, \ldots, e_k$ are *orthogonal idempotents* in A if the following conditions are satisfied

1. $e_1 + \ldots + e_k = 1$,
2. $e_i \cdot e_j = 0$, for any $1 \leqslant i < j \leqslant k$.

Furthermore we say that an idempotent e is *primitive* if it cannot be written as a sum $e = e' + e''$, where e' and e'' are idempotents such that $e' \cdot e'' = 0$.

If $\{e_1, \dots, e_k\}$ is a set of central orthogonal and primitive idempotents in a semi-simple $\mathbb{Q}$-algebra A, then simple ideals in the Wedderburn decomposition of A are principal ideals generated by the idempotents e_i. In particular decomposition of $\mathbb{Q}[G]$ from Example 2.1.3 can be obtained from the set $\{e_d \colon d \mid n\}$, where e_d acts on $\mathbb{Q}[\xi_d]$ by identity and anihilates other irreducible modules.

$$\mathbb{Q}[\mathbb{Z}/n] = \bigoplus_{d|n} \mathbb{Q}[\mathbb{Z}/n] \cdot e_d. \tag{1}$$

Example 2.1.8. The set of central orthogonal and primitive idempotents for $\mathbb{Q}[\mathbb{Z}/p^n]$ can be obtained from its characters.

$$e_1 = \frac{1}{p^n} \sum_{j=0}^{p^n-1} t^j$$
$$e_{p^{n-s}} = \frac{p-1}{p^{s+1}} \sum_{m=0}^{p^s-1} t^{m \cdot p^{n-s}} - \frac{1}{p^{s+1}} \sum_{j=1}^{p-1} \sum_{m=0}^{p^s-1} \left(t^{p^{n-s-1}}\right)^{j+m \cdot p}$$

for $0 \leqslant s < n$.

Let now $H \subset G$ be finite groups.

Definition 2.1.9. Let M be a $\mathbb{Q}[H]$-module. One can construct a $\mathbb{Q}[G]$-module, called the *induced module*, using M in the following way

$$\mathrm{Ind}_H^G M = M \otimes_{\mathbb{Q}[H]} \mathbb{Q}[G],$$

where we treat $\mathbb{Q}[G]$ as a $\mathbb{Q}[H]$-module via the map

$$\mathbb{Q}[H] \to \mathbb{Q}[G]$$

induced by the embedding of $H \hookrightarrow G$.

Remark 2.1.10. Induction can be analogously defined for other group rings $R[G]$, for R a commutative ring with unit.

If $\{g_1, \dots, g_k\}$ yield a system of representatives of the cosets of G/H, then

$$\mathbb{Q}[G] = \bigoplus_{i=1}^{k} \mathbb{Q}[H]\, g_i$$

which implies that

$$\mathrm{Ind}_H^G M = \bigoplus_{i=1}^{k} M \otimes_{\mathbb{Q}[G']} \mathbb{Q}[H]\, g_i = \bigoplus_{i=1}^{k} M g_i.$$

The action of G on $\mathrm{Ind}_H^G M$ is defined as follows. For each $g \in G$ there are unique $1 \leqslant j \leqslant k$ and $h \in H$ such that

$$g_i \cdot g = h \cdot g_j.$$

Therefore

$$(Mg_i) \cdot g = Mg_j$$

and the corresponding map of M corresponds to the action of h.

The next proposition, despite being stated only for the rational group algebra, remains true for general group ring.

Proposition 2.1.11. Let M be a left $\mathbb{Q}[G]$-module whose restriction to a subgroup H contains a $\mathbb{Q}[H]$-module L and admits a decomposition into a direct sum of vector spaces

$$M = \bigoplus_{i=1}^{k} Lg_i,$$

then M is isomorphic to $\operatorname{Ind}_H^G L$.

Definition 2.1.12. Let M be a $\mathbb{Q}[G]$-module. M can be also treated as a $\mathbb{Q}[H]$-module. This operation is called *restriction* and we denote it by

$$\operatorname{Res}_H^G M.$$

Example 2.1.13. It is not hard to derive, from Example 2.1.6, the following formulas for the restriction of characters χ_{p^{n-s},p^n}.

$$\operatorname{Res}_{\mathbb{Z}/p^m}^{\mathbb{Z}/p^n} \xi_{p^{n-s},p^n} = \begin{cases} \varphi(p^{n-s})\xi_{1,p^m}, & m \leqslant s, \\ p^{n-m}\chi_{p^{m-s},p^m}, & m > s. \end{cases}$$

2.1.2 *Integral representation theory*

Definition 2.1.14. Let A be a semi-simple and finite dimensional $\mathbb{Z}$-algebra. We say that $\Lambda \subset A$ is a $\mathbb{Z}$-*order* if it is a subring of A which containts the unit and some $\mathbb{Q}$-basis of A.

In this thesis one of the most common example of a $\mathbb{Z}$-order is the group ring $\mathbb{Z}[G]$ contained in $\mathbb{Q}[G]$.

Definition 2.1.15. Let Λ be a $\mathbb{Z}$-order in a semi-simple algebra A. We say that Λ is *maximal* if it is not contained in any other $\mathbb{Z}$-order in A.

Theorem 2.1.16. Let A be a finite dimensional semi-simple $\mathbb{Q}$-algebra.

1. Every $\mathbb{Z}$-order $\Lambda \subset A$ is contained in a some maximal order Λ'.

2. If A is commutative, then it possesses a unique maximal order.

3. If we are given a Wedderburn decomposition of A

   $$A = A_1 \oplus \ldots \oplus A_k,$$

   then every maximal order $\Lambda' \subset A$ admits a decmposition into a direct sum of ideals

   $$\Lambda' = \Lambda'_1 \oplus \ldots \oplus \Lambda'_k,$$

   where each $\Lambda'_i \subset A_i$ is a maximal order in A_i for $i = 1, \ldots, k$.

Example 2.1.17. The group algebra $\mathbb{Q}[\mathbb{Z}/n]$ admits the following Wedderburn decomposition

$$\mathbb{Q}[\mathbb{Z}/n] = \bigoplus_{d|n} \mathbb{Q}[\xi_d].$$

Maximal order in $\mathbb{Q}[\xi_d]$ is equal to the ring of cyclotomic integers $\mathbb{Z}[\xi_d]$. Therefore, the unique maximal order $\Lambda' \subset \mathbb{Q}[\mathbb{Z}/n]$ is equal to the following direct sum

$$\Lambda' = \bigoplus_{d|n} \mathbb{Z}[\xi_d].$$

It is also worth to mention that Λ' is the subring of $\mathbb{Q}[\mathbb{Z}/n]$ generated by the idempotents e_d for $d \mid n$ and

$$\mathbb{Z}[\xi_d] = \Lambda' e_d.$$

Proposition 2.1.18. Let Λ' be a maximal order in $\mathbb{Q}[\mathbb{Z}/n]$. Under this assumption, the following chain of inclusions is satisfied.

$$\mathbb{Z}[\mathbb{Z}/n] \subset \Lambda' \subset \frac{1}{n}\mathbb{Z}[\mathbb{Z}/n].$$

Therefore, there exists an exact sequence

$$0 \to \mathbb{Z}[\mathbb{Z}/n] \to \Lambda' \to M \to 0,$$

where $n \cdot M = 0$.

For the sake of the next chapter we state the following proposition giving explicit formulas for the restrictions of certain $\mathbb{Z}[\mathbb{Z}/n]$-modules.

Proposition 2.1.19. Let p be a prime and n a positive integer. Choose $0 \leqslant s, m \leqslant n$, then

$$\mathrm{Res}^{\mathbb{Z}[\mathbb{Z}/p^n]}_{\mathbb{Z}[\mathbb{Z}/p^m]} \mathbb{Z}\left[\xi_{p^{n-s}}\right] = \begin{cases} \mathbb{Z}^{\varphi(p^{n-s})}, & m \leqslant s, \\ \mathbb{Z}\left[\xi_{p^{m-s}}\right]^{p^{n-m}}, & m > s. \end{cases}$$

Proof. This follows from Example 2.1.13, because $\mathrm{Res}^{\mathbb{Z}/p^n}_{\mathbb{Z}/p^m} \mathbb{Z}\left[\xi_{p^{n-s}}\right]$ is the maximal order in $\mathrm{Res}^{\mathbb{Z}/p^n}_{\mathbb{Z}/p^m} \mathbb{Q}\left[\xi_{p^{n-s}}\right]$. □

2.2 HOMOLOGICAL ALGEBRA

Apart from tools from representation theory, some elements of homological algebra will be of great importance in the constructions performed later. The purpose of this section is to present all the necessary material from homological algebra. The exposition is based on [3], [13], [26] and [27].

For the sake of this section, assume that R is a commutative ring with unit. Furthermore, all cochain complexes in question are cochain complexes of finitely generated R-modules.

2.2.1 *Spectral sequences*

Spectral sequence is a very important computational tool in contemporary mathematics. Its manifestations are abundant in topology, geometry and algebra. Since one of the next chapters of this thesis is concerned with a construction of certain spectral sequence converging to the equivariant Khovanov homology, we briefly recall all the necessary background material.

Definition 2.2.1. Let M^* be a graded R-module. Define the shifted module $M\{n\}$, for an integer n, to be

$$M\{n\}^k = M^{k-n}.$$

Definition 2.2.2. Let H^* be a graded R-module. A *decreasing filtration* $\mathcal{F}$ on H^* is a decreasing family of submodules

$$\ldots \subset \mathcal{F}_{i+1} \subset \mathcal{F}_i \subset \mathcal{F}_{i-1} \subset \ldots.$$

Filtration $\mathcal{F}$ is called *bounded* if there are i_0, i_1 such that $\mathcal{F}_i = 0$ for $i > i_1$ and $\mathcal{F}_i = H$ for $i < i_0$. The pair $(H^*, \mathcal{F})$ is called *filtered graded module*.

All filtrations considered in this thesis will be bounded. So from now on, we assume that whenever we have a filtration on a graded module, then it is finite without further notice.

Definition 2.2.3. Let $(H^*, \mathcal{F})$ be a filtered graded R-module. The *associated bigraded module* $E_0^{*,*}(H^*)$ is defined as follows.

$$E_0^{p,q}(H^*) = \left(\mathcal{F}_p \cap H^{p+q}\right) / \left(\mathcal{F}_{p+1} \cap H^{p+q}\right).$$

Let C^* be a filtered cochain complex. Suppose that the differential of C^*, denoted by d, preserves the filtration. In that case d induces a map

$$d_0 \colon E_0^{p,q}(C^*) \to E_0^{p,q+1}(C^*)$$

on the associated bigraded module. This map squares to 0. Therefore, each column $E_0^{p,*}$ becomes a cochain complex. Define another bigraded module $E_1^{*,*} = H(E_0^{*,*}, d_0)$. The short exact sequence

$$0 \to \mathcal{F}_{p+1}/\mathcal{F}_{p+2} \to \mathcal{F}_p/\mathcal{F}_{p+2} \to \mathcal{F}_p/\mathcal{F}_{p+1} \to 0,$$

yields a map,

$$d_1 \colon E_1^{p,q} = H^{p+q}(\mathcal{F}_p/\mathcal{F}_{p+1}) \to H^{p+q+1}(\mathcal{F}_{p+1}/\mathcal{F}_{p+2}) = E_1^{p+1,q}.$$

This map also squares to 0. Hence, one can define $E_2^{*,*} = H(E_1^{*,*}, d_1)$. Proceeding further in an analogous manner we obtain a spectral sequence.

Definition 2.2.4. A *cohomological spectral sequence* is a sequence of bi-graded modules and homomorphisms $\{E_r^{*,*}, d_r\}$, for $r \geqslant 0$, such that the following conditions hold.

1. d_r is a differential of bidegree $(r, 1-r)$,
2. $E_{r+1}^{*,*} = H(E_r^{*,*}, d_r)$.

In principal, a spectral sequence can have non-trivial differentials on infinitely many pages. However, all spectral sequences considered in this thesis stop at a finite stage.

Definition 2.2.5. Let $\{E_r^{*,*}, d_r\}$ be a cohomological spectral sequence. We say that this spectral sequence *collapses at* N*-th stage* if $d_r = 0$ for $r \geqslant N$. If this is the case we define $E_\infty^{p,q} = E_N^{p,q} = E_{N+1}^{p,q} = E_{N+2}^{p,q} = \ldots$.

In general, spectral sequence is used to obtain some information about the homology of some cochain complex. Unfortunately, often the result of the computation does not determine uniquely the desired homology groups. Instead, it is defined up to extension of modules. This is not an issue when we work with vector spaces over a field, yet it might cause a lot of troubles when we work over other rings.

Definition 2.2.6. Let $(H^*, \mathcal{F})$ be a filtered graded module and let $\{E_r^{*,*}, d_r\}$ be a cohomological spectral sequence which collapses at N-th stage. Then one says that *the spectral sequence converges to* H^* if

$$E_\infty^{p,q} = E_0^{p,q}(H).$$

Many spectral sequences arise from a filtered cochain complex. The starting point is the define compatible filtration on the homology of the cochain complex.

Proposition 2.2.7. Let C^* be a cochain complex with a bounded and decreasing filtration $\mathcal{F}$. Suppose that the differential of C^* preserves the filtration i.e.

$$d(\mathcal{F}_i) \subset \mathcal{F}_i.$$

Under this assumption, there is an induced decreasing filtration on the homology module $H^*(C)$ defined in the following way. An element $x \in H^*(C)$ belongs to $\mathcal{F}_i(H^*(C))$ if it can be represented by a cycle $z \in \mathcal{F}_i$.

Theorem 2.2.8. Let C^* be a cochain complex with bounded and decreasing filtration $\mathcal{F}$. Then, there exists a cohomological spectral sequence $\{E_r^{*,*}, d_r\}$ which converges to $H^*(C)$ with the induced filtration described in the previous definition. Furthermore,

$$E_1^{p,q} = H^{p+q}(\mathcal{F}_p/\mathcal{F}_{p+1}).$$

2.2.2 *Ext groups*

In classical homological algebra, when we work with modules which are neither projective nor injective, we usually have to deal with lack of exactness of certain functors. To be more precise, functors like tensor product or Hom cease to be exact in such cases. This is remedied by substituting a module by its projective or injective resolution. The same strategy can be employed when we work with cochain complexes, thus obtaining the derived versions of the respective functors. The purpose of this section is to sketch the theory of the derived Hom functor.

To fix the notation, assume that C^*, D^* and E^* are bounded cochain complexes.

Definition 2.2.9. Let C^* be a cochain complex. For $n \in \mathbb{Z}$ denote by $C[n]^*$ a new cochain complex obtained from C^* by applying the following shift

$$
\begin{aligned}
C[n]^k &= C^{k-n}, \\
d^k_{C[n]} &= (-1)^n d^{k-n}_C.
\end{aligned}
$$

Definition 2.2.10. Let $f\colon C^* \to D^*$ be a chain map. We say that f is a *quasi-isomorphism* if $f^*\colon H^*(C) \to H^*(D)$ is an isomorphism.

The category of cochain complexes can be equipped with the structure of the differential graded category i.e. morphism sets can be made into cochain complexes themselves.

Definition 2.2.11. For two cochain complexes, C^* and D^*, define the Hom complex $\mathrm{Hom}^*_R(C^*, D^*)$ to be

$$\mathrm{Hom}^n_R(C^*, D^*) = \prod_{p \in \mathbb{Z}} \mathrm{hom}_R(C^p, D^{p+n}), \quad n \in \mathbb{Z}$$

and equip it with the following differential.

$$d^{C^*,D^*}(\psi_p)_p = \left(d_D \circ \psi_p - (-1)^p \psi_{p+1} \circ d_C\right)_p$$

Remark 2.2.12. Notice that since C^* and D^* are bounded, the Hom complex $\mathrm{Hom}^*_R(C^*, D^*)$ is also bounded.

The next proposition is a mere reformulation of the definition of the chain homotopy.

Proposition 2.2.13. The following equalities hold

$$H^n\left(\mathrm{Hom}^*_R(C^*, D^*)\right) = [C^*, D^*[-n]],$$

where the outer square brackets denote the set of homotopy classes of chain maps.

As was mentioned earlier, in order to preserve the exactness of the Hom functor, we need to replace every cochain complex by another one, which, in some sense, does not differ to much from the initial cochain complex. If we expect the new cochain complex to preserve the exactness of the Hom functor, it should preferably consist of either projective or injective modules. Therefore, let us define the injective resolution of a cochain complex.

Definition 2.2.14. Let C^* be cochain complex. Let I^* be bounded below cochain complex of injective modules. We say that I^* is an *injective resolution* of C^* if there exists a quasi-isomorphism

$$C^* \to I^*.$$

Categories of modules have enough injectives. This property enables us to construct an injective resolution for any module. It turns out that the same condition is sufficient to construct an injective resolution for a cochain complex, provided that the complex satisfies certain technical condition.

Proposition 2.2.15. Let C^* be a cochain complex. The complex possesses an injective resolution if, and only if $H^n(C) = 0$, for small enough n. In particular, if C^* is bounded, then it possesses an injective resolution.

Now we are ready to define the derived Hom functor and Ext groups.

Definition 2.2.16. Denote by I^* an injective resolution of a cochain complex D^*. Define the derived Hom

$$R\,\mathrm{Hom}_R^*(C^*, D^*) = \mathrm{Hom}_R^*(C^*, I^*).$$

Ext groups are defined as the homology of the derived Hom.

$$\mathbb{E}\mathrm{xt}_R^n(C^*, D^*) = H^n(\mathrm{Hom}_R^*(C^*, I^*)).$$

In other words,

$$\mathbb{E}\mathrm{xt}_R^n(C^*, D^*) = [C^*, I^*[-n]].$$

Remark 2.2.17. Of course, the derived Hom does not depend on the choice of I^*, up to quasi-isomorphism. Analogously, Ext groups are well defined up to isomorphism.

Properties of classical Ext groups extended to their generalized version.

Proposition 2.2.18. Let C^* and D^* be as in the previous definition.

1. If $C^{'*}$ and $D^{'*}$ are bounded cochain complexes quasi-isomorphic to C^* and D^*, respectively, then these quasi-isomorphisms induce isomorphisms,

$$\mathbb{E}\mathrm{xt}_R^n(C^*, D^*) \cong \mathbb{E}\mathrm{xt}_R^n\left(C^{'*}, D^{'*}\right), \quad n \in \mathbb{Z}.$$

2. If
$$0 \to C_1^* \to C_2^* \to C_3^* \to 0$$
is a short exact sequence of bounded cochain complexes, then there exists a long exact sequence of Ext groups
$$\ldots \to \mathbb{E}\mathrm{xt}_R^n\,(C_3^*, D^*) \to \mathbb{E}\mathrm{xt}_R^n\,(C_2^*, D^*) \to \mathbb{E}\mathrm{xt}_R^n\,(C_1^*, D^*) \to \ldots.$$

3. Analogously, if
$$0 \to D_1^* \to D_2^* \to D_3^* \to 0$$
is a short exact sequence of bounded cochain complexes, then there exists a long exact sequence of Ext groups
$$\ldots \to \mathbb{E}\mathrm{xt}_R^n\,(C^*, D_1^*) \to \mathbb{E}\mathrm{xt}_R^n\,(C^*, D_2^*) \to \mathbb{E}\mathrm{xt}_R^n\,(C^*, D_3^*) \to \ldots.$$

4. There exist bilinear maps, induced by composition of maps on the cochain level,
$$\mu\colon\ \mathbb{E}\mathrm{xt}_R^n\,(C^*, D^*) \times \mathbb{E}\mathrm{xt}_R^m\,(B^*, C^*) \to \mathbb{E}\mathrm{xt}_R^{n+m}\,(B^*, D^*)\,.$$

   Hence, $\mathbb{E}\mathrm{xt}_R^*\,(C^*, C^*)$ and $\mathbb{E}\mathrm{xt}_R^*\,(D^*, D^*)$ are graded rings. Additionally $\mathbb{E}\mathrm{xt}_R^*\,(C^*, D^*)$ can be eqquipped with the structure of a graded $(\mathbb{E}\mathrm{xt}_R^*\,(D^*, D^*)\,, \mathbb{E}\mathrm{xt}_R^*\,(C^*, C^*))$-bimodule.

Classical result of Cartan and Eilenberg gives two spectral sequences which converge to the respective Ext groups.

Theorem 2.2.19. There are two spectral sequences $\{{}_{I}E_r^{*,*}, d_r\}$, $\{{}_{II}E_r^{*,*}, d_r\}$ converging to $\mathbb{E}\mathrm{xt}_R^*\,(C^*, D^*)$ satisfying
$$\begin{aligned} {}_{I}E_2^{p,q} &= \mathbb{E}\mathrm{xt}_R^p\,(C^*, H^q(D)) \\ {}_{II}E_2^{p,q} &= H^q\,(\mathbb{E}\mathrm{xt}_R^*\,(C^*, D^p)) \end{aligned}$$

Part 4 of Proposition 2.2.18 shows that the Ext group possess a multiplicative structure. This additional structure is derived from certain bilinear maps defined on the cochain level. The bilinear map is compatible with the filtration of Cartan and Eilenberg, thus its existence is manifested in the Cartan-Eilenberg spectral sequence.

Theorem 2.2.20. For cochain complexes B^*, C^* and D^* there are bilinear maps of spectral sequences
$$\begin{aligned} \mu\colon {}_{I}E_r^{p,q}(C^*, D^*) \times {}_{I}E_r^{p',q'}(B^*, C^*) &\to {}_{I}E_r^{p+p',q+q'}(B^*, D^*) \\ \mu\colon {}_{II}E_r^{p,q}(C^*, D^*) \times {}_{II}E_r^{p',q'}(B^*, C^*) &\to {}_{II}E_r^{p+p',q+q'}(B^*, D^*) \end{aligned}$$

commuting with differentials i.e.
$$d_r^{B^*,D^*}(\mu(x,y)) = \mu(d_r^{C^*,D^*}(x), y) + (-1)^{p+q}\mu(x, d_r^{B^*,C^*}(y)),$$
and converging to bilinear maps from Proposition 2.2.18.

When a cochain complex D^* is equipped with a filtration, the Hom complex $\text{Hom}_R(C^*, D^*)$ becomes filtered. This filtration is defined by considering homomorphisms whose images are contained in the respective submodule of the filtration of D^*. Moreover, the filtration of D^* induces a filtration on the derived Hom complex. This leads to a spectral sequence.

Theorem 2.2.21. Suppose that D^* is a bounded and filtered cochain complex. Then there exists a spectral sequence $\{E_r^{*,*}, d_r\}$ converging to $\mathbb{E}\text{xt}_R^n(C^*, D^*)$ such that

$$E_1^{p,q} = \mathbb{E}\text{xt}_R^{p+q}\left(C^*, \mathcal{F}_p(D^*)/\mathcal{F}_{p+1}(D^*)\right).$$

The next three propositions supply us with certain computational tools needed later.

Proposition 2.2.22 (Eckmann-Shapiro Lemma)**.** Suppose that $H \subset G$ are finite groups and M is a $\mathbb{Z}[G]$-module and N is a $\mathbb{Z}[H]$-module.

$$\text{Ext}^n_{\mathbb{Z}[H]}\left(N, \text{Res}^G_H M\right) \cong \text{Ext}^n_{\mathbb{Z}[G]}\left(\text{Ind}^G_H N, M\right)$$
$$\text{Ext}^n_{\mathbb{Z}[H]}\left(\text{Res}^G_H M, N\right) \cong \text{Ext}^n_{\mathbb{Z}[G]}\left(M, \text{Ind}^G_H N\right).$$

Proposition 2.2.23. Let G and H be as in the previous proposition and suppose that C^*, D^* are cochain complexes of $\mathbb{Z}[G]$ and $\mathbb{Z}[H]$-modules, respectively. Under this assumptions, there exists the following isomorphism.

$$\mathbb{E}\text{xt}^n_{\mathbb{Z}[G]}\left(C^*, \text{Ind}^G_H D^*\right) \cong \mathbb{E}\text{xt}^n_{\mathbb{Z}[H]}\left(\text{Res}^G_H C^*, D^*\right).$$

Proof. The proof is very similar to the proof of Eckmann-Shapiro Lemma.

$$\text{Hom}^m_{\mathbb{Z}[G]}\left(C^*, \text{Ind}^G_H D^*\right) = \prod_{p\in\mathbb{Z}} \text{Hom}_{\mathbb{Z}[G]}\left(C^p, \text{Ind}^G_H D^{p+n}\right)$$

From [3, Prop. 5.9] it follows that

$$\text{Ind}^G_H D^{p+n} \cong \text{Hom}_{\mathbb{Z}[H]}\left(\mathbb{Z}[G], D^{p+n}\right).$$

The above isomorphism is defined in the following way. First, define $\mathbb{Z}[H]$-maps

$$\phi\colon D^{p+n} \to \text{Hom}_{\mathbb{Z}[H]}\left(\mathbb{Z}[G], D^{p+n}\right)$$

such that

$$\phi(m)(g) = \begin{cases} gm, & g \in H, \\ 0, & g \notin H. \end{cases}$$

Homomorphism ϕ admits a unique extension to the following $\mathbb{Z}[G]$-map

$$\overline{\phi}\colon \text{Ind}^G_H D^{p+n} \to \text{Hom}_{\mathbb{Z}[H]}\left(\mathbb{Z}[G], D^{p+n}\right).$$

For more details see for example [3, Chap. III.3].

From the above discussion we obtain the following isomorphisms.

$$\begin{aligned}&\operatorname{Hom}_{\mathbb{Z}[G]}\left(C^p, \operatorname{Ind}_H^G D^{p+n}\right) \cong \\ &\cong \operatorname{Hom}_{\mathbb{Z}[G]}\left(C^p, \operatorname{Hom}_{\mathbb{Z}[H]}\left(\mathbb{Z}[G], D^{p+n}\right)\right) \cong \\ &\cong \operatorname{Hom}_{\mathbb{Z}[H]}\left(\operatorname{Res}_H^G C^p, D^{p+n}\right).\end{aligned}$$

The above isomorphisms commute with the differentials $d^{C^*, \operatorname{Ind}_H^G D^*}$ and $d^{\operatorname{Res}_H^G C^*, D^*}$. Consequently,

$$\operatorname{Hom}^*_{\mathbb{Z}[G]}\left(C^*, \operatorname{Ind}_H^G D^*\right) \cong \operatorname{Hom}^*_{\mathbb{Z}[H]}\left(\operatorname{Res}_H^G C^*, D^*\right).$$

This concludes the proof. □

Proposition 2.2.24. Let n be a positive integer and let d be its divisor. Then, there exists an isomorphism

$$\operatorname{Ext}^*_{\mathbb{Z}[\mathbb{Z}/p^n]}\left(\mathbb{Z}[\xi_{p^s}], \mathbb{Z}[\xi_{p^s}]\right) \cong \mathbb{Z}[\xi_{p^s}][T]/(\Phi_{p^s,p^n}(\xi_{p^s})T),$$

where

$$\Phi_{p^s,p^n}(t) = \frac{t^{p^n}-1}{\Phi_{p^s}(t)}$$

and $T_S \in \operatorname{Ext}^2_{\mathbb{Z}[\mathbb{Z}/p^n]}\left(\mathbb{Z}[\xi_{p^s}], \mathbb{Z}[\xi_{p^s}]\right)$ is a class represented by the following Yoneda extension

$$0 \to \mathbb{Z}[\xi_{p^s}] \to \mathbb{Z}[\mathbb{Z}/p^n] \overset{\Phi_{p^s}(t)}{\to} \mathbb{Z}[\mathbb{Z}/p^n] \to \mathbb{Z}[\xi_{p^s}] \to 0.$$

Additionally, for any $\mathbb{Z}[\mathbb{Z}/p^n]$-module N, multiplication by T_s

$$-\cup T_s \colon \operatorname{Ext}^i_{\mathbb{Z}[\mathbb{Z}/p^n]}\left(\mathbb{Z}[\xi_{p^s}], N\right) \to \operatorname{Ext}^{i+2}_{\mathbb{Z}[\mathbb{Z}/p^n]}\left(\mathbb{Z}[\xi_{p^s}], N\right)$$

is an isomorphism for $i > 0$ and epimorphism for $i = 0$. In particular

$$\operatorname{Ext}^{2i}_{\mathbb{Z}[\mathbb{Z}/p^n]}\left(\mathbb{Z}[\xi_{p^s}], \mathbb{Z}[\xi_{p^s}]\right) = \begin{cases} \mathbb{Z}/p^m, & i > 0, s = 0, \\ \mathbb{Z}/p^{m-s+1}, & i > 0, s > 0, \\ \mathbb{Z}[\xi_{p^s}], & i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The first part follows from [34, Lemma 1.1]. To prove the second part, notice that-

$$\begin{aligned}\Phi_{p^s,p^m}(\xi_{p^s}) &= \lim_{z \to \xi_{p^s}} \frac{z^{p^m}-1}{\Phi_{p^s}(z)} = \lim_{z \to \xi_{p^s}} (z^{p^{s-1}}-1)\frac{z^{p^m}-1}{z^{p^s}-1} = \\ &= p^{m-s}(\xi_p - 1).\end{aligned}$$

by de L'Hospital rule. Since the algebraic norm of $\xi_p - 1$ is equal to p, it follows readilly that

$$\mathbb{Z}[\xi_{p^s}]/(\Phi_{p^s,p^n}(\xi_{p^s})) \cong \mathbb{Z}/p^{n-s+1}.$$

□

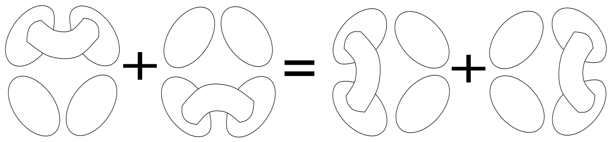


Figure 1: *4Tu* relation

2.3 BAR-NATAN'S BRACKET OF A LINK

This section is devoted to defining the most important concept of this thesis – the Khovanov homology. For more details on Khovanov homology consult [2], [10] or [32].

2.3.1 *Construction of the bracket*

Definition 2.3.1. Let $\mathrm{Cob}^3_\ell(2n)$, for a non-negative integer n, denote the category with objects and morphisms described below.

1. Objects in $\mathrm{Cob}^3_\ell(2n)$ are crossingless tangles in D^2 with exactly $2n$ endpoints lying on the boundary ∂D^2.
2. Morphisms between tangles T_1 and T_2 are formal linear combinations of isotopy classes, rel boundary, of oriented cobordisms from T_1 to T_2. These cobordisms are required to be collared near the boundary, so that glueing is well defined.
3. Composition of morphisms $\Sigma_1 \colon T_1 \to T_2$ and $\Sigma_2 \colon T_2 \to T_3$ is realised by glueing surfaces along a common boundary component thus obtaining a new cobordism $\Sigma_1 \cup_{T_2} \Sigma_2$.
4. Additionally we impose a few relations in $\mathrm{Cob}^3_\ell(2n)$.
   a) S relation – whenever a cobordism Σ has a connected component diffeomorphic to the 2-sphere this cobordism is identified with the zero morphism.
   b) T relation – whenever a cobordism Σ has a connected component diffeomorphic to the 2-torus we can erase this component and multiply the remaining cobordism Σ' by 2.
   c) *4TU*-relation – a local relation which is illustrated on Figure 1. This relation tells us how we can move one-handles attached to cobordism in question.

Definition 2.3.2. Define the additive category $\mathrm{Mat}\left(\mathrm{Cob}^3_\ell(2n)\right)$ as follows.

1. The set of objects consists of finite formal direct sums of objects of $\mathrm{Cob}^3_\ell(2n)$.

Figure 2: Positive and negative crossings

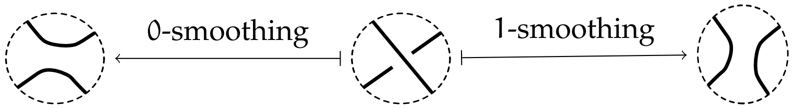


Figure 3: 0- and 1-smoothings

2. Morphisms are matrices

$$f\colon \bigoplus_{i=1}^{k} T_i \to \bigoplus_{j=1}^{n} T'_j$$

$$f = \begin{bmatrix} f_{11} & f_{12} & \dots & f_{1k} \\ f_{21} & f_{22} & \dots & f_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \dots & f_{nk} \end{bmatrix}$$

where $f_{ij} \in \mathrm{Mor}_{\mathrm{Cob}^3_\ell(2n)}(T_i, T'_j)$. Composition of morphisms corresponds to multiplication of the respective matrices.

Definition 2.3.3. Define

$$\mathrm{Kob}\,(2n) = \mathrm{Kom}\left(\mathrm{Mat}\left(\mathrm{Cob}^3_\ell\,(2n)\right)\right)$$

to be the category of finite cochain complexes over $\mathrm{Mat}\left(\mathrm{Cob}^3_\ell\,(2n)\right)$. Morphisms in $\mathrm{Kob}\,(2n)$ are chain maps between the respective complexes, which are defined in the usual way. We can define the homotopy category of $\mathrm{Kob}\,(2n)$.

$$\mathrm{Kob}_h\,(2n) = \mathrm{Kom}_h\left(\mathrm{Mat}\left(\mathrm{Cob}^3_\ell\,(2n)\right)\right)$$

with the same objects as $\mathrm{Kob}\,(2n)$ and morphisms being homotopy classes of chain maps, where the chain homotopy is defined as usual.

Let T be an oriented tangle in D^2 with $2n$ endpoints. The *Bar-Natan's bracket* $[\![T]\!]_{BN}$ is an object in $\mathrm{Kob}_h\,(2n)$ defined as follows. Denote by $n_+(T)$ and $n_-(T)$ the number of positive and negative crossings of T, respectively. For $0 \leqslant r \leqslant n_+(T) + n_-(T)$ let $[\![T]\!]^{r-n_-}_{BN}$ be the formal direct sum of crossingless tangles obtained from T by resolving exactly r crossings with the 1-smoothing and all remainder crossings with the 0-smoothing. The 0- and 1-smoothings are depicted on Figure 3.

In order to define the differential, consider two resolutions T_0 and T_1 of T, which differ only at a single crossing. To be more precise, T_i was obtained from T by applying i-smoothing at this crossing, for $i = 0, 1$, and they agree otherwise. Define a map in $\mathrm{Cob}^3_\ell(2n)$

$$\Sigma_{0\to 1} : T_0 \to T_1 \tag{2}$$

to be the elementary cobordism from T_0 to T_1 obtained from $T_0 \times [0, 1]$ by attaching 1-handle to $T_0 \times \{1\}$ where the crossing change happen. Now assemble all these cobordisms to a map

$$[\![T]\!]^{r-n_-}_{BN} \to [\![T]\!]^{r+1-n_-}_{BN}.$$

Such map is not yet a differential, because its square is not equal to zero in $\mathrm{Kob}(2n)$. Indeed, consider two crossings and resolve then in two different ways. This yields the following commutative diagram in $\mathrm{Kob}(2n)$.

$$\begin{array}{ccccc} & & T_{10} & & \\ & \nearrow & & \searrow & \\ T_{00} & & \oplus & & T_{11} \\ & \searrow & & \nearrow & \\ & & T_{01} & & \end{array}$$

The diagram is indeed commutative, because the respective cobordisms, $\Sigma_{00\to 10} \cup_{T_{10}} \Sigma_{10\to 11}$ and $\Sigma_{00\to 01} \cup_{T_{01}} \Sigma_{01\to 11}$, from T_{00} to T_{11}, are isotopic rel boundary. Hence, in order to get a differential we need to modify the definition of the map, by assigning additional signs, so that in every commutative square as above, the two maps $\Sigma_{10\to 11} \circ \Sigma_{00\to 10}$, $\Sigma_{01\to 11} \circ \Sigma_{00\to 01}$ appear a with different sign.

Let $\mathrm{Cr}\, T$ denote the set of crossings of T. Choose a linear order on $\mathrm{Cr}\, T$. Let W be a $\mathbb{Q}$-vector space spanned by the elements of $\mathrm{Cr}\, T$ and let $V = \Lambda^* W$ be the exterior algebra of W. Vector space V has a distinguished basis of the form

$$c_{i_1} \wedge c_{i_2} \wedge \ldots \wedge c_{i_k},$$

where $c_{i_j} \in \mathrm{Cr}\, T$ for $1 \leqslant j \leqslant k$ and

$$c_{i_1} < c_{i_2} < \ldots < c_{i_k},$$

with respect to the chosen ordering. Let $\mathcal{B}$ be the set of vectors from this distinguished basis. Every resolution of T can be labeled with a unique element from $\mathcal{B}$. Indeed, associate to every resolution a vector $v \in \mathcal{B}$ in such a way that

$$v = c_{i_1} \wedge c_{i_2} \wedge \ldots \wedge c_{i_r}$$

if, and only if, the resolution in question was obtained from T by application of 1-smoothing only to crossings $c_{i_1}, c_{i_2}, \ldots, c_{i_r}$. To every pair (v, w), where $v \in \mathcal{B}$ and $w \in \mathrm{Cr}(T)$, such that $v \wedge w \neq 0 \in V$, we can associate a map in $\mathrm{Cob}^3_\ell(2n)$

$$\Sigma_{(v,w)} \colon T_v \to T_{v'},$$

where $v' \in \mathcal{B}$ and $v' = \mathrm{sign}(v, w) v \wedge w$ with $\mathrm{sign}(v, w) \in \{\pm 1\}$. The map in question is the cobordism from (2).

To fix the sign issue, define the differential in $[\![T]\!]_{BN}$ by the following formula.

$$d_{r-n_-} \colon [\![T]\!]^{r-n_-}_{BN} \to [\![T]\!]^{r-n_-+1}_{BN},$$
$$d_{r-n_-} = (-1)^{n_-(T)} \sum_{(v,w)} \mathrm{sign}(v, w) \Sigma_{(v,w)},$$

where the summation extends over pairs $(v, w) \in \mathcal{B} \times \mathrm{Cr}(T)$ such that $v \wedge w \neq 0$.

Proposition 2.3.4. Bar-Natan's bracket $[\![T]\!]_{BN}$, of a tangle T, belongs to $\mathrm{Kob}_h(2n)$, for an appropriate n.

Theorem 2.3.5 (Invariance of the Bar-Natan's bracket)**.** Chain homotopy type of the Bar-Natan's bracket $[\![T]\!]_{BN}$ is an isotopy invariant of the tangle T.

Apart from that, the invariant of a tangle can be equipped with an additional structure. This additional structure utilizes the fact that morphisms have additional topological data – the genus of the respective surface. This additional data equips $\mathrm{Mat}\left(\mathrm{Cob}^3_\ell(2n)\right)$ with a grading, thus making it into a graded category.

Definition 2.3.6. Let $\mathcal{C}$ be preadditive category. We say that $\mathcal{C}$ is *graded* if the following conditions hold.

1. For any two objects $\mathcal{O}_1$ and $\mathcal{O}_2$ from $\mathcal{C}$, morphisms from $\mathcal{O}_1$ to $\mathcal{O}_2$ form a graded abelian group with composition being compatible with grading, that is

$$\deg f \circ g = \deg f + \deg g.$$

   Additionally we require that identity morphisms are of degree zero.

2. There is a $\mathbb{Z}$ action on the objects of $\mathcal{C}$

$$(m, \mathcal{O}) \mapsto \mathcal{O}\{m\}.$$

   As sets, morphisms are unchanged under this action, that is

$$\mathrm{Mor}_\mathcal{C}(\mathcal{O}_1\{m_1\}, \mathcal{O}_2\{m_2\}) = \mathrm{Mor}_\mathcal{C}(\mathcal{O}_1, \mathcal{O}_2).$$

   Gradings, however, change. If we choose $f \in \mathrm{Mor}_\mathcal{C}(\mathcal{O}_1, \mathcal{O}_2)$ such that $\deg f = d$, then f, considered as an element of the morphism set $\mathrm{Mor}_\mathcal{C}(\mathcal{O}_1\{m_1\}, \mathcal{O}_2\{m_2\})$, has $\deg f = d + m_2 - m_1$.

In order to define the grading, enlarge the collection of objects by adding formal finite direct sums of $T\{m\}$, for some $m \in \mathbb{Z}$. If Σ is a cobordism from $\mathrm{Cob}^3_\ell(2n)$ give it $\deg \Sigma = \chi(\Sigma) - n$.

Definition 2.3.7. For a tangle T in D^2, with $2n$ endpoints, define its *Khovanov bracket*

$$[\![T]\!]_{Kh} \in \mathrm{Kob}_h(2n),$$

where we treat $\mathrm{Kob}_h(2n)$ as a graded category, to be the following complex

$$[\![T]\!]_{Kh}^{r-n_-(T)} = [\![T]\!]_{BN}^{r-n_-(T)}\{r + n_+(T) - n_-(T)\}.$$

The differential remains unchanged.

Theorem 2.3.8.

1. For any tangle T, the differential in $[\![T]\!]_{Kh}$ is of degree zero.
2. Graded chain homotopy type (i.e. we consider only chain homotopy equivalences of degree 0) of $[\![T]\!]_{Kh}$ is an invariant of the isotopy class of T.

2.3.2 *Planar algebra structure*

Definition 2.3.9. A d-*input planar arc-diagram* D is a big "output" disk with d smaller "input" disks removed and equipped with a collection of oriented and disjointly embedded arcs. Each arc is either closed, or has endpoints on ∂D. Each input disk is labeled with an integer ranging from 1 to d, and a basepoint is chosen on each connected component of the boundary. Such collection of data is considered only up to isotopy rel boundary.

Definition 2.3.10. Let s be a finite string of arrows ↑ and ↓. Denote by $\mathcal{T}^0(s)$ the set of all s-ended oriented tangle diagrams in a based disk D^2, that is if we start at the chosen basepoint on ∂D^2 and proceeds in the counterclockwise direction, we will obtain s by looking at the orientation of the endpoint of the tangle met along the way. Let $\mathcal{T}(s)$ denote quotient of $\mathcal{T}^0(s)$ by Reidemeister moves.

Every d-input planar arc-diagram D determines an operations

$$D\colon \mathcal{T}^0(s_1) \times \ldots \times \mathcal{T}^0(s_d) \to \mathcal{T}^0(s),$$
$$D\colon \mathcal{T}(s_1) \times \ldots \times \mathcal{T}(s_d) \to \mathcal{T}(s)$$

which glues tangles from $\mathcal{T}^0(s_i)$, or $\mathcal{T}(s_i)$ in the i-th input disk. These operations are associative, in the sense that, if D_i was obtained by glueing D' along the boundary of the i-th input disk of D, then

$$D_i = D \circ \left(I \times \ldots \times D' \times \ldots \times I\right).$$

The diagram from Figure 4 acts as the identity.

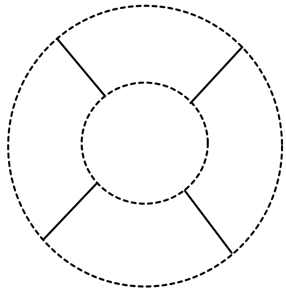


Figure 4: Identity in the planar algebra

Definition 2.3.11. A collection of sets $\mathcal{P}(s)$, indexed by string of arrows, and operations as described above, satisfying the associativity and identity relations, is called an *oriented planar algebra*. Analogously, it is possible to define an *unoriented planar algebra* by disregarding orientations of tangles.

A *morphism of planar algebras* is a collection of maps $\Psi\colon \mathcal{P}(s) \to \mathcal{P}'(s)$ satisfying

$$\Psi \circ D = D \circ (\Psi \times \ldots \times \Psi)\,.$$

Example 2.3.12. Objects from $\mathrm{Cob}^3_\ell(2n)$ can be bundled into an oriented planar algebra. This is in fact a planar subalgebra of $\mathcal{T}$ consisting of crossingless tangles.

Example 2.3.13. Analogously, morphisms from $\mathrm{Cob}^3_\ell(2n)$ can be organised into a planar algebra. To define how a given planar diagram D acts, consider $D \times [0,1]$. Every arc ℓ in D determines a rectangle $\ell \times [0,1]$. Glue cobordisms from $\mathrm{Cob}^3_\ell(2n)$ into holes of $D \times [0,1]$. These cobordisms, together with the rectangles $\ell \times [0,1]$, yield a new cobordism, which is the result of the operation.

Theorem 2.3.14.
1. $\mathrm{Kob}(2n)$ and $\mathrm{Kob}_h(2n)$ admit the structure of an oriented planar algebra, which is inherited from $\mathrm{Cob}^3_\ell(2n)$. In particular, every planar algebra operation transforms homotopy equivalent complexes into homotopy equivalent complexes.
2. The Khovanov bracket

$$[\![\cdot]\!]_{Kh}\colon\ (\mathcal{T}(s)) \to \left(\mathrm{Kom}_h\left(\mathrm{Mat}\left(\mathrm{Cob}^3_\ell(2n)\right)\right)\right)$$

is a morphism of planar algebras.

2.3.3 *Applying TQFT*

In order to obtain computable invariants from Khovanov's bracket, of a link L, it is necessary to pass to an algebraic cochain complex. This is done with the aid of a TQFT functor.

Definition 2.3.15. Let

$$\mathcal{T}\colon\ \mathrm{Cob}^3_\ell(0) \to \mathrm{Mod}_R$$

be an additive functor whose target is the category of R-modules, where R is a commutative ring with unit. We say that T is a *TQFT* if the following conditions are satisfied.

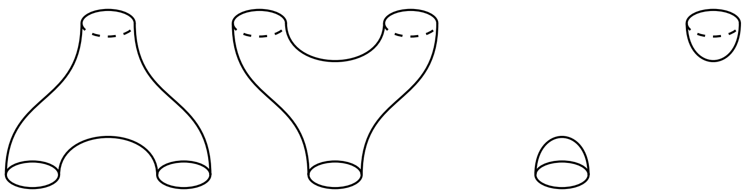
1. $\mathcal{T}$ maps disjoint unions of objects of $\mathrm{Cob}^3_\ell(0)$ to tensor products of the corresponding R-modules.
2. $\mathcal{T}$ maps disjoint unions of cobordisms into tensor products of the corresponding maps.
3. The cylinder $S \times [0,1]$ is mapped to the identity morphism.

Theorem 2.3.16. Every TQFT

$$\mathcal{T}\colon \mathrm{Cob}^3_\ell(0) \to \mathrm{Vect}_k\,.$$

is completely determined by its values on the following manifold.

1. A circle $T(S^1) = \mathcal{A}$.
2. The following elementary cobordisms (read from top to bottom).



$\Delta\colon \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ $\quad \mu\colon \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ $\quad \epsilon\colon k \to \mathcal{A}$ $\quad \eta\colon \mathcal{A} \to k$

Quintuple $(\mathcal{A}, \mu, \Delta, \epsilon, \eta)$ is a *Frobenius algebra*.

Remark 2.3.17. For more details about Frobenius algebras and TQFTs consult [11].

Example 2.3.18. Consider the following TQFT.

$$\mathcal{A} = \mathbb{Z}[X]/\left(X^2\right), \quad \deg \mathbb{1} = 1, \quad \deg X = -1,$$

$$\begin{aligned}
\mu(\mathbb{1}\otimes\mathbb{1}) &= \mathbb{1},\\
\mu(\mathbb{1}\otimes X) &= \mu(X\otimes\mathbb{1}) = X,\\
\mu(X\otimes X) &= 0,\\
\Delta(\mathbb{1}) &= \mathbb{1}\otimes X + X\otimes\mathbb{1},\\
\Delta(X) &= X\otimes X,\\
\epsilon(1) &= \mathbb{1},\\
\eta(\mathbb{1}) &= 0,\\
\eta(X) &= 1.
\end{aligned}$$

Definition 2.3.19. Let $\mathcal{T}$ be a TQFT determined by the data from Example 2.3.18. Define the *Khovanov's cochain complex*, associated to a link diagram D, to be

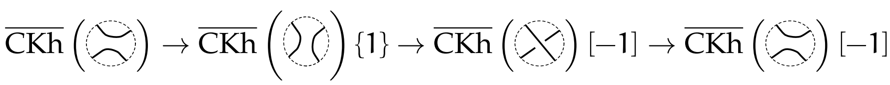
$$\mathrm{CKh}(D) = \mathcal{T}([\![D]\!]_{\mathrm{Kh}}).$$

This is a cochain complex of graded $\mathbb{Z}$-modules. Its homology, denoted by $\mathrm{Kh}(D)$, is called the *Khovanov homology* of D. Define also the *shifted Khovanov complex* of D to be

$$\overline{\mathrm{CKh}}(D) = \mathcal{T}([\![D]\!]_{\mathrm{Kh}})[n_-(D)]\{2n_-(D) - n_+(D)\}.$$

Corollary 2.3.20. If D is a link diagram, the graded chain homotopy type of $\mathrm{CKh}(D)$ is an isotopy invariant of D. Consequently, the Khovanov homology of D is also an isotopy invariant.

Theorem 2.3.21. For any link L there exists an exact triangle



which yields a long exact sequence of Khovanov homology groups.

Example 2.3.22. Let T_n denote the n-component trivial link. Its Khovanov homology is given below.

$$\mathrm{Kh}(T_n) = \mathcal{A}^{\otimes n},$$

where $\mathcal{A}$ is the algebra from Example 2.3.18.

Definition 2.3.23. Let M^* be a graded $\mathbb{Q}$-vector space of finite dimension. Define its quantum dimension to be the following Laurent polynomial.

$$\mathrm{qdim}_{\mathbb{Q}} M^* = \sum_n q^n \dim_{\mathbb{Q}} M^n \in \mathbb{Z}\left[q, q^{-1}\right].$$

Definition 2.3.24. The Khovanov polynomial of a link L is the following two variable Laurent polynomial.

$$\mathrm{KhP}(L) = \sum_{i,j} t^i \,\mathrm{qdim}_{\mathbb{Q}} \mathrm{Kh}^{i,*}(L) \otimes \mathbb{Q}.$$

Define the unreduced Jones polynomial to be the following one variable laurent polynomial.

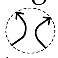
$$J(L) = \mathrm{KhP}(L)(-1, q).$$

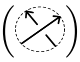
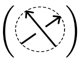
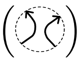
Proposition 2.3.25. The unreduced Jones polynomial satisfies the following properties.

1. If T_n denotes the n-components trivial link, then

$$J(T) = (q + q^{-1})^n.$$

2. Let $L \cup L'$ be a split link, which disjoint union of two links L and L'.
$$J(L \cup L') = J(L)\,J(L').$$

3. Let D be an oriented diagram of a link L. Choose a crossing of D. Let ,  and  denote the diagram obtained from D by cutting out a small neighbourhood of the chosen crossing and gluing in the crossing from the respective piture.

$q^{-2}\,J\Big($  $\Big) - q^{2}\,J\Big($  $\Big) = (q^{-1} - q)\,J\Big($  $\Big)$.

3 EQUIVARIANT KHOVANOV HOMOLOGY

This chapter is devoted to our construction of the equivariant Khovanov homology for periodic links. First, we recall the definition of periodic links and analyze the Khovanov complex of such links. We show that using the Bar-Natan's sign convention, it possible to define an action of the cyclic group on the Khovanov complex of a periodic link. Next, we analyze the effect of performing Reidemeister moves.

The equivariant Khovanov homology of periodic links is defined with the aid of the machinery of derived functors. Then, we describe properties of the equivariant Khovanov homology, like its relation to the classical Khovanov homology and analyze the additional torsion that it contains. We also compute the equivariant Khovanov homology of trivial links.

3.1 PERIODIC LINKS

Definition 3.1.1. Let n be a positive integer, and let L be a link in S^3. We say that L is n-periodic, if there exists an action of the cyclic group of order n on S^3 satisfying the following conditions.

1. The fixed point set, denoted by F, is the unknot.
2. L is disjoint from F.
3. L is a $\mathbb{Z}/n$-invariant subset of S^3.

Example 3.1.2. Borromean rings provide an example of a 3-periodic link. The symmetry is visualised on Figure 5. The dot marks the fixed point axis.

Example 3.1.3. Torus links constitute an infinite family of periodic links. In fact, according to [15], the torus link $T(m,n)$ is d-periodic if, and only if, d divides either m or n.

Periodic diagrams of periodic links can be described in terms of planar algebras. Take an n-periodic planar diagram D_n with n input disks, like the one on Figure 6. Choose a tangle T which possesses enough endpoints, and glue n copies of T into the input disks of D_n. In this way, we obtain a periodic link whose quotient is represented by an appropriate closure of T. See Figure 7 for an example.

Using this description of periodic links, it is possible to exhibit a cobordism which induces an action of $\mathbb{Z}/n$ on $[\![D]\!]_{Kh}$, for D a periodic link diagram. First, notice that we can assume that D represents a link

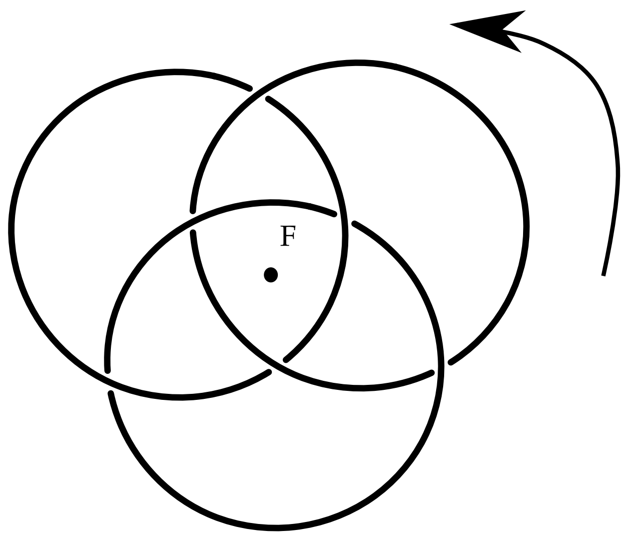


Figure 5: Borromean rings are 3-periodic. The fixed point axis F is marked with a dot.

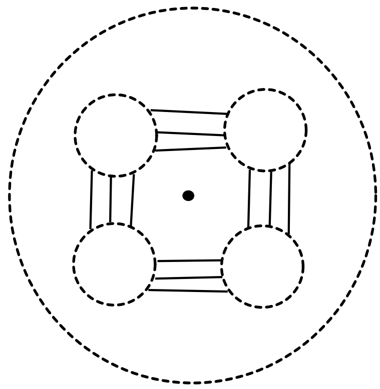


Figure 6: 4-periodic planar diagram.

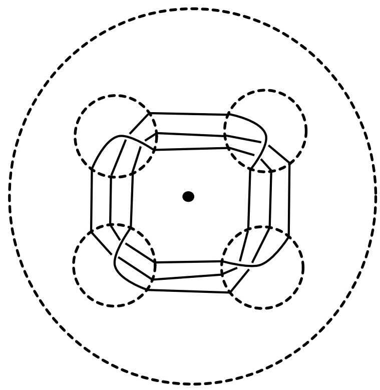


Figure 7: Torus knot $T(3,4)$ as a 4-periodic knot obtained from the planar diagram from Figure 6

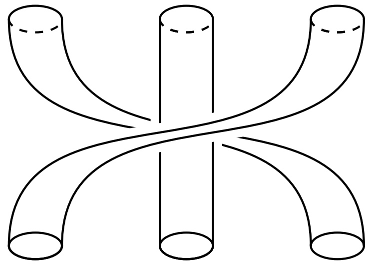


Figure 8: Periodic Kauffman state with 3 components and symmetry of order 2. Middle cylinder contains the fixed point axis F.

in $D^2 \times I$ and the symmetry comes from a rotation of the D^2 factor. In order to construct the cobordism, notice that the diffeomorphism, denote it by f, generating the $\mathbb{Z}/n$-symmetry of $D^2 \times I$, is isotopic to the identity. Indeed, this isotopy can be chosen in such a way that it changes the angle of rotation linearly from 0 to $\frac{2\pi}{n}$. Denote this isotopy by H. The cobordism in question is the trace of H.

$$\Sigma_H = \{(H(x,t),t) \in D^2 \times I \times I \colon x \in L, \quad t \in I\}.$$

Cobordism Σ_H is diffeomorphic to the cylinder $S^1 \times I$, however it is not isotopic, rel boundary, to the cylinder, which is equal to Σ_{H_0}, where H_0 denotes the constant isotopy from the identity to the identity. However, Σ_H is invertible in $\mathrm{Kob}_h(2\ell)$, because the composition $\Sigma_{\overline{H}} \circ \Sigma_H$, where $\overline{H}(\cdot,t) = H(\cdot, 1-t)$, is isotopic to Σ_{H_0}, rel boundary.

Before proceeding further, one remark is in order. During the construction of $[\![D]\!]_{Kh}$ it was necessary to multiply each summand of the differential with ± 1. This particular choice of signs forces us to do the same with maps between complexes. Recall from section 2.3.1 that we considered two vector spaces – W generated by crossings of the diagram and the exterior algebra $V = \Lambda^* W$. Each Kauffman state, which is another name for any resolution of D, was labelled with a vector from the distinguished basis of V. Choose a tangle T and let W_T be the vector space associated to T and $D = D_n(T,\ldots,T)$ with W_D defined analogously. Under this assumptions

$$W_D \cong W_T^n, \quad \text{and} \quad \Lambda^* W_D \cong (\Lambda^* W_T)^{\otimes n}.$$

Symmetry of D induces an action of $\mathbb{Z}/n$ on $\Lambda^* W_D$, which permutes factors in the above decomposition. Cobordism Σ_H discussed above induces a map

$$\Sigma_H \colon [\![D]\!]_{Kh} \to [\![D]\!]_{Kh}$$

which permutes all Kauffman states of D. This permutation is compatible with the induced action on $\Lambda^* W_D$. Geometrically, the map $\Sigma_H|_{[\![D]\!]_{Kh}^{r-n_-}}$ is induced by a "permutation" cobordisms similar to the one from Figure 8. However, additional sign is needed to assure that this map commutes with the differential. Let us define

$$\begin{aligned} \psi\colon (\Lambda^* W_T)^{\otimes n} &\to (\Lambda^* W_T)^{\otimes n} \\ \psi\colon x_1 \otimes x_2 \otimes \ldots \otimes x_n &\mapsto (-1)^{\alpha} x_2 \otimes \ldots \otimes x_n \otimes x_1, \quad x_i \in W_T, \end{aligned} \tag{3}$$

where

$$\alpha = (n-1)n_-(T) + \deg x_1 (\deg x_2 + \deg x_3 + \ldots + \deg x_n).$$

Automorphism ψ maps any vector from the distinguished basis of $\Lambda^* W_D$ to ± 1 multiplicity of some other vector from the basis.

$$\psi(v) = \mathrm{sign}(\psi, v) w$$

We can utilize these signs to change the definition of Σ_H as follows.

$$\begin{aligned} \Sigma_H|_{D_v} &\colon D_v \to D_w, \\ \Sigma_H|_{D_v} &= \mathrm{sign}(\psi, v)\Sigma_{v,w}, \end{aligned} \tag{4}$$

where $\Sigma_{v,w}$ denotes the appropriate permutation cobordism. This discussion leads to the following proposition.

Proposition 3.1.4. If D is a periodic link diagram, then $\mathrm{CKh}(D)$ is a complex of graded $\mathbb{Z}[\mathbb{Z}/n]$-modules.

Remark 3.1.5. This sign convention was implicitly described in [2].

Proof of Prop. 3.1.4. The only thing left to prove, is the commutativity of Σ_H and the differential of $\mathrm{CKh}(D)$. Geometric properties of the Khovanov bracket imply that the components of both maps commute up to sign. Hence, the only thing left to check is that all these maps really commute, provided that we choose the signs as in the discussion above.

Let $x_1, \ldots, x_n \in V$ be homogeneous vectors. Consider the following linear maps

$$\begin{aligned} d_D\colon (\Lambda^* W_T)^{\otimes n} &\to (\Lambda^* W_T)^{\otimes n}, \\ d_D\colon x_1 \otimes \ldots \otimes x_n &\mapsto \sum_{i=1}^{n} (-1)^{\alpha_i} x_1 \otimes \ldots \otimes d_T(x_i) \otimes \ldots \otimes x_n, \\ \sigma_i\colon (\Lambda^* W_T)^{\otimes n} &\to (\Lambda^* W_T)^{\otimes n}, \\ \sigma_i\colon x_1 \otimes \ldots \otimes x_n &\mapsto (-1)^{\deg x_i \cdot \deg x_{i+1}} x_1 \otimes \ldots \otimes x_{i+1} \otimes x_i \otimes \ldots \otimes x_n, \\ \widetilde{\sigma_i} &= (-1)^{n_-(T)} \sigma_i. \end{aligned}$$

where $1 \leqslant i \leqslant n-1$, $\alpha_i = (-1)^{\deg(x_n)+\ldots+\deg(x_{i+1})}$ and

$$d_T(w) = \sum_{v \in \mathrm{Cr}\, T} w \wedge v.$$

Notice that the map ψ from (3) is expressible as the composition of the maps $\widetilde{\sigma}_i$ in the following way.

$$\psi = \widetilde{\sigma}_{n-1} \circ \widetilde{\sigma}_{n-2} \circ \ldots \circ \widetilde{\sigma}_1.$$

The map d_D, on the other hand, corresponds to the differential. Let

$$d_{r-n_-(D)} \colon CKh^{r-n_-,*}(D) \to CKh^{r+1-n_-(D)}(D)$$

be the differential in the Khovanov complex. In the notation from previous chapter we have that

$$d_{r-n_-(D)} = \sum_{(v,w)} \operatorname{sign}(v,w)\Sigma_{(v,w)}.$$

It is not hard to check, that the coefficient $\operatorname{sign}(v,w)$ is equal to the coefficient of $v \wedge w$ in $d_D(v)$.

Therefore, it is sufficient to check that for $1 \leqslant i \leqslant n-1$ the following equality holds

$$\sigma_i \circ d_D = d_D \circ \sigma_i.$$

This can be verified by an elementary calculation.

□

Let us now analyze the structure of the cochain complex $CKh(D)$.

Definition 3.1.6.
1. Let $\mathcal{S}_r(D)$ denote the set of Kauffman states of D which were obtained by resolving exactly r crossings with the 1-smoothing.
2. For $d \mid n$, let $\mathcal{S}^d(D)$ denote the set of Kauffman states which inherit a symmetry of order d from the symmetry of D, that is Kauffman states of the form

   $$D_n(T_1, \ldots, T_{\frac{n}{d}}, T_1, \ldots, T_{\frac{n}{d}}, \ldots, T_1, \ldots, T_{\frac{n}{d}}),$$

   where $T_1, \ldots, T_{\frac{n}{d}}$ are distinct resolutions of T.
3. For a Kauffman state s, write $\operatorname{Iso}_D(s) = \mathbb{Z}/d$ if, and only if $s \in \mathcal{S}^d(D)$.
4. Define $\mathcal{S}^d_r(D) = \mathcal{S}^d(D) \cap \mathcal{S}_r(D)$.
5. Define $\overline{\mathcal{S}}^d_r(D)$ to be the quotient of $\mathcal{S}^d_r(D)$ by the action of $\mathbb{Z}/n$.

Remark 3.1.7. If $\mathcal{S}^d_r(D)$ is non-empty, then $d \mid \gcd(n,r)$.

Definition 3.1.8. Let $\mathbb{Z}_-$ be the following $\mathbb{Z}[\mathbb{Z}/n]$-module.

$$\mathbb{Z}_- = \begin{cases} \mathbb{Z}[\xi_2], & 2 \mid n, \\ \mathbb{Z}, & 2 \nmid n. \end{cases}$$

In other words, if n is even, the generator of the cyclic group $\mathbb{Z}/n$ acts on $\mathbb{Z}_-$ by multiplication by -1, otherwise it is the trivial module.

Lemma 3.1.9. Let $\mathcal{T}$ be any TQFT functor whose target is the category of R-modules, for R a commutative ring with unit. If $s_1, \dots, s_{\frac{n}{d}} \in \mathcal{S}_r^d(D)$, for $d \mid \gcd(n,r)$ and $d \geqslant 1$, are Kauffman states constituting one orbit, then

$$\bigoplus_{i=1}^{\frac{n}{d}} \mathcal{T}([\![s_i]\!]_{Kh}) \cong \operatorname{Ind}_{\mathbb{Z}/d}^{\mathbb{Z}/n} \left(\mathcal{T}([\![s_1]\!]_{Kh}) \otimes_{\mathbb{Z}} \mathbb{Z}_-^{\otimes s(n,r,d)} \right).$$

as $R[\mathbb{Z}/n]$-modules, where

$$s(n,r,d) = \frac{(n-1)n_-(D) + r(d-1)}{d}$$

Proof. We will prove that if $s \in \mathcal{S}_r^d(D)$, then $\Sigma_H^{\frac{n}{d}}(s) = (-1)^{s(n,r,d)} s$. The orbit of s consists of $\frac{n}{d}$ Kauffman states which are permuted by the action of $\mathbb{Z}/n$. Hence, the lemma will follow from Proposition 2.1.11 once we determine the induced action of $\mathbb{Z}/d$ on $T([\![s_1]\!]_{Kh})$. Since $T([\![s_1]\!]_{Kh})$ possesses a natural action of $\mathbb{Z}/d$, the induced action will differ from this one by a certain sign. Appeareance of this sign is a consequence of our sign convention.

The Kauffman state s_1 corresponds to a vector of the form

$$w = \underbrace{v \otimes v \otimes \ldots \otimes v}_{d},$$

where $v = v_1 \otimes v_2 \otimes \ldots \otimes v_{\frac{n}{d}}$ and $v_1, \dots, v_{\frac{n}{d}} \in \Lambda^* W_T$ belong to the distinguished basis. Consequently

$$\psi^{\frac{n}{d}}(w) = (-1)^{k(d-1) + \frac{n_-(T)n(n-1)}{d}} w = (-1)^{\frac{r(d-1)+n_-(D)(n-1)}{d}} w,$$

where $k = \deg v_1 + \deg v_2 + \ldots + \deg v_{\frac{n}{d}}$. □

Corollary 3.1.10. If $\mathcal{T}$ is as in the previous lemma and $0 \leqslant r \leqslant n_+ + n_-$, then

$$\mathcal{T}([\![D]\!]_{Kh}^{r-n_-}) = \bigoplus_{d \mid \gcd(n,r)} \bigoplus_{s \in \overline{\mathcal{S}}_r^d} \operatorname{Ind}_{\mathbb{Z}/d}^{\mathbb{Z}/n} \left(\mathcal{T}([\![s]\!]_{Kh}) \otimes_{\mathbb{Z}} \mathbb{Z}_-^{\otimes s(n,r,d)} \right) \{r + n_+(D) - n_-(D)\}.$$

Remark 3.1.11. In the above formula there is a small ambiguity, since we identified a Kauffman state belonging to an orbit with this orbit. This notational shortcut does not cause any problems since all Kauffman states belonging to the same orbit yield isomorphic summands in $[\![D]\!]_{Kh}^{r-n_-}$. We will use this convention in the remainder part of this thesis.

Proof of Cor. 3.1.10. Since for $d_1, d_2 \mid \gcd(n,r)$ the sets $\mathcal{S}_r^{d_1}(D)$, $\mathcal{S}_r^{d_2}(D)$ are disjoint, corollary follows easily from lemma 3.1.9. □

Let now $D = D_n(T, \dots, T)$ be an n-periodic link diagram obtained from a tangle T. Suppose that we choose another tangle T', which differs from T by a single application of one of the Reidemester moves. Form another link diagram $D' = D_n(T', \dots, T')$. This raises a question, what is the relationship between $[\![D]\!]_{Kh}$ and $[\![D']\!]_{Kh}$ in $\mathrm{Kob}_h(0)$. The following theorem from [2] answers this question.

Theorem 3.1.12. There exists a chain map, induced by the Reidemeisted moves,

$$\mathcal{R}\colon [\![D]\!]_{Kh} \to [\![D']\!]_{Kh},$$

which is an isomorphism in $\mathrm{Kob}_h(0)$. In particular, this map induces a chain homotopy equivalence after application of any TQFT functor.

However, in the equivariant setting we obtain a considerably weaker invariance result.

Theorem 3.1.13. If D and D' are as above and $\mathcal{T}$ is a TQFT functor whose target is the category of R-modules, then the map

$$\mathcal{R}\colon [\![D]\!]_{Kh} \to [\![D']\!]_{Kh},$$

induced by Reidemeister moves, yields a quasi-isomorphism

$$\mathcal{T}(\mathcal{R})\colon \mathcal{T}([\![D]\!]_{Kh}) \to \mathcal{T}([\![D']\!]_{Kh})$$

in the category of cochain complexes of $R[\mathbb{Z}/n]$-modules.

Proof. It is sufficient to prove, that $\mathcal{T}(\mathcal{R})$ is a morphism in the category of $R[\mathbb{Z}/n]$-modules, because Theorem 3.1.12 implies that it automatically induces an isomorphism on homology.

To check this condition, refer to the proof of [2, Thm. 2]. The bracket $[\![D]\!]_{Kh}$ is constructed along the lines of the formal tensor product of copies of the complex $[\![T]\!]_{Kh}$. Each collection of morphisms

$$f_i\colon [\![T]\!]_{Kh} \to [\![T']\!]_{Kh},$$

for $i = 1, \dots, n$, yield a morphism

$$f_1 \otimes \dots \otimes f_n\colon [\![D]\!]_{Kh} \to [\![D']\!]_{Kh}.$$

Taking into account the symmetry of D and D', we obtain the following commutative diagram

$$\begin{array}{ccc} [\![D]\!]_{Kh} & \xrightarrow{f_1 \otimes \dots \otimes f_n} & [\![D']\!]_{Kh} \\ \Sigma_D \downarrow & & \downarrow \Sigma_{D'} \\ [\![D]\!]_{Kh} & \xrightarrow{f_2 \otimes \dots \otimes f_n \otimes f_1} & [\![D']\!]_{Kh} \end{array}$$

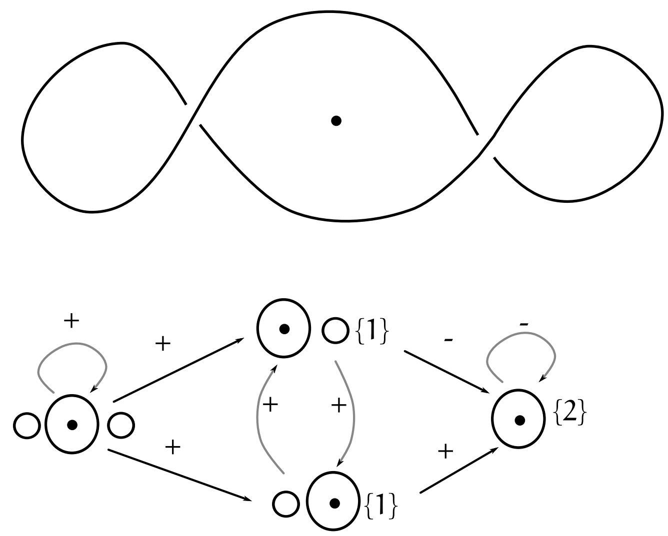


Figure 9: 2-periodic diagram of the unknot and its Khovanov bracket. Gray arrows indicate the $\mathbb{Z}/2$-action. Black dot stands for the fixed point axis.

where Σ_D and $\Sigma_{D'}$ denote the automorphisms of complexes induced by the action of $\mathbb{Z}/n$. Since R is of the form $\mathcal{R} = \mathcal{R}' \otimes \ldots \otimes \mathcal{R}'$, where

$$\mathcal{R}' \colon [\![T]\!]_{Kh} \to [\![T']\!]_{Kh}$$

is induced by a single Reidemeister move, it follows that $\mathcal{T}(\mathcal{R})$ is a morphism in the category of $R\,[\mathbb{Z}/n]$-modules.

□

3.2 EQUIVARIANT KHOVANOV HOMOLOGY

3.2.1 *Integral equivariant Khovanov homology*

Let L be an n-periodic link. It was shown earlier, that under this assumption, the Khovanov complex $CKh(D)$, where D is an n-periodic diagram of L, admits the structure of a cochain complex of graded $\mathbb{Z}\,[\mathbb{Z}/n]$-modules. We could try to obtain an invariant of L by defining the equivariant Khovanov homology to be

$$H^{*,*}(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/n]}\,(M, CKh(D))),$$

for some $\mathbb{Z}\,[\mathbb{Z}/n]$-module M. Unfortunately, Theorem 3.1.13 indicates that this approach might not work. This is indeed the case, which is illustrated by the following example.

Example 3.2.1. Consider the 2-periodic diagram D from Figure 9. We will show that due to the lack of exactness of the Hom functor, the ordinary cohomology with coefficients depends on the chosen diagram. Since

$$\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}\,(\mathbb{Z}_-, M) = \{x \in M \colon t \cdot x = -x\},$$

we obtain

$$\begin{aligned}
&\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}\left(\mathbb{Z}_{-}, \mathrm{CKh}^{1,*}(D)\right) = \\
&= \left\langle \begin{bmatrix} \mathbb{1}\otimes\mathbb{1} \\ -\mathbb{1}\otimes\mathbb{1} \end{bmatrix}, \begin{bmatrix} \mathbb{1}\otimes X \\ -X\otimes\mathbb{1} \end{bmatrix}, \begin{bmatrix} X\otimes\mathbb{1} \\ -\mathbb{1}\otimes X \end{bmatrix}, \begin{bmatrix} X\otimes X \\ -X\otimes X \end{bmatrix} \right\rangle \\
&\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}\left(\mathbb{Z}_{-}, \mathrm{CKh}^{2,*}(D)\right) = \langle \mathbb{1}, X\rangle .
\end{aligned}$$

Inspection of the differential $d\colon \mathrm{CKh}^{1,*}(D) \to \mathrm{CKh}^{2,*}(D)$ yields

$$d\colon \begin{bmatrix} \mathbb{1}\otimes\mathbb{1} \\ -\mathbb{1}\otimes\mathbb{1} \end{bmatrix} \mapsto -2\cdot\mathbb{1},$$

$$d\colon \begin{bmatrix} \mathbb{1}\otimes X \\ -X\otimes\mathbb{1} \end{bmatrix} \mapsto -2\cdot X,$$

therefore

$$H^{2,*}\left(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}\left(\mathbb{Z}_{-}, \mathrm{CKh}(D)\right)\right) = \mathbb{Z}/2\{5\} \oplus \mathbb{Z}/2\{3\}.$$

On the other hand,

$$H^{2,*}\left(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}\left(\mathbb{Z}_{-}, \mathrm{CKh}(U)\right)\right) = 0,$$

where U denotes the crossingless diagram of the unknot.

The above example shows the necessity of considering the derived functors $\mathrm{Hom}^{*}_{\mathbb{Z}[\mathbb{Z}/n]}(M, -)$ and their homology. This is due to the fact, that the Khovanov complex of a periodic link is built from permutation modules, which are in general neither projective nor injective. This causes the discrepancy visible in the previous example.

Definition 3.2.2. Define the equivariant Khovanov homology of an n-periodic diagram D to be the following triply-graded module, for which the third grading is supported only for $d \mid n$.

$$\mathrm{Kh}^{*,*,d}_{G}(L) = \mathbb{E}\mathrm{xt}^{*,*}_{\mathbb{Z}[G]}\left(\mathbb{Z}\left[\xi_d\right], \mathrm{CKh}(D)\right).$$

It is worth to notice, that since $\mathrm{CKh}(D)$ is a complex of graded modules, Ext groups become also naturally graded, provided that we regard $\mathbb{Z}\left[\xi_d\right]$ as a graded module concentrated in degree 0.

Theorem 3.2.3. Equivariant Khovanov homology groups are invariants of periodic links, that is they are invariant under equivariant Reidemeister moves, as described in the previous section.

Proof. Theorem 3.1.13 implies that application of an equivariant Reidemeister move yields a quasi-isomorphism of the corresponding Khovanov complexes. Application of Proposition 2.2.18 gives an isomorphism between the respective ext groups, which concludes the proof. □

One of the first questions, regarding the properties of the equivariant Khovanov homology, we can ask, is the question about its relation to the classical Khovanov homology. The answer is given in the following theorem.

Theorem 3.2.4. Let $p_1, \ldots, p_s$ be the collection of all prime divisors of n. Define the ring $R_n = \mathbb{Z}\left[\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_s}\right]$. There exists a natural map

$$\bigoplus_{d|r} Kh^{*,*,d}_{\mathbb{Z}/n}(L) \to Kh(L),$$

which, when tensored with R_n, becomes an isomorphism.

Proof. From Theorem 2.2.19 it follows that

$$\mathbb{E}xt^{*,*}_{\mathbb{Z}[\mathbb{Z}/n]}\left(\mathbb{Z}\left[\mathbb{Z}/n\right], CKh(D)\right) \cong Kh^{*,*}(D).$$

Indeed, entries in the E_2 page of the Cartan-Eileberg spectral sequence ${}_{I}E^{*,*}_{*}$ are equal to

$$\begin{aligned}
&Ext^{p,*}_{\mathbb{Z}[\mathbb{Z}/n]}\left(\mathbb{Z}\left[\mathbb{Z}/n\right], Kh^{q,*}(D)\right) = \\
&= \begin{cases} Hom_{\mathbb{Z}[\mathbb{Z}/n]}\left(\mathbb{Z}\left[\mathbb{Z}/n\right], Kh^{q,*}(D)\right), & p = 0, \\ 0, & p > 0. \end{cases}
\end{aligned}$$

Hence, the spectral sequence collapses at E_2. Since

$$Hom_{\mathbb{Z}[\mathbb{Z}/n]}\left(\mathbb{Z}\left[\mathbb{Z}/n\right], Kh^{q,*}(D)\right) \cong Kh^{q,*}(D)),$$

we obtain the desired conclusion.

The short exact sequence from proposition 2.1.18 implies that

$$\Lambda' \otimes_{\mathbb{Z}} R_n = \mathbb{Z}\left[\mathbb{Z}/n\right] \otimes_{\mathbb{Z}} R_n = R_n\left[\mathbb{Z}/n\right],$$

where Λ' denotes the maximal order in $\mathbb{Q}\left[\mathbb{Z}/n\right]$. Therefore,

$$\begin{aligned}
&\mathbb{E}xt^{r,*}_{\mathbb{Z}[\mathbb{Z}/n]}\left(\Lambda', CKh(D)\right) \otimes_{\mathbb{Z}} R_n \cong \\
&\cong \mathbb{E}xt^{r,*}_{R_n[\mathbb{Z}/n]}\left(\Lambda' \otimes_{\mathbb{Z}} R_n, CKh(D) \otimes_{\mathbb{Z}} R_n\right) \cong \\
&\cong \mathbb{E}xt^{r,*}_{R_n[\mathbb{Z}/n]}\left(R_n\left[\mathbb{Z}/n\right], CKh(D) \otimes_{\mathbb{Z}} R_n\right) \cong Kh^{r,*}(D) \otimes_{\mathbb{Z}} R_n,
\end{aligned}$$

because R_n is flat over $\mathbb{Z}$. The last step of the proof consist of noticing, that since $\Lambda' = \bigoplus_{d|n} \mathbb{Z}\left[\xi_d\right]$, hence

$$\mathbb{E}xt^{*,*}_{\mathbb{Z}[\mathbb{Z}/n]}\left(\Lambda', CKh(D)\right) = \bigoplus_{d|r} Kh^{*,*,d}_{G}(L).$$

□

Until the end of this section we will restrict our attention to the case of p^n-periodic links, where p is an odd prime. This restriction is needed to perform the computations of the Khovanov homology of trivial links.

Proposition 3.2.5. Let T_f be an f-component trivial link. It is p^n-periodic for any prime p and $n > 0$. Indeed, we can put components of T_f disjointly in such a way that all of them are rotated by the angle of $\frac{2\pi}{p^n}$. The equivariant Khovanov homology of T_f is expressible in terms of the group cohomology of the cyclic group $\mathbb{Z}/p^n$ in the following way.

$$Kh^{*,*,p^s}_{\mathbb{Z}/p^n}(T_f) = \bigoplus_{i=0}^{f} H^*\left(\mathbb{Z}/p^n, \mathbb{Z}\left[\xi_{p^s}\right]\right)^{\binom{f}{i}}\{2i-f\}.$$

Proof. Since $CKh(T_f)$ is a complex of trivial $\mathbb{Z}[\mathbb{Z}/p^n]$-modules, it follows readilly that

$$\begin{aligned} Kh^{*,*,p^s}_{\mathbb{Z}/p^n}(T_f) &= \mathbb{E}xt^{*,*}_{\mathbb{Z}[\mathbb{Z}/p^n]}\left(\mathbb{Z}\left[\xi_{p^s}\right], CKh(T_f)\right) \cong \\ &\cong H^{*,*}\left(\mathbb{Z}/p^n, Kh^{0,*}(T_f) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\xi_{p^s}\right]\right) \cong \\ &\cong \bigoplus_{i=0}^{f} H^*\left(\mathbb{Z}/p^n, \mathbb{Z}\left[\xi_{p^s}\right]\right)^{\binom{f}{i}}\{2i-f\}, \end{aligned}$$

because according to Example 2.3.22

$$Kh^{0,*}(T_f) = \mathcal{A}^{\otimes f}.$$

□

Computation of the equivariant Khovanov homology of the trivial link T_{kp^n}, whose components are freely permuted by the action of the cyclic group is a little more involved. In order to do that, let us define the following family of polynomials.

Definition 3.2.6. Define a sequence of Laurent polynomials i.e. elements of the ring $\mathbb{Z}\left[q, q^{-1}\right]$.

$$\begin{aligned} \mathcal{P}_0(q) &= q + q^{-1} \\ \mathcal{P}_n(q) &= \frac{1}{p^n} \sum_{\substack{1 \leqslant k \leqslant p^n-1 \\ \gcd(k,p^n)=1}} \binom{p^n}{k} q^{2k-p^n} + \\ &+ \frac{1}{p^n} \sum_{1 \leqslant s < n} \sum_{\substack{1 \leqslant k \leqslant p^n-1 \\ \gcd(k,p^n)=p^s}} \left(\binom{p^n}{k} - \binom{p^{n-s}}{k'}\right) q^{2k-p^n}, \end{aligned}$$

where $k' = k/p^s$ and $n \geqslant 1$.

Proposition 3.2.7. The Khovanov complex $CKh(T_{kp^n})$ can be decomposed into a direct sum of permutation modules in the following way

$$CKh^{0,*}(T_{kp^n}) = \bigoplus_{s=0}^{n} \mathrm{Ind}^{\mathbb{Z}/p^n}_{\mathbb{Z}/p^{n-s}} M^k_s,$$

where M_s^k is a free abelian group and a trivial $\mathbb{Z}/p^{n-s}$-module satisfying

$$\operatorname{qdim} M_s^k = \sum_{\ell=1}^{k} p^{s\cdot(\ell-1)} \mathcal{P}_s(q^{p^{n-s}})^{\ell} \sum_{\substack{0\leqslant i_0,\ldots,i_{s-1}\leqslant k \\ i_0+\ldots+i_{s-1}=k-\ell}} \prod_{j=0}^{s-1} \left(p^j \mathcal{P}_j(q^{p^{n-j}})\right)^{i_j}.$$

Proof. Since the induced action on

$$CKh^{*,*}(T_{kp^n}) = CKh^{0,*}(T_{kp^n}) = \underbrace{\mathcal{A}^{\otimes k} \otimes \mathcal{A}^{\otimes k} \otimes \ldots \otimes \mathcal{A}^{\otimes k}}_{p^n}$$

permutes the factors in the tensor product above, thus it is sufficient to consider the restriction of this action to the basis of $\mathcal{A}^{\otimes kp^n}$ consisting of tensor products of vectors $\mathbb{1}$ and X, where $\mathcal{A}$, $\mathbb{1}$ and X are defined in Example 2.3.18.

Let us start with the case $k = 1$. Denote by ℓ the following map

$$\ell\colon \mathcal{A}^{\otimes p^n} \to \mathcal{A}^{\otimes p^n},$$
$$\ell\colon x_1 \otimes x_2 \otimes \ldots \otimes x_{p^n} \mapsto x_2 \otimes \ldots \otimes x_{p^n} \otimes x_1.$$

In order to obtain the desired decomposition of $CKh^{0,*}(T_{p^n})$ it is sufficient to decompose the basis of $\mathcal{A}^{\otimes p^n}$ into a disjoint union of orbits. Observe that if $v = x_1 \otimes \ldots \otimes x_{p^n}$ satisfies $\ell^{p^s}(v) = v$, for some $s \leqslant n$, then v is completely determined by its first p^s factors $x_1, \ldots, x_{p^s}$.

Lemma 3.2.8. The set of basis vectors satisfying the following conditions, for fixed $s \leqslant n$,

1. $\operatorname{Iso}(v) = \mathbb{Z}/p^{n-s}$,
2. $\deg v = p^{n-s}(2k - p^s)$, for $1 \leqslant k \leqslant p^s - 1$

has cardinality

$$= \begin{cases} \binom{p^s}{k}, & \gcd(k, p^s) = 1, \\ \binom{p^s}{k} - \binom{p^{s-u}}{k'}, & \gcd(k, p^s) = p^u, \end{cases}$$

where $k' = k/p^u$. In particular, $\operatorname{qdim} M_s^1 = \mathcal{P}_s(q^{p^{n-s}})$.

Proof. Notice first, that if a vector v is fixed by ℓ^{p^s}, then necessarily $p^{n-s} \mid \deg v$. If k is not divisible by p, then v automatically satisfies the first condition and cardinality of the set of such vectors is equal to $\binom{p^s}{k}$.

On the other hand, when $\gcd(k, p^s) = p^u$, there are vectors v such that $\deg v = p^{n-s}(2k - p^n)$ and $\operatorname{Iso}(v)$ contains properly $\mathbb{Z}/p^{n-s}$. There are exactly $\binom{p^{s-u}}{k/p^s}$ such vectors. Therefore, the overall cardinality in this case is equal to $\binom{p^s}{k} - \binom{p^{s-u}}{k'}$. □

To perform the inductive step, notice that if $v \in \mathcal{A}^{(k-1)p^n}$ and $w \in \mathcal{A}^{p^n}$ satisfy $\mathrm{Iso}(v) = \mathbb{Z}/p^{n-s}$ and $\mathrm{Iso}(w) = \mathbb{Z}/p^{n-s'}$, then $\mathrm{Iso}(v \otimes w) = \mathbb{Z}/p^{\min(n-s,n-s')}$. When $\mathrm{Iso}(v) = \mathbb{Z}/p^{n-s}$, then the orbit of v can be identified with $\mathbb{Z}/p^s$, with the action coming from the following quotient map

$$\mathbb{Z}/p^n \to \mathbb{Z}/p^s.$$

The product of two orbits $\mathbb{Z}/p^s$ and $\mathbb{Z}/p^{s'}$, with the diagonal action, consists of several orbits. This decomposition is given below.

$$\mathbb{Z}/p^s \times \mathbb{Z}/p^{s'} = \bigsqcup_{i=1}^{p^{s'}} \mathbb{Z}/p^s, \tag{5}$$

if $s \geqslant s'$. Thus,

$$M_s^k = \bigoplus_{0 \leqslant s' < s} \left(\left(M_s^{k-1} \otimes_{\mathbb{Z}} M_{s'}^1\right)^{p^{s'}} \oplus \left(M_{s'}^{k-1} \otimes_{\mathbb{Z}} M_s^1\right)^{p^{s'}} \right) \\ \oplus \left(M_s^{k-1} \otimes_{\mathbb{Z}} M_s^1\right)^{p^s}.$$

Consequently,

$$\operatorname{qdim} M_s^k = \sum_{0 \leqslant s' < s} p^{s'} (\operatorname{qdim} M_s^{k-1} \cdot \operatorname{qdim} M_{s'}^1 \\ + \operatorname{qdim} M_{s'}^{k-1} \cdot \operatorname{qdim} M_s^1) + \operatorname{qdim} M_s^{k-1} \cdot \operatorname{qdim} M_s^1.$$

Expanding this inductive formula yields the desired result. □

Corollary 3.2.9. Polynomials $\mathcal{P}_n$ satisfy the following equality.

$$(q + q^{-1})^{p^n} = \sum_{s=0}^{p^n} p^s \mathcal{P}_s(q^{p^{n-s}}).$$

Proof. This follows readilly from the previous proposition applied to the case $k = 1$. Indeed, since

$$\mathrm{CKh}^{0,*}(T_{p^n}) = \bigoplus_{s=0}^{n} \mathrm{Ind}_{\mathbb{Z}/p^{n-s}}^{\mathbb{Z}/p^n} M_s^1,$$

it follows that

$$(q + q^{-1})^{p^n} = \operatorname{qdim} \mathrm{CKh}^{0,*}(T_{p^n}) = \sum_{s=0}^{n} p^s \operatorname{qdim} M_s^1 = \\ = \sum_{s=0}^{n} p^s \mathcal{P}_s(q^{p^{n-s}}).$$

□

Corollary 3.2.10. Let T_{kp^n+f} be the p^n-periodic trivial link, whose components are divided into k free orbits and f fixed circles.

$$\mathrm{CKh}^{0,*}(T_{kp^n+f}) = \bigoplus_{s=0}^{n} \mathrm{Ind}_{\mathbb{Z}/p^{n-s}}^{\mathbb{Z}/p^n} M_s^{k,f},$$

where $M_s^{k,f}$ is a trivial $\mathbb{Z}/p^{n-s}$-module such that

$$\operatorname{qdim} M_s^{k,f} = (q+q^{-1})^f \operatorname{qdim} M_s^k.$$

Proof. Corollary follows from Proposition 3.2.7. Indeed, because

$$\mathrm{CKh}^{0,*}(T_{kp^n+f}) = \mathrm{CKh}^{0,*}(T_{kp^n}) \otimes_{\mathbb{Z}} \mathrm{CKh}^{0,*}(T_f).$$

Arguing as in the proof of Proposition 3.2.7 we can consider orbits of vectors which are tensor products of $\mathbb{1}$ and X. Every such orbits is a tensor product of an orbit from $\mathrm{CKh}^{0,*}(T_{kp^n})$ and a trivial orbit from $\mathrm{CKh}^{0,*}(T_f)$. This additional trivial orbits preserves the isotropy of the orbit, therefore

$$M_s^{k,f} = M_s^k \otimes_{\mathbb{Z}} \mathcal{A}^{\otimes f},$$

which concludes the proof. □

Analysis of the Khovanov complex of the trivial link yields the following result giving explicit formulas for the equivariant Khovanov homology of trivial links.

Proposition 3.2.11.

$$\mathrm{Kh}_{\mathbb{Z}/p^n}^{0,*,p^{n-u}}(T_{kp^n+f}) = \bigoplus_{s=0}^{n} W_{s,u}^{k,f},$$

where

$$W_{s,u}^{k,f} = \begin{cases} \bigoplus_i H^*(\mathbb{Z}/p^{n-s}, \mathbb{Z}^{d_i})^{\varphi(p^{n-u})}\{i\}, & n-s \leqslant u, \\ \bigoplus_i H^*(\mathbb{Z}/p^{n-s}, \mathbb{Z}\left[\xi_{p^{n-s-u}}\right]^{d_i})^{p^s}\{i\}, & n-s > u. \end{cases}$$

and

$$\operatorname{qdim} M_s^{k,f} = \sum_i d_i q^i,$$

for some non-negative integers d_i.

Proof. From corollary 3.2.10 one obtains equality

$$\mathrm{Ext}_{\mathbb{Z}/p^n}^{*,*}\left(\mathbb{Z}\left[\xi_{p^{n-u}}\right], \mathrm{CKh}^{0,*}(T_{kp^n+f})\right) =$$
$$= \bigoplus_{s=0}^{n} \mathrm{Ext}_{\mathbb{Z}/p^n}^{*,*}\left(\mathbb{Z}\left[\xi_{p^{n-u}}\right], \mathrm{Ind}_{\mathbb{Z}/p^{n-s}}^{\mathbb{Z}/p^n} M_s^{k,f}\right).$$

From Shapiro's Lemma and Proposition 2.1.19

$$\begin{aligned}
&\mathrm{Ext}^{*,*}_{\mathbb{Z}/p^n}\left(\mathbb{Z}\left[\xi_{p^{n-u}}\right], \mathrm{Ind}^{\mathbb{Z}/p^n}_{\mathbb{Z}/p^{n-s}} M^{k,f}_s\right) \cong \\
&\cong \mathrm{Ext}^{*,*}_{\mathbb{Z}/p^s}\left(\mathrm{Res}^{\mathbb{Z}/p^n}_{\mathbb{Z}/p^{n-s}} \mathbb{Z}\left[\xi_{p^{n-u}}\right], M^{k,f}_s\right) \cong \\
&\cong \begin{cases} \mathrm{Ext}^{*,*}_{\mathbb{Z}/p^s}\left(\mathbb{Z}^{\varphi(p^{n-u})}, M^{k,f}_s\right), & n-s \leqslant u \\ \mathrm{Ext}^{*,*}_{\mathbb{Z}/p^s}\left(\mathbb{Z}\left[\xi_{p^{n-s-u}}\right]^{p^s}, M^{k,f}_s\right), & n-s > u. \end{cases}
\end{aligned}$$

The rest follows from [3, Prop. 2.2]. □

Theorem 3.2.4 along with Propositions 3.2.7 and 3.2.11 imply that the equivariant Khovanov homology contains an abundance of torsion not present in the classical Khovanov homology. However, we can use the additional algebraic structure of the equivariant Khovanov homology to obtain some information about this extra torsion. To be more precise, Proposition 2.2.18 implies that the equivariant Khovanov homology $Kh^{*,*,p^s}_{\mathbb{Z}/p^n}(D)$ is a module over the ext ring $\mathrm{Ext}^*_{\mathbb{Z}[\mathbb{Z}/p^n]}(\mathbb{Z}[\xi_{p^s}], \mathbb{Z}[\xi_{p^s}])$. This is the key to understand the additional torsion.

Proposition 3.2.12. For any $r > 0$, the ring $\mathrm{Ext}^*_{\mathbb{Z}[\mathbb{Z}/p^n]}(\mathbb{Z}[\xi_{p^s}], \mathbb{Z}[\xi_{p^s}])$ acts on ${}_{II}E_r$, where $({}_{II}E_r, d_r)$ is the Cartan-Eilenberg spectral sequence converging to $Kh^{*,*,p^s}_{\mathbb{Z}/p^n}(D)$. To be more precise, the ring

$$\mathrm{Ext}^*_{\mathbb{Z}[\mathbb{Z}/p^n]}(\mathbb{Z}[\xi_{p^s}], \mathbb{Z}[\xi_{p^s}])$$

acts on the columns of the spectral sequence.

$${}_{II}E^{i,j}_r \otimes_{\mathbb{Z}} \mathrm{Ext}^k_{\mathbb{Z}[\mathbb{Z}/p^n]}(\mathbb{Z}[\xi_{p^s}], \mathbb{Z}[\xi_{p^s}]) \to {}_{II}E^{i,j+k}_r.$$

Moreover, on E_1 page this action agrees with the natural action of the ext ring on the module $\mathrm{Ext}^{*,*}_{\mathbb{Z}[\mathbb{Z}/p^n]}\left(\mathbb{Z}[\xi_{p^s}], CKh^{i,*}(D)\right)$.

Proof. It is a direct consequence of theorem 2.2.20, because the appropriate version of the Cartan-Eilenberg spectral sequence for

$$\mathrm{Ext}^*_{\mathbb{Z}[\mathbb{Z}/p^n]}(\mathbb{Z}[\xi_{p^s}], \mathbb{Z}[\xi_{p^s}])$$

collapses at E_1. □

Corollary 3.2.13. Let T_s denote the cohomology class in the ext ring

$$T_s \in \mathrm{Ext}^2_{\mathbb{Z}[\mathbb{Z}/p^n]}(\mathbb{Z}[\xi_{p^s}], \mathbb{Z}[\xi_{p^s}])$$

from proposition 2.2.24. Multiplication by T_s

$$- \cup T_s \colon Kh^{i,*,p^s}_{\mathbb{Z}/p^n}(D) \to Kh^{i+2,*,p^s}_{\mathbb{Z}/p^n}(D)$$

is an epimorphism for $i = n_+$ and isomorphism for $i > n_+$.

Remark 3.2.14. This is an analogue of [34, Lemma 1.1].

Proof. Proposition 2.2.24 implies that on the E_1 page of the Cartan-Eilenerg spectral sequence, T_s acts by epimorphisms for $j = 0$ and by isomorphisms for $j > 0$. First, we will show that this is also the case for $r \geqslant 1$.

Lemma 3.2.15. For $r \geqslant 1$, then multiplication by T_s

$$_{II}E_r^{i,j} \to {}_{II}E_r^{i,j+2}.$$

is an epimorphism for $j = 0$ and an isomorphism for $j > 0$.

Proof. The argument is inductive. As it was pointed out earlier, the case of $r = 1$ follows readilly from Proposition 2.2.24. To perform the inductive step, notice that there exists the following commutative diagram of short exact sequences,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_r^{i,j} & \longrightarrow & K_r^{i,j} & \longrightarrow & E_{r+1}^{i,j} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle - \cup T_s} & & \downarrow{\scriptstyle - \cup T_s} & & \downarrow{\scriptstyle - \cup T_s} & & \\ 0 & \longrightarrow & B_r^{i,j+2} & \longrightarrow & K_r^{i,j+2} & \longrightarrow & E_{r+1}^{i,j+2} & \longrightarrow & 0 \end{array}$$

where

$$B_r^{i,j} = \operatorname{Im} d_r \cap E_r^{i,j},$$
$$K_r^{i,j} = \ker d_r \cap E_r^{i,j}.$$

From the inductive hypothesis we know that the two leftmost vertical homomorphisms are epimorphisms or isomorphisms, respectively, thus the third vertical homomorphism is an epimorphism or isomorphism, respectively, by the Five Lemma. $\square$

Since the filtration, from which the Cartan-Eilenberg spectral sequence originates, is finite, it guarantees that the maps of the spectral sequence induced by T_s converges to the right target. To retrieve the map on the equivariant Khovanov homology it is necessary to analyze the map on each step of the filtration. This is also done inductively. The starting point is the consideration of

$$\mathcal{F}^{n_+(D)} \operatorname{Kh}^{n,*,p^s}_{\mathbb{Z}/p^n}(D) = E_\infty^{n_+(D), n-n_+(D)+1}.$$

The indcutive step is performed with the aid of the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}^p \operatorname{Kh}^{n,*,p^s}_{\mathbb{Z}/p^n}(D) & \longrightarrow & \mathcal{F}^{p-1} \operatorname{Kh}^{n,*,p^s}_{\mathbb{Z}/p^n}(D) & \longrightarrow & E_\infty^{p-1,n-p+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle - \cup T_s} & & \downarrow{\scriptstyle - \cup T_s} & & \downarrow{\scriptstyle - \cup T_s} & & \\ 0 & \longrightarrow & \mathcal{F}^p \operatorname{Kh}^{n+2,*,p^s}_{\mathbb{Z}/p^n}(D) & \longrightarrow & \mathcal{F}^{p-1} \operatorname{Kh}^{n+2,*,p^s}_{\mathbb{Z}/p^n}(D) & \longrightarrow & E_\infty^{p-1,n-p+3} & \longrightarrow & 0 \end{array}$$

The inductive step consists of noticing that if $n \geqslant n_+$ the Corollary follows by application of the Five Lemma. □

Corollary 3.2.16. For $i > n_+$, $Kh^{i,*,1}_{\mathbb{Z}/p^n}(D)$ is annihilated by p^n, and for $1 \leqslant s \leqslant n$, $Kh^{i,*,p^s}_{\mathbb{Z}/p^n}(D)$ is annihilated by p^{n-s+1}.

Proof. This follows from previous corollary and the fact that $p^n T_0 = 0$ and $p^{n-s+1} T_s = 0$, for $1 \leqslant s \leqslant n$. □

3.2.2 *Rational equivariant Khovanov homology*

Since the rational group algebra $\mathbb{Q}[\mathbb{Z}/n]$ is semi-simple and artinian, the algebraic structure of the equivariant Khovanov homology with rational coefficients simplifies considerably.

Proposition 3.2.17. If D is an n-periodic link diagram, then

$$CKh^{*,*}(D;\mathbb{Q}) \cong \bigoplus_{d|n} CKh^{*,*,d}_{\mathbb{Z}/n}(D;\mathbb{Q}),$$

where

$$CKh^{*,*,d}_{\mathbb{Z}/n}(D;\mathbb{Q}) = \operatorname{Hom}_{\mathbb{Q}[\mathbb{Z}/n]}\left(\mathbb{Q}[\xi_d], CKh^{*,*}(D;\mathbb{Q})\right).$$

Moreover,

$$CKh^{*,*,d}_{\mathbb{Z}/n}(D;\mathbb{Q}) = CKh^{*,*}(D;\mathbb{Q}) \cdot e_d,$$

and

$$Kh^{*,*,d}_{\mathbb{Z}/n}(D;\mathbb{Q}) = Kh^{*,*}(D;\mathbb{Q}) \cdot e_d = H^{*,*}(CKh(D;\mathbb{Q}) \cdot e_d),$$

where e_d is one of the central idempotents from Example 1.

Proof. The proposition is a consequence of the Wedderburn decomposition and Schur's Lemma. For more details refer to chapter 2.1.1 or [6, Chap. 1]. □

The above proposition has the following corollary.

Corollary 3.2.18. Suppose that D is an n-periodic diagram of a link. Choose $d \mid n$. If for any i, j we have $\dim_{\mathbb{Q}} Kh^{i,j}(D;\mathbb{Q}) < \varphi(d)$, wher φ denotes the Euler's totient function, then $Kh^{*,*,d}_{\mathbb{Z}/n}(D;\mathbb{Q}) = 0$.

Proof. Indeed, since $Kh^{*,*,d}_{\mathbb{Z}/n}(D;\mathbb{Q})$ is a $\mathbb{Q}[\xi_d]$ vector space, it follows that $\dim_{\mathbb{Q}} Kh^{*,*,d}_{\mathbb{Z}/n}(D;\mathbb{Q})$ is divisible by $\dim_{\mathbb{Q}} \mathbb{Q}[\xi_d] = \varphi(d)$. □

The above corollary can be used to compute the equivariant Khovanov homology in some cases.

Corollary 3.2.19. Let $T(n,2)$ be the torus link. Let $d > 2$ be a divisor of n. According to Example 3.1.3, $T(n,2)$ is d-periodic. Let $d' > 2$ and $d' \mid d$.

$$Kh^{*,*,d'}_{\mathbb{Z}/d}(T(n,2);\mathbb{Q}) = 0.$$

Proof. Indeed, because according to [10, Prop. 35] for all i, j we have

$$\dim_{\mathbb{Q}} \mathrm{Kh}^{i,j}(T(n,2);\mathbb{Q}) \leqslant 1$$

and $\varphi(d') > 1$ if $d' > 2$. □

Corollary 3.2.20. Let $\gcd(3,n) = 1$. The 3-equivariant Khovanov homology $\mathrm{Kh}^{*,*,3}_{\mathbb{Z}/3}(T(n,3);\mathbb{Q})$ of the torus knot $T(n,3)$ vanishes.
If $d > 2$ divides n, $d' > 2$ and $d' \mid d$, then $\mathrm{Kh}^{*,*,d'}_{\mathbb{Z}/d}(T(n,3);\mathbb{Q}) = 0$.

Proof. Indeed, because [33, Thm. 3.1] implies that for all i, j we have

$$\dim_{\mathbb{Q}} \mathrm{Kh}^{i,j}(T(n,3);\mathbb{Q}) \leqslant 1,$$

provided that $\gcd(3,n) = 1$. □

4 THE SPECTRAL SEQUENCE

Computation of the classical Khovanov homology is usually done with the aid of the exact sequence from Theorem 2.3.21, applied to a chosen link crossing, which is then resolved in two different ways. However, resolution of a single crossing kills the symmetry of the periodic link diagram in question. Instead, we need to resolve a whole orbit of crossings to obtain another periodic diagram. But this is not sufficient to recover the equivariant Khovanov homology. We have to take into account all possible resolutions of a chosen orbit of crossings. These data are organized into a spectral sequence the construction of which is the main goal of this chapter. During the construction, we obtain a filtration which can be applied to get the spectral sequence from [8]. The spectral sequence that we obtain is applied to the computation of the 2-equivariant Khovanov homology of torus links $T(n, 2)$.

4.1 CONSTRUCTION OF THE SPECTRAL SEQUENCE

Start with a link L and its n-periodic diagram D. Choose a subset of crossings $X \subset \operatorname{Cr} D$.

Definition 4.1.1. Let $\alpha\colon \operatorname{Cr}(D) \to \{0, 1, x\}$ be a map.

1. If $i \in \{0, 1, x\}$ define $|\alpha|_i = \#\alpha^{-1}(i)$.
2. Define the support of α to be $\operatorname{supp} \alpha = \alpha^{-1}(\{0, 1\})$.
3. Define also the following family of maps
$$\mathcal{B}_k(X) = \{\alpha\colon \operatorname{Cr}(D) \to \{0, 1, x\} \quad | \quad \operatorname{supp} \alpha = X, \quad |\alpha|_1 = k\}.$$
4. Denote by D_α the diagram obtained from D by resolving crossings from $\alpha^{-1}(0)$ by 0- and from $\alpha^{-1}(1)$ by 1-smoothing.

First, we will work with $\overline{\mathrm{CKh}}(D)$, as defined in 2.3.19, and construct a filtration on this complex. The filtration on $\mathrm{CKh}(D)$ will be obtained from this one by an appropriate shift in degree.

Fix a crossing $c \in \operatorname{Cr}(D)$ and consider three functions
$$\alpha_0, \alpha_1, \alpha_x\colon \operatorname{Cr}(D) \to \{0, 1, x\},$$
which attain different value at c, i.e. $\alpha_1(c) = 1$, $\alpha_x(c) = x$ and $\alpha_0(c) = 0$, and are identical otherwise. This data yields the following short exact sequence of complexes.
$$0 \to \overline{\mathrm{CKh}}(D_{\alpha_1})[1]\{1\} \to \overline{\mathrm{CKh}}(D_{\alpha_x}) \to \overline{\mathrm{CKh}}(D_{\alpha_0}) \to 0, \tag{6}$$

as in [33]. However, according to Theorem 2.3.21 there exists a chain map

$$\delta_c\colon \overline{\mathrm{CKh}}(D_{\alpha_0}) \to \overline{\mathrm{CKh}}(D_{\alpha_1})\{1\},$$

such that $\overline{\mathrm{CKh}}(D_{\alpha_x}) = \mathrm{Cone}(\delta_c)$, where $\mathrm{Cone}(\delta_c)$ denotes the algebraic mapping cone of δ_c, and (6) is the corresponding short exact sequence of complexes. The map δ_c is obtained as follows. We identify $\overline{\mathrm{CKh}}(D_{\alpha_0})$ and $\overline{\mathrm{CKh}}(D_{\alpha_1})$ with submodules of $\overline{\mathrm{CKh}}(D_{\alpha_x})$ "generated" by Kauffman states with c resolved by 0- or 1-smoothing, respectively. As a graded module $\overline{\mathrm{CKh}}(D_{\alpha_x})$ splits in the following way

$$\overline{\mathrm{CKh}}(D_{\alpha_x}) = \overline{\mathrm{CKh}}(D_{\alpha_0}) \oplus \overline{\mathrm{CKh}}(D_{\alpha_1}).$$

Additionally, $\overline{\mathrm{CKh}}(D_{\alpha_1})$ is a subcomplex. If π_1 denotes the projection of $\overline{\mathrm{CKh}}(D_{\alpha_x})$ onto $\overline{\mathrm{CKh}}(D_{\alpha_1})$ and i_0 denote the inclusion of $\overline{\mathrm{CKh}}(D_{\alpha_0})$, then

$$\delta_c = \pi_1 \circ d \circ i_0.$$

When we consider two crossings c and c', we obtain the following bicomplex

$$\overline{\mathrm{CKh}}(D_{\alpha_{00}}) \longrightarrow \begin{matrix} \overline{\mathrm{CKh}}(D_{\alpha_{10}})\{1\} \\ \oplus \\ \overline{\mathrm{CKh}}(D_{\alpha_{01}})\{1\} \end{matrix} \longrightarrow \overline{\mathrm{CKh}}(D_{\alpha_{11}})\{2\}$$

where $\alpha_{00}, \alpha_{10}, \alpha_{01}, \alpha_{11}$ differ only at c or c' and

$$\begin{aligned} &\alpha_{00}(c) = \alpha(c') = 0, \\ &\alpha_{10}(c) = 1, \quad \alpha_{10}(c') = 0, \\ &\alpha_{01}(c) = 0, \quad \alpha_{01}(c') = 1, \\ &\alpha_{11}(c) = \alpha_{11}(c') = 1. \end{aligned}$$

The horizontal maps are defined analogously as in the previous case. The total complex of the above bicomplex is equal to the shifted Khovanov complex of $D_{\alpha_{xx}}$, where α_{xx} agrees with α_{00}, α_{10}, α_{01} and α_{11} outside c and c' and

$$\alpha_{xx}(c) = \alpha_{xx}(c') = x.$$

Continuing this procedure we obtain the following bicomplex

$$N^{i,j,k} = \begin{cases} \bigoplus_{\alpha \in \mathcal{B}_i(X)} \overline{\mathrm{CKh}}^{j,k}(D_\alpha)\{i\}, & 0 \leqslant i \leqslant \#X, \\ 0, & \text{otherwise.} \end{cases}$$

The total complex of N, denoted by $\mathrm{Tot}(N)^{*,*}$ and defined by

$$\mathrm{Tot}(N)^{i,j} = \bigoplus_{k+l=i} N^{k,l,j},$$

is equal to the shifted Khovanov complex $\overline{\mathrm{CKh}}(D)$.

Definition 4.1.2. Let $M^{*,*,*}$ be a bicomplex of graded $\mathbb{Z}$-modules such that

$$M^{i,j,k} = \begin{cases} \bigoplus_{\alpha \in \mathcal{B}_i(X)} \mathrm{CKh}^{j,k}(D_\alpha)[c(D_\alpha)]\{i + 3c(D_\alpha) + \#X\}, & 0 \leqslant i \leqslant \#X, \\ 0, & \text{otherwise,} \end{cases}$$

where $c(D_\alpha) = n_-(D_\alpha) - n_-(D)$. Vertical differentials

$$M^{i,j,*} \to M^{i,j+1,*}$$

are sums of ± 1 multiplicities of differentials in the respective Khovanov complexes. Horizontal differentials

$$M^{i,j,*} \to M^{i+1,j,*},$$

on the other hand, are induced from the appropriate horizontal maps in the bicomplex $N^{*,*,*}$.

Proposition 4.1.3. The total complex of $M^{*,*,*}$ is equal to the Khovanov complex $\mathrm{CKh}(D)$.

Proof. Since the total complex of $N^{*,*,*}$ is equal to $\overline{\mathrm{CKh}}(D)$ and

$$\mathrm{CKh}(D) = \overline{\mathrm{CKh}}(D)[-n_-(D)]\{n_+(D) - 2n_-(D)\},$$

we only need to check, that the application of the proper shift to N results in M.

$$\begin{aligned} &N^{i,j,k}[-n_-(D)]\{n_+(D) - 2n_-(D)\} = \\ &= \bigoplus_{\alpha \in \mathcal{B}_i(X)} \overline{\mathrm{CKh}}^{j,k}(D_\alpha)[-n_-(D)]\{i + n_+(D) - 2n_-(D)\} = \\ &= \bigoplus_{\alpha \in \mathcal{B}_i(X)} \mathrm{CKh}^{j,k}(D_\alpha)[c(D_\alpha)]\{i + 2c(D_\alpha) + n_+(D) - n_+(D_\alpha)\} = \\ &= \bigoplus_{\alpha \in \mathcal{B}_i(X)} \mathrm{CKh}^{j,k}(D_\alpha)[c(D_\alpha)]\{i + 3c(D_\alpha)\#X\} = M^{i,j,k}, \end{aligned}$$

because

$$\begin{aligned} &n_+(D) - n_+(D_\beta) = \\ &= \#\mathrm{Cr}(D) - n_-(D) - (\#\mathrm{Cr}(D) - \#X - n_-(D_\beta)) = \\ &= c(D_\beta) + \#X. \end{aligned}$$

□

Definition 4.1.4. Let

$$\mathcal{F}_i(X) = \mathrm{Tot}(\bigoplus_{j \geqslant i} M^{j,*,*}),$$

for $0 \leqslant i \leqslant \#X$. The family $\{\mathcal{F}_i(D)\}_i$ is a filtration of the Khovanov complex. This filtration is sometimes called the the column filtration of the bicomplex $M^{*,*,*}$, see [13, Thm. 2.15].

The following theorem was first proved in [8].

Theorem 4.1.5. Let D be a link diagram and let $X \subset \mathrm{Cr}(D)$. The pair (D, X) determines a spectral sequence

$$\{E_r^{*,*,*}, d_r\}$$

of graded modules converging to $\mathrm{Kh}^{*,*}(D)$ such that

$$E_1^{i,j,*} = \bigoplus_{\beta \in \mathcal{B}_i(X)} \mathrm{Kh}^{j,*}(D_\beta)[c(D_\beta)]\{i + 3c(D_\beta) + \#X\},$$

where $c(D_\beta) = n_-(D_\beta) - n_-(D)$.

Proof. This is the spectral sequence associated to the column filtration of the bicomplex M as in [13, Thm. 2.15]. □

Suppose now, that D is an n-periodic link diagram. If $X \subset \mathrm{Cr}\, D$ is invariant, under the action of $\mathbb{Z}/n$, then for any $0 \leqslant k \leqslant \#X$ there exists an induced action on $\mathcal{B}_k(X)$. Hence, each member $\mathcal{F}_k(X)$, for $0 \leqslant k \leqslant \#X$, of the filtration is invariant under the action of $\mathbb{Z}/n$. This discussion leads to the following conclusion.

Proposition 4.1.6. If $X \subset \mathrm{Cr}\, D$ is an invariant subset, then every member of the filtration, from Definition 4.1.4 is a $\mathbb{Z}[\mathbb{Z}/n]$-subcomplex of $\mathrm{CKh}(D)$.

From now on, we will assume that X consists of a single orbit of crossings. We will perform analysis of the quotients $\mathcal{F}_i(X)/\mathcal{F}_{i+1}(X)$ to determine their structure as $\mathbb{Z}[\mathbb{Z}/n]$-modules.

Definition 4.1.7. Let $0 \leqslant i \leqslant n$ and $d \mid \gcd(n, i)$. Analogously as in the previous chapter denote by $\mathcal{B}_i^d(X)$ the subset of $\mathcal{B}_i(X)$ consisting of maps satisfying $\mathrm{Iso}(\alpha) = \mathbb{Z}/d$. Also denote by $\overline{\mathcal{B}}_i^d(X)$ the quotient of $\mathcal{B}_i^d(X)$ by $\mathbb{Z}/n$.

Lemma 4.1.8. If $\alpha \in \mathcal{B}_i^d(X)$, then D_α is $\mathbb{Z}/d$-periodic.

Proof. The Lemma follows readily, because such diagrams have similar structure as the Kauffman states belonging to $\mathcal{S}_i^d(D)$, as in Definition 3.1.6. □

Proposition 4.1.9. Suppose, that n is odd, then for $0 \leqslant i \leqslant n$

$$\mathcal{F}_i(X)/\mathcal{F}_{i+1}(X) = \\ = \bigoplus_{d \mid \gcd(n,i)} \bigoplus_{\alpha \in \overline{\mathcal{B}}_i^d(X)} \mathrm{Ind}_{\mathbb{Z}/d}^{\mathbb{Z}/n} \left(\mathrm{CKh}(D_\alpha)[t(\alpha)]\{q(\alpha)\}\right),$$

where

$$t(\alpha) = c(D_\alpha) + i,$$
$$q(\alpha) = i + 3c(D_\alpha) + n.$$

Proof. To proof of the Lemma uses an adaptation of the argument from the proof of Lemma 3.1.9 and Corollary 3.1.10. □

Proposition 4.1.10. Let $n = 2$. Under this assumption we have

$$\mathcal{F}_0(X)/\mathcal{F}_1(X) = CKh(D_{\alpha_{00}})[t(\alpha_{00})]\{q(\alpha_{00})\} \otimes_{\mathbb{Z}} \mathbb{Z}_-^{\otimes \frac{c(D_{\alpha_{00}})}{2}},$$
$$\mathcal{F}_1(X)/\mathcal{F}_2(X) = CKh(D_{\alpha_{10}})[t(\alpha_{10})]\{q(\alpha_{10})\} \otimes_{\mathbb{Z}} \mathbb{Z}\,[\mathbb{Z}/2]\,,$$
$$\mathcal{F}_2(X) = CKh(D_{\alpha_{11}})[t(\alpha_{11})]\{q(\alpha_{11})\} \otimes_{\mathbb{Z}} \mathbb{Z}_-^{\otimes \frac{c(D_{\alpha_{11}})}{2}+1}.$$

Proof. The middle equality follows easily, because complexes corresponding to $D_{\alpha_{10}}$ and $D_{\alpha_{01}}$ are freely permuted by $\mathbb{Z}/2$.

To check the first and third inequality, recall from Equation 3 from Chapter 3 the sign convention. We have two distinguished crossings $c_1, c_2 \in \operatorname{supp} \alpha_{00}$. Let us denote by T, as in Chapter 3, the tangle from which D was constructed. Let $x_1, x_2 \in \Lambda^* W_T$ be such that $x_1 \otimes x_2$ corresponds to certain Kauffman state of $D_{\alpha_{00}}$. According to our convention, the permutation map acts on $CKh(D_{\alpha_{00}})$ as follows

$$x_1 \otimes x_2 \mapsto (-1)^{\frac{n_-(D_{\alpha_{00}})}{2} + \deg x_1 \deg x_2} x_2 \otimes x_1,$$

whereas on $CKh(D)$ it acts as follows.

$$x_1 \otimes x_2 \mapsto (-1)^{\frac{n_-(D)}{2} + \deg x_1 \deg x_2} x_2 \otimes x_1.$$

Comparison of the two coefficients yields, that we need to twist the action of $\mathbb{Z}/2$ on $CKh(D_{\alpha_{00}})$ by $\mathbb{Z}_-^{\otimes \frac{c(D_{\alpha_{00}})}{2}}$.

Analogous argument applied to D_{11} yields the third equality. □

Let p be an odd prime and $n > 0$ an integer. We state the next theorem only for 2-periodic and p^n-periodic links, since these cases will be of importance in the remainder part of the thesis. The statement in other cases can be analogously derived, however we omit it due to its technical complication, which dims the whole idea of the spectral sequence.

Theorem 4.1.11. Let L be a p^n-periodic link, where p is an odd prime, and let $X \subset \operatorname{Cr} D$ consists of a single orbit. Then for any $0 \leqslant s \leqslant n$ there exists a spectral sequence $\{{}_{p^{n-s}}E_r^{*,*}, d_r\}$ of graded modules converging to $Kh^{*,*,p^{n-s}}_{\mathbb{Z}/p^n}(D)$ with

$${}_{p^{n-s}}E_1^{0,j} = Kh^{j,*,p^{n-s}}_{\mathbb{Z}/p^n}(D_{\alpha_0})[c(D_{\alpha_0})]\{q(\alpha_0)\},$$
$${}_{p^{n-s}}E_1^{p^n,j} = Kh^{j,*,p^{n-s}}_{\mathbb{Z}/p^n}(D_{\alpha_1})[c(D_{\alpha_1})]\{q(\alpha_1)\},$$
$${}_{p^{n-s}}E_1^{i,j} = \bigoplus_{0 \leqslant \nu \leqslant u_i} \bigoplus_{\alpha \in \overline{\mathcal{B}}_i^{p^\nu}(X)} Kh^{j,*,k(\nu,s)}_{\mathbb{Z}/p^\nu}(D_\alpha)[c(D_\alpha)]\{q(\alpha)\}^{\ell(\nu,s)}$$

for $0 < i < p^n$, and 0 otherwise, $i = p^{u_i} g$, where $\gcd(p, g) = 1$ and α_0, α_1 the unique elements of $\mathcal{B}_0(X)$ and $\mathcal{B}_{p^n}(X)$, respectively. Above we used

$$q(\alpha) = i + 3c(D_\alpha) + p^n,$$
$$k(s,\nu) = \begin{cases} 1, & \nu \leqslant s, \\ p^{\nu-s}, & \nu > s, \end{cases}$$
$$\ell(s,\nu) = \begin{cases} \varphi(p^{n-s}), & \nu \leqslant s, \\ p^{n-\nu}, & \nu > s, \end{cases}$$

The E_1 pages of the respective spectral sequences for 2-periodic links are given below.

$$\begin{aligned}
{}_1E_1^{0,j} &= \mathrm{Kh}^{j,*,s(D_{00})}_{\mathbb{Z}/2}(D_{00})\{3c(D_{00}) + 2\},\\
{}_1E_1^{1,j} &= \mathrm{Kh}^{j,*}(D_{01})\{3c(D_{01}) + 3\},\\
{}_1E_1^{2,j} &= \mathrm{Kh}^{j,*,3-s(c(D_{11}))}_{\mathbb{Z}/2}(D_{11})\{3c(D_{11}) + 4\},\\
{}_2E_1^{0,j} &= \mathrm{Kh}^{j,*,3-s(D_{00})}_{\mathbb{Z}/2}(D_{00})\{3c(D_{00}) + 2\},\\
{}_2E_1^{1,j} &= \mathrm{Kh}^{j,*}(D_{01})\{3c(D_{01}) + 3\},\\
{}_2E_1^{2,j} &= \mathrm{Kh}^{j,*,s(c(D_{11}))}_{\mathbb{Z}/2}(D_{11})\{3c(D_{11}) + 4\},
\end{aligned}$$

where $s \in \{1, 2\}$ and

$$s(D_\alpha) \equiv \frac{c(D_\alpha)}{2} \pmod 2$$

Proof. In the odd case apply Theorem 2.2.21 to the filtration $\mathcal{F}_*(X)$. Use Proposition 4.1.9 and Eckamann-Shapiro Lemma to compute the entries in the E_1 page as in the proof of Proposition 3.2.11.

In the even case apply Theorem 2.2.21 and Proposition 4.1.10. □

4.2 SAMPLE COMPUTATIONS

The purpose of this section is to compute the rational 2-equivariant Khovanov homology of torus links $T(n, 2)$. Before we start, however, let us define the equivariant Khovanov and Jones polynomials. Although we study their properties in the next chapter, we define them here to simplify the statements of the results presented in this section.

Definition 4.2.1. Let L be an n-periodic link. For $d \mid n$, define the d-th equivariant Khovanov polynomial of L as follows

$$\mathrm{KhP}_{n,d}(L)(t, q) = \sum_{i,j} t^i q^j \dim_{\mathbb{Q}[\xi_d]} \mathrm{Kh}^{i,j,d}(L; \mathbb{Q})$$

and the d-th equivariant Jones polynomial of L as

$$J_{n,d}(L)(q) = \mathrm{KhP}_{n,d}(L)(-1, q).$$

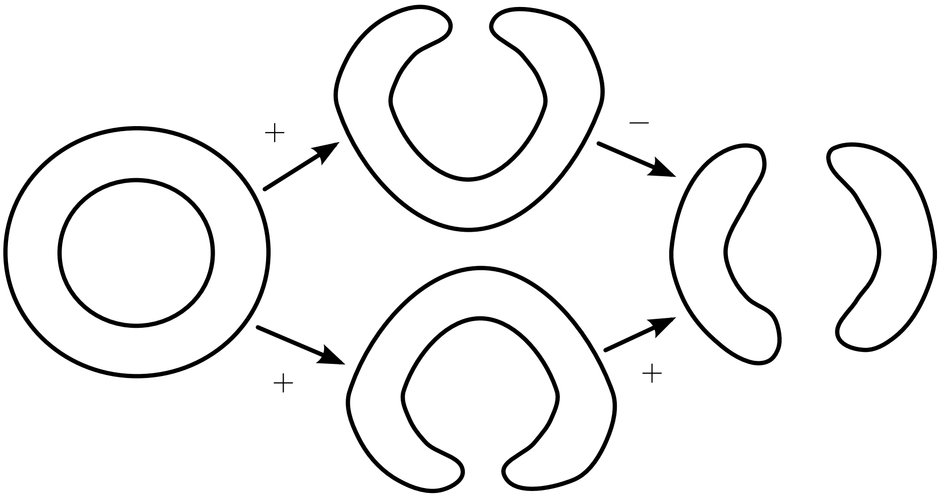


Figure 10: Anticommutative cube for $T(2,2)$

Let us, however, start with $T(2,2)$, which serves as a basis for further calculations.

Example 4.2.2. Consider the Hopf link as on Figure 13 for $n = 2$. Its Khovanov bracket is depicted on Figure 10. Figures 11 and 12 depict $CKh^{*,*,1}_{\mathbb{Z}/2}(T(2,2))$ and $CKh^{*,*,2}_{\mathbb{Z}/2}(T(2,2))$, respectively. Equivariant Khovanov and Jones polynomials of $T(2,2)$ are given below.

$$\begin{aligned} KhP_{2,1}(T(2,2))(t,q) &= 1 + q^2 + t^2q^4 \\ KhP_{2,2}(T(2,2))(t,q) &= t^2q^6 \\ J_{2,1}(T(2,2))(q) &= 1 + q^2 + q^4 \\ J_{2,2}(T(2,2))(q) &= q^6 \end{aligned}$$

Let us also state the following proposition from [10], which describes the Khovanov homology of torus links $T(n,2)$. We will use it extensively throughout this section.

Proposition 4.2.3. The Khovanov polynomial of $Kh(T(n,2))$ is equal to

$$\begin{aligned} KhP(T(2k,2)) &= q^{2k-2} + q^{2k} + t^2q^{2k+2}(1+tq^4)\sum_{j=0}^{k-2} t^{2j}q^{4j} \\ &\quad + t^{2k}q^{6k-2} + t^{2k}q^{6k}, \\ KhP(T(2k+1,2)) &= q^{2k-1} + q^{2k+1} + t^2q^{2k+3}(1+tq^4)\sum_{j=0}^{k-1} t^{2j}q^{4j}, \end{aligned}$$

for $k > 1$.

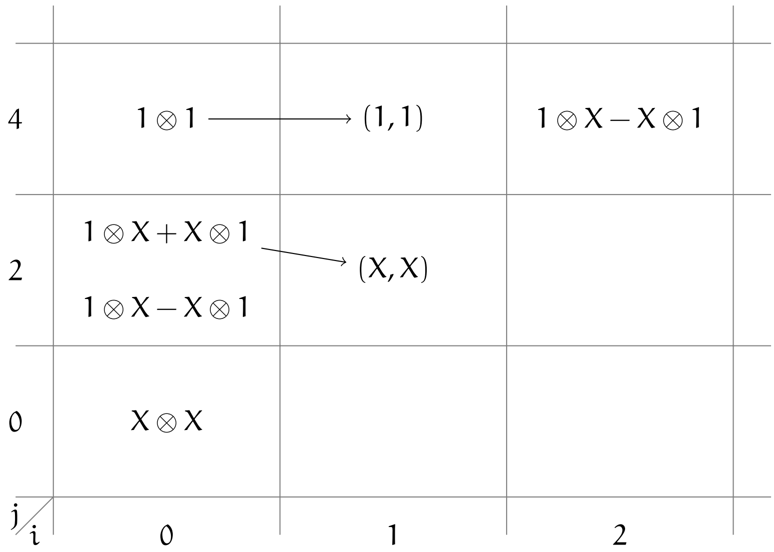


Figure 11: Computation of $\mathrm{Kh}^{*,*,1}_{\mathbb{Z}/2}(T(2,2);\mathbb{Q})$.

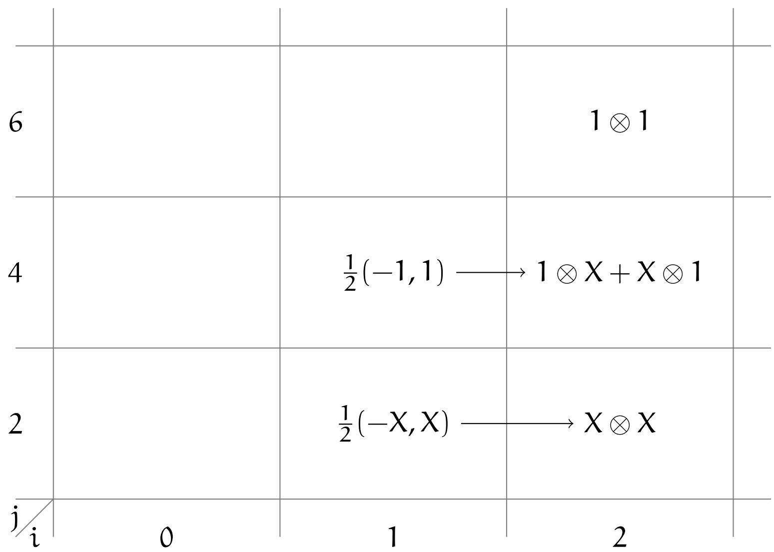


Figure 12: Computation of $\mathrm{Kh}^{*,*,2}_{\mathbb{Z}/2}(T(2,2);\mathbb{Q})$.

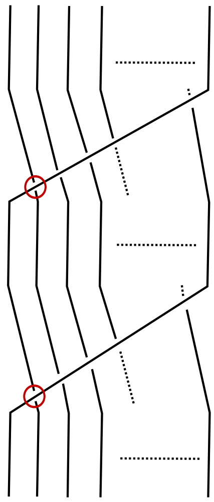


Figure 13: The 2-periodic diagram of $T(n,2)$. The chosen orbit of crossings is marked with red circles.

Theorem 4.2.4. Khovanov polynomials of the 2-equivariant Khovanov homology of torus links $T(n,2)$ are given below.

$$\begin{aligned} KhP_{2,1}(T(2n+1,2)) &= KhP(T(2n+1,2)) \\ KhP_{2,2}(T(2n+1,2)) &= 0 \\ KhP_{2,1}(T(2n,2)) &= KhP(T(2n,2)) - t^{2n}q^{6n} \\ KhP_{2,2}(T(2n,2)) &= t^{2n}q^{6n} \end{aligned}$$

Consider the 2-periodic diagram D of $T(n,2)$ from Figure 13 with the chosen orbit marked with red circles. Orient the diagram so that all crossings are positive. The associated bicomplex in $Kob(0)$ is depicted on Figure 14.

Lemma 4.2.5. The 0-th column of the E_1 page of the spectral sequence from Theorem 4.1.11 applied to the 2-periodic diagram D, has the following form.

$$\begin{aligned} {}_1E_1^{0,j,k} &= Kh_{\mathbb{Z}/2}^{j,k-1,1}(T(n-1,2)) \oplus Kh_{\mathbb{Z}/2}^{j,k-3,1}(T(n-1,2)), \\ {}_2E_1^{0,j,k} &= Kh_{\mathbb{Z}/2}^{j,k-1,2}(T(n-1,2)) \oplus Kh_{\mathbb{Z}/2}^{j,k-3,2}(T(n-1,2)). \end{aligned}$$

Proof. From figure 14 it is not hard to see, that the diagram D_{00} represents the split sum $T(n-1,2) \sqcup U$, where U denotes the unknot. Additionally, D_{00} inherits orientation from D, therefore $c(D_{00}) = 0$, because D was oriented so that all crossings are positive. This concludes the proof. □

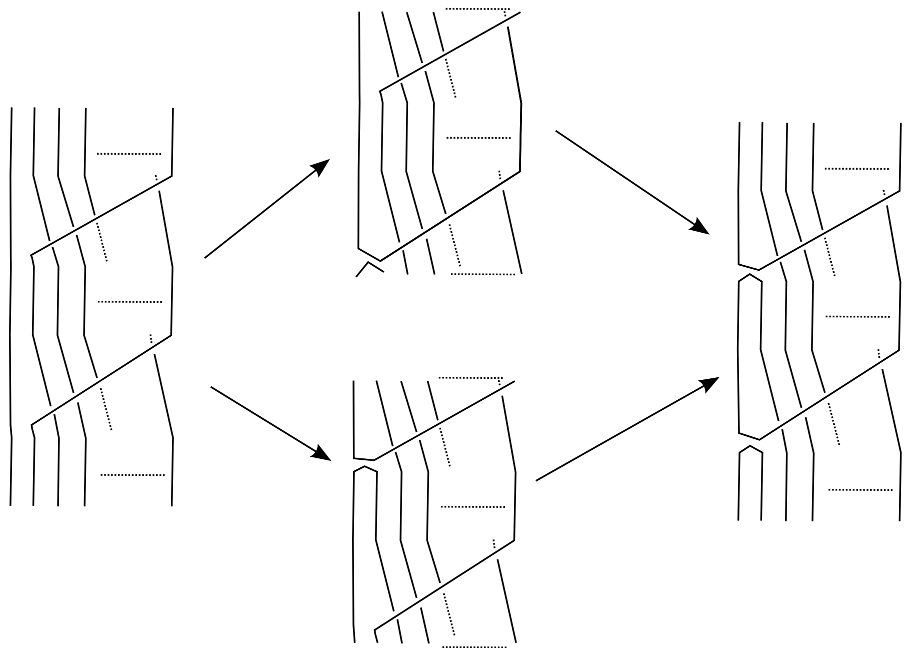


Figure 14: Bicomplex associated to the 2-periodic diagram of $T(n, 2)$ from figure 13.

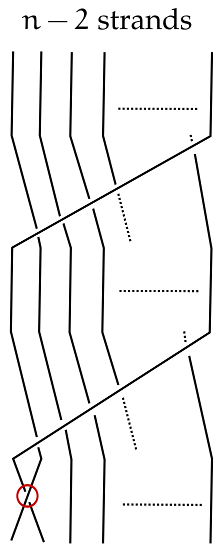


Figure 15: Diagram D' isotopic to the diagram of the D_{01}.

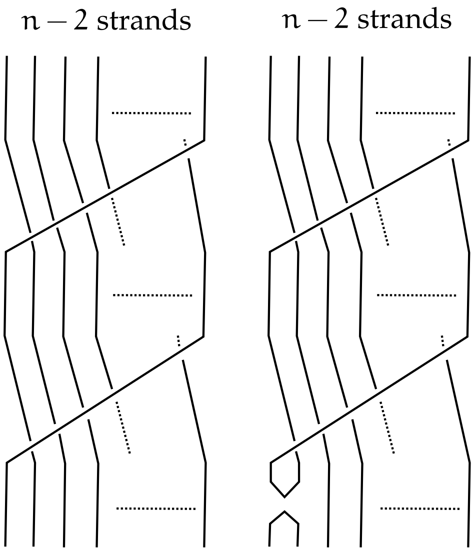


Figure 16: 0- and 1- smoothings of the diagram D', respectively.

Lemma 4.2.6. The 2-nd column of the E_1 page of the spectral sequence from theorem 4.1.11 applied to the 2-periodic diagram D, has teh following form.

$$\begin{aligned} {}_1E_1^{2,j,k} &= Kh_{\mathbb{Z}/2}^{j,k-4,2}(T(n-2,2)) \\ {}_2E_1^{2,j,k} &= Kh_{\mathbb{Z}/2}^{j,k-4,1}(T(n-2,2)) \end{aligned}$$

Proof. From figure 14 it follows that $D_{11} = T(n-2,2)$. It is not hard to check that we can orient D_{11} in such a way that all crossings are positive, therefore $c(D_{11}) = 0$. This finishes the proof. □

Lemma 4.2.7. The 1-st column of the E_1 page of the spectral sequence from theorem 4.1.11 applied to the 2-periodic diagram D has the following form.

$$ {}_1E_1^{1,j,k} = {}_2E_1^{1,j,k} = \begin{cases} Kh^{j,k-4}(T(2k-2,2)), & j < 2k-2, \\ \mathbb{Q}\{6k-4\}, & j = 2k-2, \\ \mathbb{Q}\{6k\}, & j = 2k-1, \\ 0, & j > 2k-1, \end{cases} $$

if $n = 2k$ and

$$ {}_1E_1^{1,j,k} = {}_2E_1^{1,j,k} = \begin{cases} Kh^{j,k-4}(T(2k-1,2)), & j < 2k, \\ \mathbb{Q}\{6k+1\} \oplus \mathbb{Q}\{6k+3\}, & j = 2k, \\ 0, & j > 2k, \end{cases} $$

if $n = 2k+1$.

Proof. Let us denote by D_{01} one of the diagrams in the middle column of Figure 14. First, let us compute $c(D_{01})$. It is not hard to see, that D_{01} can be oriented in such a way that all crossings are positive. Therefore $c(D_{01}) = 0$.

Let us denote by D' the diagram from Figure 15. Orient it, so that all crossings are positive. Quick inspection shows that D_{01} and D' are isotopic. In order to prove the lemma, let us compute $Kh(D')$. To do this we will use the exact triangle from 2.3.21 with respect to the crossing marked on Figure 15. Let D'_0 and D'_1 denote the 0- and 1-smoothing of D', respectively, as on Figure 16. The 0-smoothing is the torus link $T(n-2, 2)$ with $c(D'_0) = 0$. On the other hand, D'_1 is a diagram of the unknot with $c(D'_1) = n-2$.

Consider first the case $n = 2k$, for $k > 1$. In the long exact sequence derived from Theorem 2.3.21 almost all terms corresponding to $Kh(D'_1)$ vanish. There are only two non-vanishing terms. Further inspection of the long exact sequence yields that there can be only one possibly non-vanishing morphism in this sequence.

$$\mathbb{Q} = Kh^{2k-2,6k-6}(T(2k-2,2)) \xrightarrow{\delta} Kh^{0,-1}(U) = \mathbb{Q}$$

Suppose that $\delta = 0$ and notice that if $n = 2k$, then D' represents a knot. It is not hard to see, that the Khovanov homology of this knot is concentrated only on two diagonals $j - 2i = 2k-3, 2k-1$, regardless of whether δ vanishes or not. Further, if δ vanishes, then

$$KhP(D') = q\, KhP(T(2k-2,2)) + t^{2k-1}q^{6k-7} + t^{2k-1}q^{6k-5}. \qquad (7)$$

On the other hand, [12, Thm. 4.4] and [22, Prop. 3.3] imply that

$$KhP(D') \equiv q^{s(D')}(q+q^{-1}) \pmod{(1+tq^4)}, \qquad (8)$$

for some integer $s(D')$. However, from (7) it follows that

$$\begin{aligned} KhP(D') &\equiv q^{2k-3} + q^{2k-1} + \\ &+ q^{-8k+8}(q^{6k-7} + q^{6k-5})(1-q^{-4}) \pmod{(1+tq^4)}, \end{aligned}$$

which contradicts (8). Thus, δ must be non-trivial.

If $n = 2k+1$, there is also only one case to consider. Namely

$$\mathbb{Q} = Kh^{2k-1,6k-3}(T(2k-1,2)) \to Kh^{0,-1}(U) = \mathbb{Q}.$$

Notice that D' represents a 2-component link, whose Khovanov homology is concentrated on two diagonals. Therefore, analogously as in the previous case, [12, Thm. 4.4] and [22, Prop. 3.3] imply that

$$KhP(D') \equiv q^{s}(q+q^{-1}) + t^{\ell}q^{s'}(q+q^{-1}) \pmod{(1+tq^4)}, \qquad (9)$$

where ℓ denotes the linking number of the components of D'. Argument analogous as in the even case yields that now δ must vanish. □

Proof of Thm. 4.2.4. The proof is inductive. The first case was done in Example 4.2.2.

Consider first $T(2n+1,2)$. From lemmas 4.2.5, 4.2.6 and 4.2.7 we can derive the E_1 page spectral sequence ${}_2E_*^{*,*,*}$, which is depicted on Figures 17. In order to finish the computation we need to apply Proposition 3.2.17 and Proposition 4.2.3. Since

$$\begin{aligned} Kh^{2n+1,*}(T(2n+1,2)) &= \mathbb{Q}\{6n+3\}, \\ Kh^{2n,*}(T(2n+1,2)) &= \mathbb{Q}\{6n-1\}, \end{aligned}$$

if follows easily from Proposition 4.2.3 that the differential

$$d_1^{0,2n} \colon {}_2E_1^{0,2n,*} \to {}_2E_1^{1,2n,*}$$

is an isomorphism. Analogous comparisons of grading of ${}_2E_1^{1,k}$ and $Kh^{k+1,*}(T(2n+1,2))$ yield that ${}_2E_2^{*,*,*}$ is zero. Thus,

$$Kh_{\mathbb{Z}/2}^{*,*,2}(T(2n+1,2)) = 0,$$

and consequently

$$Kh_{\mathbb{Z}/2}^{*,*,1}(T(2n+1,2)) = Kh^{*,*}(T(2n+1,2)).$$

Consider now $T(2n,2)$. The E_1 page of the spectral sequence is presented on Figure 17. Comparison of gradings of ${}_2E_1^{*,*,*}$ and gradings of $Kh^{*,*}(T(2n-2,2))$ yields that the only non-zero entry of ${}_2E_2^{*,*,*}$ is

$${}_2E_2^{1,2n-1,6n} = \mathbb{Q}.$$

Therefore,

$$Kh_{\mathbb{Z}/2}^{*,*,2}(T(2n,2)) = \mathbb{Q}[2n]\{6n\}.$$

□

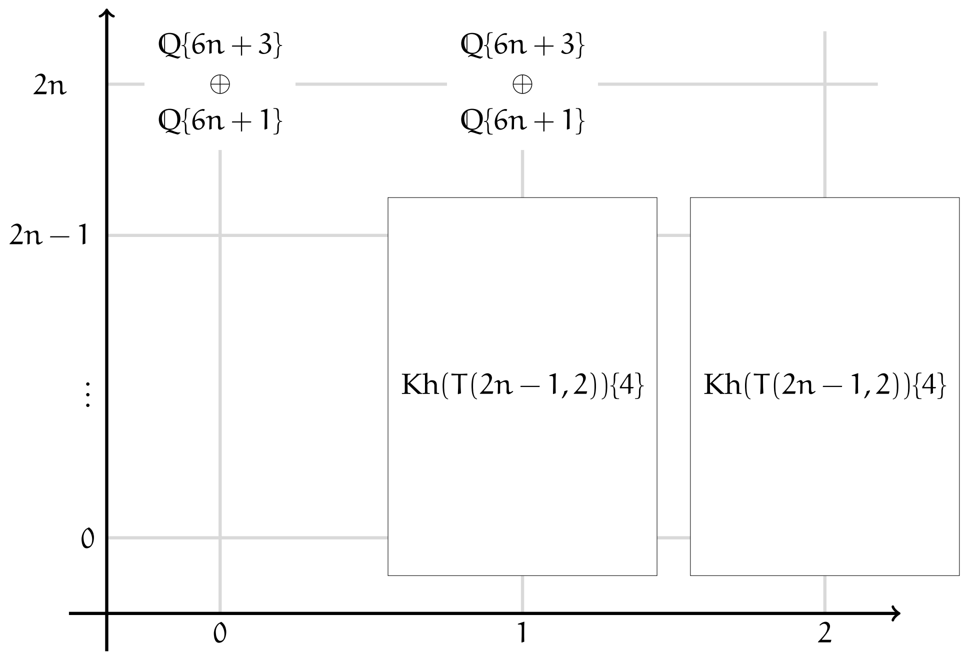


Figure 17: ${}_2E_1^{*,*,*}$ of $T(2n+1,2)$.

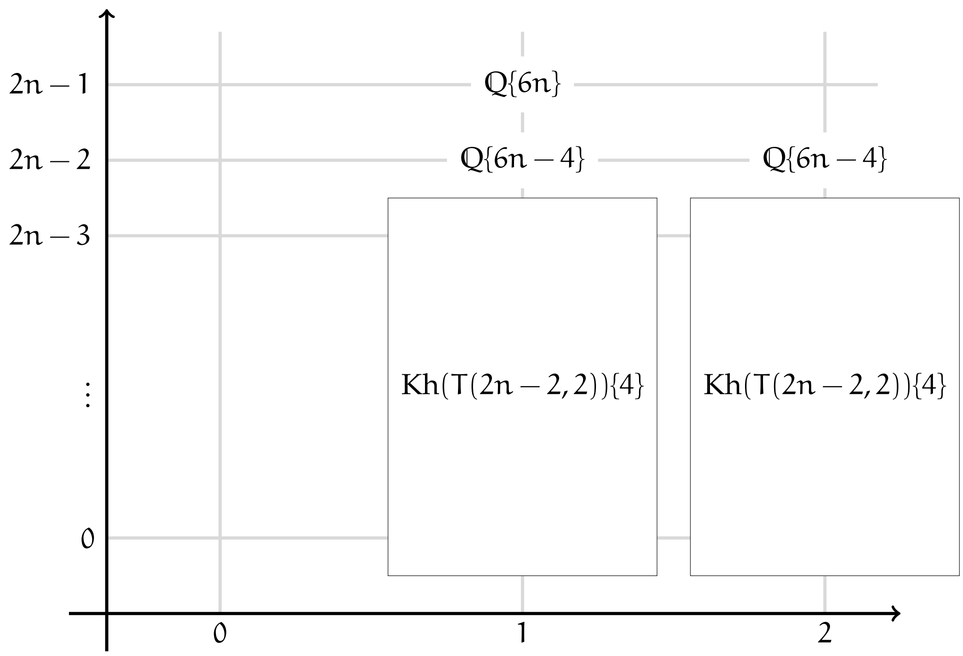


Figure 18: ${}_2E_1^{*,*,*}$ of $T(2n,2)$.

5 EQUIVARIANT JONES POLYNOMIALS

The aim of this chapter is to investigate properties of equivariant analogues of the Jones polynomial. We analyze their relations to the classical Jones and Khovanov polynomials. Next, we define the difference polynomials and show that they satisfy an analogue of the skein relation. This leads us to a new criterion for periodicity of links, which is a stronger version of the criterion given by J.H. Przytycki. Further, we present a state sum expansion for the difference polynomials, which leads us to a proof of the classical congruence of Murasugi. We conclude this chapter with proofs of the skein relation and the state sum formula.

5.1 BASIC PROPERTIES

First, let us recall the definition of the equivariant polynomials.

Definition 4.2.1. Let L be an n-periodic link. For $d \mid n$ define a d-th equivariant Khovanov polynomial by

$$\mathrm{KhP}_{n,d}(L)(t,q) = \sum_{i,j} t^i q^j \dim_{\mathbb{Q}[\xi_d]} \mathrm{Kh}^{i,j,d}(L;\mathbb{Q})$$

and d-th equivariant Jones polynomial

$$J_{n,d}(L)(q) = \mathrm{KhP}_{n,d}(L)(-1,q).$$

The next theorem describes basic properties of equivariant Jones polynomials.

Theorem 5.1.1. Let L be an n-periodic link and let $d \mid n$.

1. Equivariant Khovanov and Jones polynomials are invariants of periodic links i.e. they are invariant under Reidemeister moves.
2. If $J(L)$ denotes the ordinary unreduced Jones polynomial, then the following equality holds.

$$J(L) = \sum_{d|n} \phi(d)\, J_{n,d}(L),$$

where ϕ denotes the Euler's totient function.

3. If $d \mid n$ and for all i, j we have $\dim_{\mathbb{Q}} \mathrm{Kh}^{i,j}(L;\mathbb{Q}) < \varphi(d)$, then

$$\mathrm{KhP}_{n,d}(L) = 0,$$
$$J_{n,d}(L) = 0.$$

Proof.
1. First part of the theorem follows from theorem 3.2.3.
2. This follows readily from Proposition 3.2.17.
3. This is a reformulation of Corollary 3.2.18.

□

From now on we assume that all links are p^n-periodic, for p an odd prime and $n > 0$.

Definition 5.1.2. Suppose that D is a p^n-periodic link diagram. Define the difference Jones polynomials

$$DJ_{n,s}(D) = J_{p^n,p^s}(D) - J_{p^n,p^{s+1}}(D)$$

for $0 \leqslant s \leqslant n$.

Corollary 5.1.3. The following equality holds.

$$J(D) = \sum_{s=0}^{n} p^s \, DJ_{n,s}(D)$$

Proof. Proof follows from theorem 5.1.1 and a simple fact that

$$\phi(p^s) = p^s - p^{s-1}.$$

□

Example 5.1.4. Let $T_{k\cdot p^n+f}$ denote the p^n-periodic diagram of the trivial link with $k \cdot p^n + f$ components. Assume that this diagram has k free orbits of components and f fixed circles. From Proposition 3.2.7 we obtain that the equivariant and difference Jones polynomials of $T_{k\cdot p^n+f}$ can be expressed in terms of polynomials $\mathcal{P}_s$ defined in 3.2.6. Indeed, Proposition 3.2.11 implies that

$$\begin{aligned} Kh^{0,*,p^{n-u}}_{\mathbb{Z}/p^n}(T_{k\cdot p^n+f};\mathbb{Q}) &= \bigoplus_{s=n-u}^{n} \bigoplus_{i} H^0(\mathbb{Z}/p^{n-s}, \mathbb{Q}^{d_i})^{\varphi(p^{n-u})}\{i\} = \\ &= \bigoplus_{s=n-u}^{n} (M_s^{k,f})^{\varphi(p^{n-u})}, \end{aligned}$$

because

$$H^0(\mathbb{Z}/p^{n-s}, \mathbb{Q}\left[\xi_{p^{n-s-u}}\right]) = \hom_{\mathbb{Q}[\mathbb{Z}/p^n]}(\mathbb{Q}, \mathbb{Q}\left[\xi_{p^{n-s-u}}\right]) = 0,$$

when $n - s - u > 1$, by Schur's Lemma. Consequently

$$\begin{aligned} J_{p^n,p^{n-u}}(T_{k\cdot p^n+f}) &= \sum_{s=n-u}^{n} \operatorname{qdim} M_s^{k,f}, \\ DJ_{n,n-u}(T_{k\cdot p^n+f}) &= \operatorname{qdim} M_{n-u}^{k,f}. \end{aligned}$$

One of the most important properties of the Jones polynomials is the skein relation from Proposition 2.3.25. As it turns out, the difference polynomials satisfy an analogue of this property.

Theorem 5.1.5. The difference Jones polynomials have the following properties

1. DJ_0 satisfies the following version of the skein relation

$$q^{-2p^n} DJ_{n,0}\left(\times_+ \ldots \times_+\right) - q^{2p^n} DJ_{n,0}\left(\times_- \ldots \times_-\right) =$$
$$= \left(q^{-p^n} - q^{p^n}\right) DJ_{n,0}\left()(_0 \ldots)(_0\right),$$

   where $\times_+ \ldots \times_+$, $\times_- \ldots \times_-$ and $)(_0 \ldots)(_0$ denote the orbit of positive, negative and orientation preserving resolutions of crossing, respectively.

2. for any $0 \leqslant s \leqslant n$, DJ_s satisfies the following congruences

$$q^{-2p^n} DJ_{n,n-s}\left(\times_+ \ldots \times_+\right) - q^{2p^n} DJ_{n,n-s}\left(\times_- \ldots \times_-\right) \equiv$$
$$\equiv \left(q^{-p^n} - q^{p^n}\right) DJ_{n,n-s}\left()(_0 \ldots)(_0\right) \pmod{q^{p^s} - q^{-p^s}}.$$

Remark 5.1.6. We defer the proof of Theorem 5.1.5 to Section 5.3.

The above theorem has a number of consequences regarding the Jones polynomial of a periodic link. For example, it enables us to write down a few criterions for the periodicity of a link in terms of its Jones polynomial. One such example was given by J.H. Przytycki in [17].

Theorem 5.1.7. Suppose that L is a p^n-periodic link. Then the following congruence holds

$$J(L)(q) \equiv J(L)(q^{-1}) \pmod{\mathcal{I}_{p^n}},$$

where $\mathcal{I}_{p^n}$ is an ideal generated by the following monomials

$$p^n, p^{n-1}\left(q^p - q^{-p}\right), \ldots, p\left(q^{p^{n-1}} - q^{p^{-n-1}}\right), q^{p^n} - q^{-p^n}.$$

Proof. Notice that

$$J\left(\times_+ \ldots \times_+\right) - J\left(\times_- \ldots \times_-\right) \equiv$$
$$q^{2p^n} J\left(\times_+ \ldots \times_+\right) - q^{-2p^n} J\left(\times_- \ldots \times_-\right) \equiv$$
$$\sum_{s=0}^{n} p^{n-s}\left(q^{2p^n} DJ_{n,n-s}\left(\times_+ \ldots \times_+\right) - q^{-2p^n} DJ_{n,n-s}\left(\times_- \ldots \times_-\right)\right) \equiv$$
$$\equiv 0 \pmod{\mathcal{I}_{p^n}}.$$

Hence, switching crossings from a single orbit does not change the Jones polynomial modulo $\mathcal{I}_{p^n}$. Since we can pass from L to its mirror image $L^!$ by switching one orbit at at time, it follows that

$$J(L) \equiv J(L^!) \pmod{\mathcal{I}_{p^n}}.$$

Taking into account the relation between the Jones polynomials of L and $L^!$

$$J(L^!)(q) = J(L)(q^{-1})$$

concludes the proof □

The above theorem can be considerably strengthen with the aid of Theorem 5.1.1.

Theorem 5.1.8. Suppose that L is a p^n-periodic link and for all i, j we have $\dim_{\mathbb{Q}} Kh^{i,j}(L; \mathbb{Q}) < \varphi(p^s)$, then the following congruence holds

$$J(L)(q) \equiv J(L)(q^{-1}) \pmod{\mathcal{I}_{p^n,s}},$$

where $\mathcal{I}_{p^n,s}$ is the ideal generated by the following monomials

$$q^{p^n} - q^{-p^n}, p\left(q^{p^{n-1}} - q^{-p^{n-1}}\right), \ldots, p^{s-1}\left(q^{p^{n-s+1}} - q^{-p^{n-s+1}}\right).$$

Proof. First notice that Theorem 5.1.1 implies that for $s' \geqslant s$ the equivariant Jones polynomials $J_{n,p^{s'}}(L)$ vanish. Therefore $DJ_{s'} = 0$. Corollary 5.1.3 implies that

$$J(L) = \sum_{i=0}^{s-1} p^i \, DJ_i \, .$$

Now argue as in the proof of the previous Theorem to obtain the desired result. □

Example 5.1.9. Consider the 10_{61} knot from the Rolfsen table [23]. If we are interested in the symmetry of order 5, then according to SAGE [25], the following congruence holds

$$J(10_{61})(q) - J(10_{61})(q^{-1}) \equiv 0 \pmod{q^5 - q^{-5}, 5(q - q^{-1})}.$$

Hence, Przytycki's Theorem does not obstruct 10_{61} to have symmetry of order 5. However, if we notice that, as depicted on Figure 19, dimension of $Kh^{i,j}(10_{61})$ is always smaller that $\varphi(5) = 4$ and apply Theorem 5.1.8 we obtain

$$J(10_{61})(q) - J(10_{61})(q^{-1}) \not\equiv 0 \pmod{q^5 - q^{-5}}$$

Consequently 10_{61} is not 5-periodic.

j \ i	−4	−3	−2	−1	0	1	2	3	4	5	6
17											1
15										1	
13									2	1	
11								2	1		
9							3	2			
7						2	2				
5					2	3					
3				3	3						
1				1							
−1		1	3								
−3											
−5	1										

Kh

Figure 19: Ranks of $Kh^{i,j}(10_{61})$ according to [24].

5.2 STATE SUM FOR THE DIFFERENCE POLYNOMIALS

One of the most import property of the Jones polynomial is that it can be written as a certain sum over all Kauffman states obtained from the given diagram. This point of view on the Jones polynomial was pioneered in [9]. In particular, if D is a link diagram, then the Jones polynomial of the corresponding links expands into the following sum

$$J(D) = (-1)^{n_-(D)} q^{n_+(D) - 2n_-(D)} \sum_{s \in \mathcal{S}(D)} (-q)^{r(s)} (q + q^{-1})^{k(s)},$$

where $k(s)$ denotes the number of components of the Kauffman state s and $r(s)$ denotes the number of 1-smoothings used to get the Kauffman state s, see [32, Lect. 1]. The purpose of this section is to show that the difference Jones polynomials admit analogous expansion.

Theorem 5.2.1. Let D be a p^n-periodic diagram of a link. Then for any $0 \leqslant m \leqslant n$ we have the following equality

$$DJ_{n,n-m}(D) = (-1)^{n_-} q^{n_+ - 2n_-} \sum_{m \leqslant \nu \leqslant n} \sum_{s \in \mathcal{S}^{p^\nu}(D)} (-q)^{r(s)} \, DJ_{p^\nu, p^{\nu - s}}(s).$$

For a Kauffman state s we write $r(s) = r$ if $s \in \mathcal{S}_r(D)$, compare Definition 3.1.6.

Remark 5.2.2. The proof of the above Theorem is deffered to Section 5.3.

First application of the state sum formula of the difference Jones polynomials is concerned with the following criterion for a knot to be periodic. This is the classical criterion of Murasugi from [16].

Theorem 5.2.3. Let D be a p^n-periodic link diagram. Let D_* denote the quotient diagram.

$$J(D) \equiv J(D_*)^{p^n} \pmod{p, (q+q^{-1})^{\alpha(D)(p^n-1)} - 1},$$

where

$$\alpha(D) = \begin{cases} 1, & 2 \nmid \operatorname{lk}(D,F), \\ 2, & 2 \mid \operatorname{lk}(D,F). \end{cases}$$

Above, F denotes the fixed point set.

Proof. Let us analyze the relation between $DJ_{n,0}(D)$ and $J(D_*)$.

Proposition 5.2.4. The following congruence holds.

$$DJ_{n,0}(D) \equiv J(D_*)^{p^n} \pmod{p, (q+q^{-1})^{\alpha(D)(p^n-1)} - 1}.$$

Proof. Notice that the state formula for $DJ_{n,0}(D)$ involves only Kauffman states which inherit $\mathbb{Z}/p^n$-symmetry. Such Kauffman states correspond bijectively to the Kauffman states of the quotient diagram.

Let s be a Kauffman state obtained from D such that $\operatorname{Iso}(s) = \mathbb{Z}/p^n$. Assume that s consists of k free orbits and f fixed circles. Thus, according to Example 5.1.4 and Corollary 3.2.10 the Kauffman state contributes

$$\begin{aligned}(-1)^{n_-(D)+r(s)} q^{n_+(D)-2n_-(D)+r(s)} (q^{p^n} + q^{-p^n})^k (q+q^{-1})^f \equiv \\ \equiv (-1)^{n_-(D)+r(s)} q^{n_+(D)-2n_-(D)+r(s)} (q+q^{-1})^{kp^n+f} \pmod{p}.\end{aligned}$$

to the state sum for $DJ_{n,0}$. The quotient Kauffman state s_* consists of $k+f$ components. Hence, it contributes

$$(-1)^{p^n(n_-(D_*)+r(s_*))} q^{p^n(n_+(D_*)-2n_-(D_*)+r(s_*))} (q+q^{-q})^{p^n(k+f)}$$

to $J(D_*)^{p^n}$ mod p. The difference of both contributions is divisible by $(q+q^{-1})^{f(p^n-1)} - 1$. Since

$$f \equiv \operatorname{lk}(D,F) \pmod{2},$$

the proposition follows. □

We can conclude the proof once we note that

$$J(D) \equiv DJ_{n,1}(D) \pmod{p}$$

by Corollary 5.1.3. □

5.3 PROOFS

This section is entirely devoted to the presentation of proofs of Theorems 5.1.5 and 5.2.1. These two proofs are very alike in principle, therefore we only perform detailed calculations in the first proof, because the very same calculations are present in the second proof as well.

5.3.1 *Proof of Theorem 5.1.5*

Let us begin with a definition.

Definition 5.3.1. Let $\{E_s^{*,*}, d_s\}$ be a spectral sequence of graded finite-dimensional $\mathbb{F}$-modules, where $\mathbb{F}$ is a field, converging to some doubly-graded finite-dimensional $\mathbb{F}$-module $H^{*,*}$. Suppose that the spectral sequence collapses at some finite stage. Define the Poincaré polynomial of the E_s page to be the following polynomial.

$$P(E_s)(t,q) = \sum_{i,j} t^{i+j} \operatorname{qdim}_{\mathbb{F}} E_s^{i,j}$$

Poincaré polynomial admits the following decomposition

$$P(E_s) = \sum_i t^i P_i(E_s),$$

where

$$P_i(E_s) = \sum_j t^j \operatorname{qdim}_{\mathbb{F}} E_s^{i,j}.$$

Lemma 5.3.2. For any $s > 0$ the following equality holds, whenever it makes sense,

$$P(E_s)(-1,q) = P(E_\infty)(-1,q).$$

Proof. This is a direct consequence of [13, Ex. 1.7]. □

Let us now analyze E_1 pages of the spectral sequences from Theorem 4.1.11, for odd p. Let us make the following notation. If $1 \leqslant i \leqslant p^n - 1$, then $i = p^{u_i} g$, where $\gcd(p,g) = 1$. Poincaré polynomials of columns are given below.

$$P_0({}_{p^{n-s}}E_1) = t^{c(D_{\alpha_0})} q^{3c(D_{\alpha_0})} \mathrm{KhP}_{p^n,p^{n-s}}(D_{\alpha_0}), \tag{10}$$

$$P_{p^n}({}_{p^{n-s}}E_1) = t^{c(D_{\alpha_1})} q^{3c(D_{\alpha_1})+2p^n} \mathrm{KhP}_{p^n,p^{n-s}}(D_{\alpha_1}), \tag{11}$$

$$\begin{aligned} P_i({}_{p^{n-s}}E_1) = & \sum_{0 \leqslant \nu \leqslant \min(s,u_i)} \sum_{\alpha \in \overline{\mathcal{B}}_i^{p^\nu}(X)} t^{c(D_\alpha)} q^{i+3c(D_\alpha)+p^n} \mathrm{KhP}_{p^\nu,p^1}(D_\alpha) + \\ & + \sum_{\min(s,u_i) < \nu \leqslant u_i} \sum_{\alpha \in \overline{\mathcal{B}}_i^{p^\nu}} t^{c(D_\alpha)} q^{i+3c(D_\alpha)+p^n} \mathrm{KhP}_{p^\nu,p^{\nu-s}}(D_\alpha). \end{aligned} \tag{12}$$

In order to make further computations more manageable let us introduce the following notation.

$$G_i(v,w) = \sum_{\alpha \in \overline{\mathcal{B}}_i^{p^v}} t^{c(D_\alpha)} q^{3c(D_\alpha)} \mathrm{KhP}_{p^v,p^w}(D_\alpha),$$

$$DJG_i(v,w) = G_i(v,w)(-1,q) = \sum_{\alpha \in \overline{\mathcal{B}}_i^{p^v}} (-1)^{c(D_\alpha)} q^{3c(D_\alpha)} DJ_{v,w}(D_\alpha)$$

so for $1 \leqslant i \leqslant p^n - 1$ the Poincaré polynomial can be expressed as the following more compact sum.

$$P_i({}_{p^{n-s}}E_1) = q^{i+p^n} \sum_{v=0}^{\min(s,u_i)} G_i(v,0) + \tag{13}$$

$$= q^{i+p^n} \sum_{v=\min(s,u_i)+1}^{u_i} G_i(v,v-s). \tag{14}$$

Lemma 5.3.3. Following formula holds for the Poincaré polynomials

$$P({}_{p^{n-s}}E_1) - P({}_{p^{n-s+1}}E_1) =$$

$$\sum_{1 \leqslant j \leqslant p^{n-s}-1} t^{j \cdot p^s} q^{j \cdot p^s + p^n} \sum_{v=s}^{u_i} \left(G_{jp^s}(v, s-v) - G_{jp^s}(v, v-s+1) \right)$$

$$+ t^{c(D_{\alpha_0})} q^{3c(D_{\alpha_0})+p^n} \left(\mathrm{KhP}_{p^n,p^{n-s}}(D_{\alpha_0}) - \mathrm{KhP}_{p^n,p^{n-s+1}}(D_{\alpha_0}) \right)$$

$$+ t^{c(D_{\alpha_1})+p^n} q^{3c(D_{\alpha_1})+2p^n} \left(\mathrm{KhP}_{p^n,p^{n-s}}(D_{\alpha_1}) - \mathrm{KhP}_{p^n,p^{n-s+1}}(D_{\alpha_1}) \right).$$

Proof. Indeed, because

$$P_i({}_{p^{n-s}}E_1) - P_i(E_{p^{n-s+1}}E_1) =$$

$$= \begin{cases} \sum_{v=s}^{u_i} \left(G_i(v, s-v) \right) - G_i(v, v-s+1)), & p^s \mid i, \\ 0, & p^s \nmid i, \end{cases}$$

which can be easily verfied using formula (13). □

Corollary 5.3.4. The following formula holds for the difference polynomials

$$DJ_{n,n-s}(D) =$$

$$\sum_{1 \leqslant j \leqslant p^{n-s}-1} (-1)^{j \cdot p^s} q^{j \cdot p^s + p^n} \sum_{v=s}^{u_i} DJG_{jp^s}(v, v-s)$$

$$+ (-1)^{c(D_{\alpha_0})} q^{3c(D_{\alpha_0})+p^n} DJ_{n-s}(D_{\alpha_0})$$

$$+ (-1)^{c(D_{\alpha_1})+p^n} q^{3c(D_{\alpha_1})+2p^n} DJ_{n-s}(D_{\alpha_1}).$$

Proof. It follows easily from previous Lemma by substituting $t = -1$ and noting that $P({}_{p^{n-s}}E_1)(-1,q) = J_{p^n,p^{n-s}}(D)$, by Lemma 5.3.2. □

Definition 5.3.5. For any $0 \leqslant i \leqslant p^n$ define a map

$$\kappa \colon \mathcal{B}_i(X) \to \mathcal{B}_{p^n-i}(X)$$
$$\kappa(\beta)(c) = \begin{cases} 1-\beta(c), & c \in X, \\ \beta(c), & c \notin X. \end{cases}$$

Proposition 5.3.6. Let D be p^n-periodic link diagram and let $X \subset \operatorname{Cr} D$ be a chosen orbit of crossings. Suppose that all crossings from X are positive and let $D^!$ denote invariant link diagram obtained from D by changing all crossings from X to negative ones. Then the following equalities hold

$$D_\alpha = D^!_{\kappa(\alpha)}$$
$$|\kappa(\alpha)|_u = p^n - |\alpha|_u, \text{ for } u = 0, 1,$$
$$c(D_\alpha) = c(D^!_{\kappa(\alpha)}) + p^n$$

Proof. The first two equalities are direct consequences of definitions. To prove the third one notice that

$$n_-(D) = n_-(D^!) - p^n.$$

Therefore

$$c(D_\alpha) = n_-(D_\alpha) - n_-(D) = n_-(D^!_{\kappa(\alpha)}) - n_-(D^!) + p^n = \\ = c(D^!_{\kappa(\alpha)}) + p^n.$$

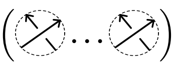
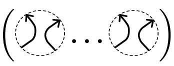
□

Proof of Theorem 5.1.5. To prove the first part notice that from corollary 5.3.4 it follows that

$$DJ_{n,0}(D) = (-1)^{c(D_{\alpha_0})} q^{3c(D_{\alpha_0})+p^n} DJ_{n,0}(D_{\alpha_0}) \\ + (-1)^{c(D_{\alpha_1})+p^n} q^{3c(D_{\alpha_1})+2p^n} DJ_{n,0}(D_{\alpha_1}),$$

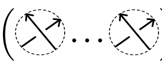
because D_{α_0} and D_{α_1} are the only diagrams with isotropy group equal to $\mathbb{Z}/p^n$, because these are the only invariant diagrams.

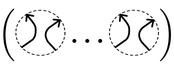
Now without loss of generality assume that the chosen orbit of crossings consists of positive crossings. Then D_{α_0} inherits orientations from D and therefore $c(D_{\alpha_0}) = 0$. Therefore

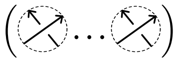
$DJ_{n,0}$ () $= q^{p^n} DJ_{n,0}$ ()

$$+ (-1)^{c(D_{\alpha_1})+p^n} q^{3c(D_{\alpha_1})+2p^n} DJ_{n,0}(D_{\alpha_1}),$$

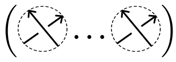
On the other hand for $D^!$ as in the previous proposition $D^!_{\alpha_1}$ inherits its orientation. Furthermore $c(D^!_{\alpha_1}) = -p^n$. This gives the following equality

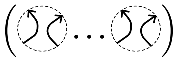
$DJ_{n,0}$ () $= (-1)^{c(D^!_{\alpha_0})} q^{3c(D^!_{\alpha_0})+p^n} DJ_{n,0}(D^!_{\alpha_0})$

$+ q^{-p^n} DJ_{n,0}$ (), Denote $c = c(D_{\alpha_1})$, then $D_{\alpha_1} = D^!_{\alpha_0}$ and $c(D^!_{\alpha_0}) = c - p^n$ by Proposition 5.3.6. Therefore

$DJ_{n,0}$  $= q^{p^n} DJ_0$ 

$$+ (-1)^{c+p^n} q^{3c+2p^n} DJ_{n,0}(D_{\alpha_1}),$$

$DJ_{n,0}$  $= (-1)^{c-p^n} q^{3c-2p^n} DJ_{n,0}(\overline{D}_{\alpha_0})$

$+ q^{-p^n} DJ_0$  .

From the above equalities the first part of the theorem follows easily.

To prove the second part, notice that Proposition 5.3.6 implies that for $\nu \geqslant s$ the following equality holds.

$$(-1)^i q^{i-p^n} DJG_i(\nu, \nu - s)(D) - (-1)^{p^n - i} q^{4p^n - i} DJG_i(\nu, \nu - s)(D^!) =$$
$$= \sum_{\alpha \in \overline{\mathcal{B}}_i^{p^\nu}} (-1)^{c+p^n+1} q^{3c+3p^n} DJ_{\nu,\nu-s} \left(q^{i-p^n} - q^{p^n - i}\right)(D_\alpha).$$

Consequently, if $p^s \mid i$, then the above difference is divisible by $q^{p^s} - q^{-p^s}$. To finish the proof combine formula 5.3.4 and with the discussion above. □

5.3.2 *Proof of Theorem 5.2.1*

Proof of Theorem 5.2.1. According to the definition, the equivariant Jones polynomials, can be written as the following sum.

$$q^{-n_+(D)+n_-(D)} J_{p^n, p^{n-s}}(D) =$$
$$= \sum_{r=n_-(D)}^{n_+(D)} (-q)^r \operatorname{qdim}_{\mathbb{Q}[\xi_{p^{n-s}}]} \operatorname{Hom}_{\mathbb{Q}[\mathbb{Z}/p^n]} \left(\mathbb{Q}\left[\xi_{p^{n-s}}\right], CKh^{r-n_-(D),*}(D)\right). \tag{15}$$

Thus, the only thing we need to do is to determine the quantum dimension of the following graded module

$$\operatorname{Hom}_{\mathbb{Q}[\mathbb{Z}/p^n]} \left(\mathbb{Q}\left[\xi_{p^{n-s}}\right], CKh^{r-n_-(D),*}(D)\right)$$

for $0 \leqslant r \leqslant n_+(D) + n_-(D)$. Performing calculations as in the proof of Theorem 4.1.11. Let $r = p^{u_r} g$, where $\gcd(p, g) = 1$. We obtain the following formula.

$$\operatorname{qdim} \operatorname{Hom}_{\mathbb{Q}[\mathbb{Z}/p^n]} \left(\mathbb{Q}\left[\xi_{p^{n-s}}\right], CKh^{r-n_-(D),*}(D)\right) =$$
$$= \sum_{\nu=0}^{\min(s,u_r)} \sum_{s \in \mathcal{S}_r^{p^\nu}(D)} J_{p^\nu,1}(s) + \sum_{\nu=\min(s,u_r)+1}^{u_r} \sum_{s \in \mathcal{S}_r^{p^\nu}} J_{p^\nu, p^{\nu-s}}(s).$$

Plugging the above formula into (15) and taking the difference

$$J_{p^n,p^{n-s}}(D) - J_{p^n,p^{n-s+1}}(D)$$

yields the desired formula. □

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