ORTHONORMAL POLYNOMIAL BASES IN FUNCTION SPACES

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Abstract. We construct polynomial orthonormal bases in various function spaces. Our bases have linear order of growth of degrees of polynomials. We show that this order is optimal.

Section 1 - Introduction

In this paper we construct bases consisting of trigonometric polynomials of small degree in spaces of functions on the circle $\mathbf{T}$. It is well known that $C(\mathbf{T})$ has a Schauder basis, so by a classical stability results it has a basis consisting of trigonometric polynomials. Two questions appear naturally

(1) how small the degrees of the polynomials in the basis can be
(2) can we construct such a basis to be orthonormal, and if so what are the degrees?

Probably [Fab] is the first paper dealing with this question. Various forms of those questions were asked in many papers (see [Ul1], [F-S]) and a lot of effort was spent on various partial solutions. A good and relatively complete survey is [Ul2]. The only significant omission is that the important paper [Pri] is not mentioned there.

Given any system of trigonometric polynomials $(\Phi_k)_{k=0}^{\infty}$ on $\mathbf{T}$ we define $v_n(\Phi_k) = v_n = \max_{k\leq n} \deg \Phi_k$. Clearly $(v_n)_{n=0}^{\infty}$ is an increasing sequence. A simple algebra shows that if $(\Phi_k)_{k=0}^{\infty}$ is a Schauder basis then $v_{2n+1} \geq n$ and Faber in [Fab] have shown that equality for infinitely many n's is impossible for any Schauder basis in $C(\mathbf{T})$. A recent result of Privalov [Pri] says that in $C(\mathbf{T})$ we always have $v_n \geq \frac{1}{2}(1+\varepsilon)n$ for some positive ε. On the other hand Bočkariov [Boč] have constructed a basis (but not orthonormal) in $C(\mathbf{T})$ with $v_n \leq 4n$. For every $\varepsilon > 0$ orthonormal bases in $C(\mathbf{T})$ are known with $v_n \leq n^{1+\varepsilon}$ (see [Ča1]). Similar question for unconditional bases in $L_p(\mathbf{T})$, $1 < p < \infty$ $p \neq 2$ have been considered in [Ča2] where for every $\varepsilon > 0$ an orthonormal unconditional basis in $L_p(\mathbf{T})$ with $v_n \leq n^{1+\varepsilon}$ have been constructed. The main result of this note can be stated as follows:

Theorem A. *There exists an orthonormal system of trigonometric polynomials $(\Phi_k)_{k=0}^{\infty}$ on $\mathbf{T}$ such that*

(1) $v_n(\Phi_k) \leq \frac{4}{3} \cdot n$
(2) *$(\Phi_k)_{k=0}^{\infty}$ is a Schauder basis in $C(\mathbf{T})$ (and so in $L_1(\mathbf{T})$)*

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(3) $(\Phi_k)_{k=0}^{\infty}$ *is an unconditional basis in* ReH_1 *and thus in all* $L_p(\mathbf{T})$ *for* $1 < p < \infty$.

We also give a simple proof of the result of [Boč] and extend the result of [Pri] to other function spaces.

It should be pointed out that our methods are direct (we do not use any approximation of good bases by trigonometric polynomials) and are a direct outgrowth of a construction of J. Bourgain [Bou] of a Schauder bases with uniformly bounded basis constants in spaces of trigonometric polynomials of a fixed degree.

The organisation of the paper is as follows. Section 2 contains some known Lemmas and estimates to be used later. In Section 3 we construct (for each ε) an interpolating basis in $C(\mathbf{T})$ with $v_n \leq (1+\varepsilon)n$. Better result have been obtained in [Pri1] (see also **Remark 4** in Section 3). This Section is included for two reasons; our construction is very simple and it may serve as an introduction to a more technically complicated but basically similar construction of an orthonormal system with properties described in Theorem A. This construction and the proof of i) and ii) of Theorem A are contained in Section 4. In this Section we also construct a polynomial basis in the disc algebra A with $v_n \leq 8/3 \cdot n$. This basis is an "analytical" version of the system $(\Phi_k)_{k=0}^{\infty}$. Section 5 contains a proof of property iii) of Theorem A. It is a modification of a proof that the Franklin system is unconditional in ReH_1, given in [Wo1]. Section 6 contains the extension of the Theorem of Privalov [Pri]. We show that his result holds also in spaces $L_1(\mathbf{T})$, A and H_1. It solves some problems left open in [Sh1] and [Sh2].

Hidden in our constructions are projections in $C(\mathbf{T})$ with uniformly bounded norms whose ranges are contained in spaces of polynomials of small degree. Such examples are related to the well known "finite dimensional π_λ-problem" (see also Remark 3 in Section 4). For more detailed investigation of ranges of such projections and construction of polynomial orthonormal bases with v_n as small as possible we refer the reader to the forthcoming paper of the second named author.

Clearly analogous questions can be considered for algebraic polynomials on an interval. They seem to require different techniques. We do not discuss them in this paper.

Section 2 - Definitions, notations and some Lemmas

We will freely use the basic and elementary facts about bases in Banach spaces. Those facts and definitions can be found in any text on Banach spaces. For obvious personal reasons we suggest [Wo2] which contains everything we will need. Naturally we will assume some familiarity with trigonometric polynomials. In particular we will freely use the natural identification of the circle $\mathbf{T}$ with the interval $[0, 2\pi]$ with identified endpoints. Let us also point out that for a function $f \in L_1(\mathbf{T})$ its n-th Fourier coefficient will be denoted by $\hat{f}(n)$. Let us recall some well known facts.

By $\mathcal{F}_M$ we will denote the Fejer kernel

$$\mathcal{F}_M(t) = \sum_{k=-M}^{M} (1 - \tfrac{|k|}{M+1}) e^{ik} = \frac{\sin^2 (M+1)t/2}{(M+1)\sin^2 t/2}. \tag{2.1}$$

The function $\mathcal{V}_{M,N}$, $M > N$ is defined by the formula

$$\mathcal{V}_{M,N} = \frac{M+1}{M-N}\mathcal{F}_M - \frac{N+1}{M-N}\mathcal{F}_N. \tag{2.2}$$

Simple calculation gives that

$$\mathcal{V}_{M,N}(t) = \frac{\sin(M+N+2)t/2 \cdot \sin(M-N)t/2}{(M-N)\sin^2 t/2}. \tag{2.3}$$

The functions $\mathcal{V}_{M,N}$ are well known de la Vallée Poussin kernels. From (2.2) and the fact that $\|\mathcal{F}_M\|_1 = 1$ we get

$$\|\mathcal{V}_{M,N}\|_1 \leq \frac{2(M+1)}{M-N}. \tag{2.4}$$

Remark. As a matter of fact $\frac{2(M+1)}{M-N}$ in (2.4) can be replaced by $C \log \frac{M+1}{M-N}$. This is known and can be obtained from Lemma 2.2.

Now we can prove

Lemma 2.1. *If f is a polynomial of degree at most N then the following holds*

$$\sum_{j=1}^{M} |f(t_0 - x_j)| \leq C \cdot M \|f\|_1 \max(\tfrac{N}{M}, 1) \tag{2.5}$$

where C is a constant and

$$x_j = \tfrac{2\pi j}{M} \quad \text{for} \quad j = 1, 2, \ldots, M. \tag{2.6}$$

Proof. For the trigonometric polynomial $\mathcal{F}_N$ we have

$$\sum_{j=1}^{M} |\mathcal{F}_N(t_0 - x_j)| = \sum_{j=1}^{M} \mathcal{F}_N(t_0 - x_j) = \sum_{j=1}^{M} \sum_{k=-N}^{N} \hat{\mathcal{F}}_N(k) e^{ik(t_0 - x_j)} =$$
$$\sum_{k=-N}^{N} \hat{\mathcal{F}}_N(k) e^{ikt_0} \sum_{j=1}^{M} e^{-i2\pi kj/M} \leq M \cdot \operatorname{card} D$$

where $D = \{k : |k| \leq N \text{ and } k = l \cdot M \text{ for some integer } l\}$. This gives (2.5) for the polynomial $\mathcal{F}_N$. Now let us observe that $f = f * \mathcal{V}_{2N,N}$ so

$$\sum_{j=1}^{M} |f(t_0 - x_j)| = \sum_{j=1}^{M} \Big| \int_{\mathbf{T}} \mathcal{V}_{2N,N}(s + x_j - t_0) f(s)\, ds \Big| \leq \int_{\mathbf{T}} \Big(\sum_{j=1}^{M} \big| \mathcal{V}_{2N,N}(s - t_0 + x_j) \big| \Big) |f(s)|\, ds.$$

Since we know already that (2.5) is valid for $\mathcal{F}_N$ and $\mathcal{F}_{2N}$, from (2.2) we obtain that the expression between round brackets is estimated by $CM \max(\frac{N}{M}, 1)$. This gives (2.5).

We will also need the following Lemma, which in full generality can be found in [S-W].

Lemma 2.2. *Let $f(t)$ be a complex-valued function defined on the real line such that $\|f\|_1 < \infty$. Then for every integer n there exists a function F on the circle $\mathbf{T}$ such that*

$$\hat{F}(k) = h(\tfrac{k}{n}) \quad \text{for all } k \tag{2.7}$$

where h is the Fourier transform of f. This function F belongs to $L_1(\mathbf{T})$ and satisfies

$$\|F\|_1 \leq (2\pi)^{-1} \|f\|_1. \tag{2.8}$$

Section 3 - Interpolating basis

In this Section we construct a polynomial interpolating basis in $C(\mathbf{T})$. Let us fix a natural number $N>0$ and consider the sequence of functions $\{W^k(t)\}_{k=0}^{\infty}$ defined by

$$\hat{W}^k(l)=\begin{cases} 1, & \text{if} \quad |l|\leq N2^k \\ \frac{(N+1)2^k-|l|}{2^k}, & \text{if} \quad N2^k<|l|<(N+1)2^k \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$

Those are de la Vallée Poussin kernels $\mathcal{V}_{(N+1)2^k-1,N2^k-1}$. Thus from (2.2) we see that

$$W^k=(N+1)\mathcal{F}_{(N+1)2^k-1}-N\mathcal{F}_{N2^k-1}. \tag{3.2}$$

and from (2.3) we get

$$W^k(t)=\frac{1}{2^k}\frac{\sin(2N+1)2^{k-1}t\ \cdot\sin 2^{k-1}t}{\sin^2 t/2} \tag{3.3}$$

Let us define

(3.4)

$$d(N,k)=N2^{k+1}+2^k$$

$$x_j^k=\frac{2\pi j}{d(N,k)} \qquad \text{for} \quad j=0,1,\dots,d(N,k)-1 \tag{3.5}$$

$$V_j^k(t)=\frac{W^k(t-x_j^k)}{d(N,k)} \qquad k=0,1,2,\dots \quad j=0,1,\dots d(N,k)-1 \tag{3.6}$$

Theorem 3.1. *The system $(\Phi_n)_{n=0}^{\infty}$ given explicitely as*

$$V_0^0,\dots,V_{d(N,0)-1}^0,V_1^1,V_3^1,V_5^1,\dots,V_{d(N,1)-1}^1,V_1^2,V_3^2,\dots$$

is an interpolating basis in $C(\mathbf{T})$. Each Φ_n is a trigonometric polynomial and $\deg\Phi_n\leq\left(1+\frac{1}{2N+1}\right)n$ *for $n>d(N,0)$.*

Proof. From (3.3) we see that

$$V_j^k(x_i^k)=\delta_{i,j}. \tag{3.7}$$

Let us denote $\mathbf{B}_k=\operatorname{span}\left(V_j^k\right)_{j=0}^{d(N,k)-1}$. Spaces $\mathbf{B}_k$, $k=0,1,\dots$ have two remarkable properties :

$$e^{ilt}\in\mathbf{B}_k \qquad \text{for} \qquad |l|\leq 2^kN \tag{3.8}$$

$$\max_{0\leq j\leq d(N,k)-1}|a_j|\leq\Big\|\sum_{j=0}^{d(N,k)-1}a_jV_j^k\Big\|_\infty\leq C\max_{0\leq j\leq d(N,k)-1}|a_j| \tag{3.9}$$

where the constant C depends on N but not on k.

To check (3.8) we write for $|l| \leq 2^k N$

$$\begin{aligned}
&\sum_{j=0}^{d(N,k)-1} e^{\frac{2\pi i l j}{d(N,k)}} V_j^k(t) \\
&= d(N,k)^{-1} \sum_{j=0}^{d(N,k)-1} e^{\frac{2\pi i l j}{d(N,k)}} \sum_{|s|<(N+1)2^k} \hat{W}^k(s) e^{i(t-x_j^k)s} \qquad (3.10)\\
&= d(N,k)^{-1} \sum_{|s|<(N+1)2^k} \hat{W}^k(s) e^{its} \sum_{j=0}^{d(N,k)-1} e^{\frac{2\pi i(l-s)j}{d(N,k)}} = e^{ilt}
\end{aligned}$$

To see the last equality note that $\sum_{j=0}^{d(N,k)-1} e^{\frac{2\pi i(l-s)j}{d(N,k)}}$ equals 0 if $(l-s)$ is not an integer multiple of $d(N,k)$ and equals $d(N,k)$ if $(l-s)$ is an integer multiple of $d(N,k)$. But this happens only when $l = s$. In this case $\hat{W}^k(s) = 1$.

The left hand side of (3.9) follows from (3.7) and the right hand side follows from Lemma 2.1 and (2.4). From (3.8) we see that $\mathbf{B}_k \subset \mathbf{B}_{k+1}$ $k = 0, 1, 2, \ldots$. We can also define natural projections P_k from $C(\mathbf{T})$ onto $\mathbf{B}_k$ by the formula

$$P_k(f) = \sum_{j=0}^{d(N,k)-1} f(x_j^k) V_j^k.$$

It follows from (3.9) that

$$\|P_k\| \leq C. \qquad (3.11)$$

From (3.5) we see that $P_k P_r = P_r P_k = P_{\min(k,r)}$. From (3.5) we infer that $x_j^k = x_{2j}^{k+1}$ so (3.7) gives $P_k(V_{2r-1}^{k+1}) = 0$ for $r = 1, 2, \ldots, \frac{d(N,k+1)}{2}$. Since the space $\mathbf{B}_{k+1} = \operatorname{span}\left(\mathbf{B}_k, \{V_{2r-1}^{k+1}\}_{r=1}^{d(N,k+1)/2}\right)$ (count dimensions) and $\bigcup_{k\geq 0} \mathbf{B}_k$ is dense in $C(\mathbf{T})$ (use (3.8)) we see that the system $(\Phi_n)_{n=0}^{\infty}$ is a Schauder basis in $C(\mathbf{T})$. From the definition of the projections P_k and (3.7) we see that it is an interpolating basis. Since each V_j^k is a trigonometric polynomial of degree $(N+1)2^k$ we see that for $n > d(N,0)$ we have $\deg \Phi_n \leq (1 + \frac{1}{2N+1})n$.

Remark 1 The basis constant of the basis $(\Phi_n)_{n=1}^{\infty}$ is at most $3C$ where C is a constant appearing in (3.9). This constant can be majorised by $c \ln N$ (cf. the Remark after (2.4)).

Remark 2 The estimate for the degree of the polynomial Φ_n looks a little better than it really is. One has to remember that the dimension of the space of trigonometric polynomials of degree at most n is $2n+1$. Thus our basis really takes a bit more than twice as big a degree as is needed algebraically.

Remark 3 A very similar basis was constructed by S.V. Bočkariov [Boč]. In the construction the difference seems to be that we use de la Vallée Poussin kernels while Bočkariov uses Fejer kernels. Our proof, however is much simpler.

Remark 4 Before we started the work on this paper we have been informed by S.V. Bočkariov that polynomial bases for $C(\mathbf{T})$ with $v_n \leq (\varepsilon + 1/2)n$ were constructed earlier by Al. A. Privalov modyfying the construction from [Boč]. After the present paper was submitted the paper of Privalov [Pri1] have appeared. He uses de la Vallée Poussin kernels and our basis is a special case of his construction. His proof however contains many technical complications which are superflous in our simple case.

Section 4 - Construction of an orthogonal system

In this section V_k $k = 0, 1, 2, \ldots$ will always denote the classical de la Vallée Poussin kernel $\mathcal{V}_{2^{k+1}-1, 2^k-1}$ (see (2.2)). We consider the functions $F_k = V_k - V_{k-1}$ $k = 0, 1, 2, \ldots$ (by V_{-1} we mean 0). In terms of Fourier coefficients functions F_k are given by

(4.1)
$$\hat{F}_k(l) = \begin{cases} \frac{(2^{k+1}-|l|)}{2^k}, & \text{for } 2^k \le |l| \le 2^{k+1} \\ \frac{(|l|-2^{k-1})}{2^{k-1}}, & \text{for } 2^{k-1} \le |l| \le 2^k \\ 0 & \text{otherwise,} \end{cases} \qquad \text{if } k > 0$$
$$\hat{F}_0(l) = \begin{cases} 1, & \text{for } |l| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let us note also that from (2.3) and definitions of F_k we get

$$F_k\left(\frac{2\pi(j-j')}{3\cdot 2^{k-1}}\right) = 3\cdot 2^{k-1}\delta j, j' \ . \tag{4.2}$$

We define polynomials P_0^k for $k = 0, 1, 2, \ldots$ by the formula

$$\hat{P}_0^k(n) = \begin{cases} \sqrt{\hat{F}_k(n)} & \text{for } n \le 0 \\ (-1)^k\sqrt{\hat{F}_k(n)} & \text{for } n \ge 0 \end{cases} \tag{4.3}$$

The desired orthogonal basis consists of polynomials $P_0^0(t)$, $P_1^0(t)$, $P_2^0(t)$ and $P_j^k(t)$ for $k = 1, 2, \ldots$ and $j = 0, 1, \ldots, 3\cdot 2^{k-1} - 1$ given by

$$P_j^k(t) = P_0^k\left(t - \frac{2\pi j}{3\cdot 2^{k-1}}\right). \tag{4.4}$$

Let us start our investigation of this system with the observation that for a fixed k functions $\left(P_j^k\right)_{j=0}^{3\cdot 2^{k-1}-1}$ are orthogonal. This follows from (4.4), (4.3) and (4.2) because

$$\begin{aligned} \langle P_j^k, P_{j'}^k\rangle &= \sum_{s=-\infty}^{\infty} \hat{P}_j^k(s)\overline{\hat{P}_{j'}^k(s)} \\ &= \sum_{s=-\infty}^{\infty} \hat{P}_0^k(s)e^{-\frac{2\pi i j s}{3\cdot 2^{k-1}}}\hat{P}_0^k(s)e^{\frac{2\pi i j' s}{3\cdot 2^{k-1}}} \\ &= \sum_{s=-\infty}^{\infty} \hat{F}_k(s)e^{\frac{2\pi i s(j'-j)}{3\cdot 2^{k-1}}} \\ &= F_k\left(\frac{2\pi(j'-j)}{3\cdot 2^{k-1}}\right) = 3\cdot 2^{k-1}\delta_{j,j'}. \end{aligned} \tag{4.5}$$

For $k>0$ consider spaces
(4.6)

$$\begin{aligned}\mathbf{B}_k^0&=\operatorname{span}\left\{\hat P_0^k(n)e^{int}+\hat P_0^k(n+3\cdot 2^k)e^{i(n+3\cdot 2^k)t}\ :\ -2^{k+1}\le n\le -2^k-1\right\}\\ \mathbf{B}_k^1&=\operatorname{span}\left\{\hat P_0^k(n)e^{int}+\hat P_0^k(n+3\cdot 2^{k-1})e^{i(n+3\cdot 2^{k-1})t}\ :\ -2^k\le n\le -2^{k-1}-1\right\}\end{aligned}$$

Clearly $\mathbf{B}_k^0$ is orthogonal to $\mathbf{B}_k^1$ for each k. Since for $-2^{k+1}\le n\le -2^k-1$ we have

$$\sum_{j=0}^{3\cdot 2^{k-1}-1} e^{\frac{2\pi inj}{3\cdot 2^{k-1}}}P_j^k=3\cdot 2^{k-1}\left(\hat P_0^k(n)e^{int}+\hat P_0^k(n+3\cdot 2^k)e^{i(n+3\cdot 2^k)t}\right)\tag{4.7}$$

and for $-2^k\le n\le 2^{k-1}-1$ we have

$$\sum_{j=0}^{3\cdot 2^{k-1}-1} e^{\frac{2\pi inj}{3\cdot 2^{k-1}}}P_j^k=3\cdot 2^{k-1}\left(\hat P_0^k(n)e^{int}+\hat P_0^k(n+3\cdot 2^{k-1})e^{i(n+3\cdot 2^{k-1}t}\right).\tag{4.8}$$

The calculation for (4.7) and (4.8) are similar to (3.10). Counting dimensions we infer that the space $\mathbf{B}_k=\operatorname{span}\left(P_j^k\right)_{j=0}^{3\cdot 2^{k-1}-1}$ is (for $k>0$) an orthogonal sum of $\mathbf{B}_k^0$ and $\mathbf{B}_k^1$. Observe also that

$$\mathbf{B}_0=\operatorname{span}(P_j^0)_{j=0}^2=\operatorname{span}(e^{int})_{n=-1}^1\tag{4.9}$$

From (4.7), (4.8) and (4.9) we see that spaces $\mathbf{B}_k^0$, $\mathbf{B}_k^1$ for $k=1,2,\dots$ and $\mathbf{B}_0$ are mutually orthogonal. The only thing which is not obvious here is to check that $\mathbf{B}_k^0$ is orthogonal to $\mathbf{B}_{k+1}^1$. This follows from (4.6) and the following equalities

$$\begin{aligned}\hat P_0^{k+1}(n)&=(-1)^k\hat P_0^k(n+3\cdot 2^k)\\ \hat P_0^k(n)&=(-1)^{k+1}\hat P_0^{k+1}(n+3\cdot 2^k)\end{aligned}\tag{4.10}$$

valid for $-2^{k+1}\le n\le -2^k-1$, which are easy consequences of (4.1) and (4.3). Note also that (4.10) and simple induction implie that

$$\operatorname{span}(\mathbf{B}_k)_{k=0}^n=\operatorname{span}\left(\{e^{ist}\}_{s=-2^n}^{2^n-1}\cup\mathbf{B}_n^0\right)\qquad\text{for } n=1,2,\dots\tag{4.11}$$

Thus we have

Lemma 4.1. *The system of polynomials defined by* (4.4) *is a complete orthogonal set.*

In order to investigate the properties of this set in the space $C(\mathbf{T})$ we will need the following estimate

Lemma 4.2. *There exists a constant C such that $\|P_j^k\|_1\le C$ for $k=0,1,2,\dots$ and $j=0,1,2,\dots,3\cdot 2^{k-1}-1$.*

Proof.. First of all note that (4.4) implies $\|P_j^k\|_1=\|P_0^k\|_1$ for all admissible j's. To estimate $\|P_0^k\|_1$ note that $P_0^k(t)=\sum_{n=-\infty}^{\infty}g(n/2^{k+1})e^{int}$ for

$$g(s)=\begin{cases}\pm\varphi(s), & \text{for } s\ge 0\\ \varphi(-s), & \text{for } s\le 0\end{cases}\tag{4.12}$$

where the choice of sign depends on the parity of k and

$$\varphi(s) = \begin{cases} \sqrt{4s-1}, & \text{for } 1/4 \le s \le 1/2 \\ \sqrt{2-2s}, & \text{for } 1/2 \le s \le 1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.13}$$

We see from Lemma 2.2 that it is enough to show that $\check{\varphi}(t) = \int_{-\infty}^{\infty} e^{ist}\varphi(s)ds$ is in $L_1(R)$. This follows from the following Lemma which we state explicitely because we will need it also in Section 5.

Lemma 4.3. *Let φ be defined by* (4.13). *Then there exists a C such that*

$$|\check{\varphi}(t)| = \left|\int_{-\infty}^{\infty} e^{ist}\varphi(s)\,ds\right| \le \begin{cases} Ct^{-3/2} & \text{if } |t| > \frac{4\pi}{3} \\ C & \text{if } |t| \le \frac{4\pi}{3} \end{cases} \tag{4.14}$$

Proof.. Integrating by parts and substituting we get

$$\left|\int_{-\infty}^{\infty} e^{ist}\varphi(s)ds\right| = \frac{1}{t}\left|\int_{-\infty}^{\infty} e^{ist}\varphi'(s)ds\right| = \frac{1}{t^2}\left|\int_{-\infty}^{\infty} e^{iu}\varphi'(\tfrac{u}{t})du\right|.$$

Since $\varphi'(s)$ has only two singularities, in $1/4$ and 1, and at those it behaves like $\frac{1}{\sqrt{s}}$, we get

$$|\check{\varphi}(t)| \le t^{-2}C\int_0^{\infty} \frac{1}{\sqrt{\frac{u}{t}}}du = ct^{-3/2}.$$

Since $\check{\varphi}$ is clearly bounded by $\|\varphi\|_1$ we get the desired estimate.

Lemma 4.4. *There exist constants C_1 and C_2 such that for every $k = 0, 1, 2, \dots$ we have*

$$C_1 3\cdot 2^{k-1}\sup_j |\alpha_j| \le \Big\|\sum_{j=0}^{3\cdot 2^{k-1}-1} \alpha_j P_j^k\Big\|_\infty \le C_2 3\cdot 2^{k-1}\sup_j|\alpha_j| \tag{4.15}$$

for all sequences of scalars $(\alpha_j)_{j=0}^{3\cdot 2^{k-1}-1}$.

Proof. From (2.1) and Lemma 4.2. we get

$$\sum_{j=0}^{3\cdot 2^{k-1}-1} |P_j^k(x)| = \sum_{j=0}^{3\cdot 2^{k-1}-1}\left|P_0^k(x - \frac{2\pi j}{3\cdot 2^{k-1}})\right| \le C3\cdot 2^{k-1}\|P_0^k\|_1 \le C\cdot 3\cdot 2^{k-1}$$

so the right hand side inequality of (4.15) follows. On the other hand, orthogonality of P_j^k's gives

$$\begin{aligned} \|P_{j_0}^k\|_1 \cdot \Big\|\sum_{j=0}^{3\cdot 2^{k-1}-1}\alpha_j P_j^k\Big\|_\infty &\ge \Big|\langle P_{j_0}^k, \sum_{j=0}^{3\cdot 2^{k-1}-1}\alpha_j P_j^k\rangle\Big| \\ &= |\alpha_{j_0}|\cdot\|P_{j_0}^k\|_2^2 = |\alpha_{j_0}|\cdot\|P_0^k\|_2^2. \end{aligned} \tag{4.16}$$

It follows from (4.5) that $\|P_0^k\|_2 = \sqrt{3\cdot 2^{k-1}}$, so Lemma 4.2 and (4.16) give the left hand side inequality in (4.15).

Theorem 4.5. *Let $(\Phi_n)_{n=0}^{\infty}$ be the orthogonal system $\left(P_j^k\right)$ for $k = 0, 1, 2, \ldots$ and $j = 0, 1, \ldots, 3 \cdot 2^{k-1} - 1$ ordered as*

$$P_0^0, P_1^0, P_2^0, P_0^1, P_1^1, P_2^1, P_0^2, \ldots, P_5^2, P_0^3, \ldots.$$

The system $(\Phi_n)_{n=0}^{\infty}$ is a Schauder basis in the space $C(\mathbf{T})$, and each Φ_n is a trigonometric polynomial of degree at most $4/3 \cdot n$.

Proof. The estimate for the degree of Φ_n follows from (4.1) and (4.3). Let us consider an orthogonal projection Q_k onto the span$(\mathbf{B}_s)_{s=0}^{k}$ (see (4.11)). It easily follows from (4.11), (4.6), (4.3) and (4.1) that Q_k can be explicitely written as
(4.17)

$$Q_k(e^{ist}) = \begin{cases} e^{ist}, & \text{if } |s| \leq 2^k \\ 0, & \text{if } |s| > 2^{k+1} \\ \hat{P}_0^k(s)\left(\hat{P}_o^k(s)e^{ist} + \hat{P}_0^k(s + 3 \cdot 2^k)e^{i(s+3\cdot 2^k)t}\right), & \text{if } -2^{k+1} \leq s \leq -2^k - 1 \\ \hat{P}_0^k(s)\left(\hat{P}_0^k(s)e^{ist} + \hat{P}_0^k(s - 3 \cdot 2^k)e^{i(s-3\cdot 2^k)t}\right), & \text{if } 2^k - 1 \leq s \leq 2^{k+1} \end{cases}$$

so

$$Q_k = V_k + A_1^k + A_2^k \tag{4.18}$$

where V_k is the operator of convolution with the de la Vallée Poussin kernel V_k and

$$\begin{aligned} A_1^k(e^{ist}) &= \hat{P}_0^k(s) \cdot \hat{P}_0^k(s + 3 \cdot 2^k)e^{i(s+3\cdot 2^k)t} \\ A_2^k(e^{ist}) &= \hat{P}_0^k(s) \cdot \hat{P}_0^k(s - 3 \cdot 2^k)e^{i(s-3\cdot 2^k)t}. \end{aligned} \tag{4.19}$$

It is well known that norms of V_k in $C(\mathbf{T})$ are uniformly bounded. The same for operators A_1^k and A_2^k follows from Lemma 4.2. The uniform boundedness of projections $(Q_k)_{k=0}^{\infty}$ and Lemma 4.4 show that $(\Phi_n)_{n=0}^{\infty}$ is indeed a Schauder basis in $C(\mathbf{T})$.

Now let us consider the space A_r of all functions $f \in C(\mathbf{T})$ such that its trigonometric conjugate $\tilde{f} \in C(\mathbf{T})$. The norm in A_r is defined by $|||f||| = \|f\|_\infty + \|\tilde{f}\|_\infty$. We have

Theorem 4.6. *The system $(\Phi_n)_{n=0}^{\infty}$ considered in Theorem 4.5 is a Schauder basis in A_r.*

Proof. We will use results obtained in the course of proof of Theorem 4.5. Note that the action of the trigonometric conjugation $\tilde{}$ on each $\mathbf{B}_k$ $k \geq 1$ can be represented as a convolution with the difference of two appropriately shifted de la Vallée Poussin kernels. This shows that $\tilde{}$ acts on each $\mathbf{B}_k$ with the uniformly bounded norm. This implies that norm $\|\cdot\|_\infty$ and $|||\cdot|||$ are on $\mathbf{B}_k$'s uniformly equivalent, so Lemma 4.4 holds for $|||\cdot|||$ as well. The final step is to observe that projections Q_k are uniformly bounded in $|||\cdot|||$. This can be easily checked for each of the operators in the decomposition (4.18).

The importance of the space A_r stems from the fact that it is naturally isomorphic to the disc algebra A (see [Wo2, III.E.16]). Let us recall that

$$A = \{f(z) : f \text{ is continuous for } |z| \leq 1 \text{ and analytic for } |z| < 1\}$$

equipped with the supremum norm. One look at the isomorphism given in [Wo2, III.E.16] convinces us that the following holds:

Corollary 4.7. *The disc algebra A has a Schauder basis $(\Psi_n)_{n=0}^{\infty}$ consisting of polynomials such that $deg\Psi_n \leq 8/3 \cdot n$. When considered on the unit circle this basis is orthogonal with respect to the Lebesque measure.*

Remark 1 One can easily see that the proof of Theorem 4.5 works as well for the system $(\tilde{\Phi}_n)_{n=0}^{\infty}$. Using this observation one can easily construct a basis in the space $C(T)$ of continuous functions on the unit circle such that a subsequence of this basis is a basis for the disc algebra. The periodic Franklin system also has this property (see [BoČ3]).

Remark 2 Let us explain in some detail the relation between our constructions of Sections 3 and 4 and the work of J. Bourgain [Bou]. In [Bou] (among other things) a basis in the space $\operatorname{span}\{e^{ikt}\}_{k=1}^{3\cdot 2^q}$ is constructed. The first half of the basis consists of functions of the form $\sum_{k=2^{q-1}}^{3\cdot 2^{q-1}} a_k e^{ikt}$ and next he constructs functions whose non-zero Fourier coefficients are contained in two intervals, in each step shorter and closer to numbers 0 and $3 \cdot 2^q$. In our construction we start from the "middle" and next expand the size of the support of the Fourier transform to get it further away from the number 0. This is the basic change made in Section 3 while in Section 4 further modifications are made to get orthogonality. It is possible to carry on a "shrinking" version of the construction of Section 4 to get an orthogonal basis with uniformly bounded basis constant in spaces $\operatorname{span}\{e^{ikt}\}_{k=1}^{N}$ for $N = 3\cdot 2^q$. Without any restriction on N such bases were constructed by S. V. Bočkariov in [Boč2]. His construction uses a discrete version of the Franklin system and is difficult to visualise in terms of Fourier coefficients.

Remark 3. The paper of J. Bourgain [Bou] have its fundamental motivation in the so called finite dimensional π_λ-problem. This problem asks the following : does there exist a function $f(\lambda)$ such that $\mathrm{d}(X, \ell_\infty^{\dim X}) \leq f(\lambda)$ for every finite dimensional subspace $X \subset C(\mathbf{T})$ such that there exists a projection P from $C(\mathbf{T})$ onto X with $\|P\| \leq \lambda$. The symbol $\mathrm{d}(\cdot,\cdot)$ denotes the classical Banach-Mazur distance [Wo2, II.E.6]. Using Lemma 4.1 from [Bou] we can show that spaces $\operatorname{span}\{\Phi_j\}_{j=N}^{M} \subset C(\mathbf{T})$ (where Φ_j's are as in Theorem 4.5) are uniformly isomorphic to $\ell_\infty^{d(n)}$-spaces. This isomorphism, however, is not constructive. It would be interesting to exhibit explicitely in those spaces a basis equivalent to the unit vector basis in $\ell_\infty^{d(n)}$. In some cases this is done in the forthcoming paper of the second named author. A related problem is to establish if the basis we obtained in Theorem 4.5. is equivalent to some classical basis like Haar or Franklin basis.

Section 5 - Unconditionality

Our aim in this Section is to show that the orthogonal system constructed in Section 4 is an unconditional basis in $L_p(\mathbf{T})$ for $1 < p < \infty$. Our approach will be via the space BMO, so let us recall appropriate definitions. We say that a function on $\mathbf{T}$ belongs to the space BMO if and only if for every interval $I \subset \mathbf{T}$ there exists a constant c_I such that

$$\sup_{I\subset \mathbf{T}} \frac{1}{|I|}\int_I |f - c_I|^2 < \infty \tag{5.1}$$

It is easy and well known that (5.1) is equivalent to the condition

$$\sup_{I\subset \mathbf{T}} \frac{1}{|I|}\int_I \left| f - \frac{1}{|I|}\int_I f \right|^2 < \infty. \tag{5.2}$$

It is also well known that the space BMO (after obvious identification of functions equal almost everywhere) is a Banach space when equipped with the norm

$$\|f\|_{BMO} = \|f\|_1 + \sup_{I\subset \mathbf{T}} \left(\frac{1}{|I|} \int_I \left| f - \frac{1}{|I|}\int_I f \right|^2 \right)^{1/2}. \tag{5.3}$$

We will also need the space ReH_1. This is the space of all functions $f \in L_1(\mathbf{T})$ such that its trigonometric conjugate $\tilde{f}$ also belongs to $L_1(\mathbf{T})$. Recall that the well known Fefferman duality Theorem asserts that $(\mathrm{Re}H_1)^* = BMO$ with the natural duality. This and much more about BMO and H_1 can be found in almost any modern text on harmonic analysis. As an example we point out [Tor].

In this Section it will be convenient to change the notation in order to introduce a more geometric picture. Let $f_j^k = \|P_j^k\|_2^{-1} P_j^k$ where polynomials P_j^k were defined in (4.4). This is a complete orthonormal system. For f_j^k we will use the notation $f_{\mathbf{v}}$ where

$$\mathbf{v} = \left[\frac{(2j-1)\pi}{3\cdot 2^{k-1}}, \frac{(2j+1)\pi}{3\cdot 2^{k-1}} \right]. \tag{5.4}$$

We treat $\mathbf{v}$ as a subinterval of $\mathbf{T}$ using the obvious covering map from real numbers onto $\mathbf{T}$. The set of all intervals as in (5.4) for $k = 0,1,2,\dots$ and $j = 0,1,\dots,3\cdot 2^{k-1}-1$ will be denoted by $\mathcal{J}$. Observe that for a fixed k all such intervals form an almost disjoint covering of $\mathbf{T}$. Also two intervals from $\mathcal{J}$ are either disjoint or one is contained in the other.

Convention. In this Section symbols $\mathbf{v}$, $\mathbf{w}$ (with subscripts as needed) will always denote intervals from $\mathcal{J}$. An arbitrary subinterval of $\mathbf{T}$ will be denoted by I (with subscripts when needed).

Now we can formulate the main result of this Section.

Theorem 5.1. *A function $f \in L_1(\mathbf{T})$ belongs to BMO if and only if*

$$\sup_{\mathbf{w}\in\mathcal{J}} \left(\frac{1}{|\mathbf{w}|} \sum_{\mathbf{v}\subset\mathbf{w}} \left| \int_{\mathbf{T}} f f_{\mathbf{v}} \right|^2 \right) < \infty. \tag{5.5}$$

Before we proceed with the proof let us note some Corollaries of this result.

Corollary 5.2. *The orthogonal system $(\Phi_n)_{n=0}^\infty$ considered in Theorem 4.5 is an unconditional basis in $L_p(\mathbf{T})$ for $1 < p < \infty$.*

Proof. Clearly the difference between the system $(\Phi_n)_{n=0}^\infty$ and the system $(f_{\mathbf{v}})_{\mathbf{v}\in\mathcal{J}}$ is only in the normalisation. From (5.5) we see that the system $(f_{\mathbf{v}})_{\mathbf{v}\in\mathcal{J}}$ is unconditional in BMO and being orthonormal it is unconditional in $L_2(\mathbf{T})$. Since $L_p(\mathbf{T})$ for $2 < p < \infty$ are interpolatiion spaces between BMO and $L_2(\mathbf{T})$ (see [C-W]) we get that $(f_{\mathbf{v}})_{\mathbf{v}\in\mathcal{J}}$ is unconditional in $L_p(\mathbf{T})$ for $2 \le p < \infty$. Since it is linearly dense (see Theorem 4.5) it is an unconditional basis. The case $1 < p < 2$ follows by duality.

Corollary 5.3. *The system $(f_{\mathbf{v}})_{\mathbf{v}\in\mathcal{J}}$ is an unconditional basis in the space ReH_1.*

Proof. This follows immediately from the well known Fefferman duality theorem that $(ReH_1)^* = BMO$.

Remarks. The formula (5.5) gives a characterisation of BMO functions. It is the same characterisation as the one given in [Wo1] using the Franklin system. As a consequence we see that the system $(f_{\mathbf{v}})_{\mathbf{v}\in\mathcal{J}}$ is equivalent in BMO with the Franklin system. In $L_p(\mathbf{T})$ for $1 < p < \infty$ it is equivalent to both Haar and Franklin system.

Now we return to the proof of Theorem 5.1. We start with the following preliminary Lemma.

Lemma 5.4. *For each function $f_j^k = f_{\mathbf{v}}$ we have*

$$|f_j^k(t)| = |f_{\mathbf{v}}(t)| \le \begin{cases} C|\mathbf{v}|^{-1/2} & \text{for } t \in \mathbf{v} \\ C|\mathbf{v}| \left| t - \frac{2\pi j}{3\cdot 2^{k-1}} \right|^{-3/2} & \text{for } t \notin \mathbf{v} \end{cases} \tag{5.6}$$

Note that $|\mathbf{v}| = (3\cdot 2^{k-1})^{-1}$. We will denote this quantity by $L(k)$

Proof. From the construction we see that it is enough to consider $f_0^k = \frac{1}{\sqrt{3}} 2^{-\frac{k-1}{2}} P_0^k$. Like in the beginning of the proof of Lemma 4.2 we observe that $P_0^k(t) = \sum_{-\infty}^{\infty} g(n2^{k+1})e^{int}$ where g is given by (4.12) and (4.13). From this and the standard transference or the Poisson's summation formula (see [S-W]) we infer that

$$P_0^k(t) = 2^{k+1} \sum_{s=-\infty}^{\infty} \check{g}(2^{k+1}t + 2\pi 2^{k+1}s) \tag{5.7}$$

This formula together with (4.12), (4.13) and Lemma 4.3 immediately yields (5.6).

Remark. The formula (5.7) can be easily checked directly. We have to compute Fourier coefficients of both sides and check that they are the same.

Proof of Theorem 5.1 is contained in two following Propositions.

Proposition 5.5. *If a sequence of complex numbers $(a(\mathbf{v}))_{\mathbf{v}\in\mathcal{J}}$ for some constant C satisfies*

$$\sum_{\mathbf{v}\subset\mathbf{w}} |a(\mathbf{v})|^2 \le C|\mathbf{w}| \qquad \text{for all } \mathbf{w}\in\mathcal{J} \tag{5.8}$$

then

$$f = \sum_{\mathbf{v}\in\mathcal{J}} a(\mathbf{v}) f_{\mathbf{v}} \in BMO \tag{5.9}$$

and $\|f\|_{BMO} \le C'$.

Proposition 5.6. *There exists a constant C such that for every function $f \in BMO$ with $\|f\|_{BMO} \le 1$ and $f = \sum_{\mathbf{v}\in\mathcal{J}} a(\mathbf{v}) f_{\mathbf{v}}$ we have*

$$\sum_{\mathbf{v}\subset\mathbf{w}} |a(\mathbf{v})|^2 \le C|\mathbf{w}| \quad \text{for all} \quad \mathbf{w}\in\mathcal{J}. \tag{5.10}$$

Proof of Proposition 5.5. From (5.8) we see that $f \in L_2(\mathbf{T})$. Let us take an arbitrary interval $I \subset \mathbf{T}$. Without loss of generality we can assume that $|I| < \frac{2\pi}{12}$. Let us fix two adjacent intervals $\mathbf{v}_1$ and $\mathbf{v}_2$ from $\mathcal{J}$ such that

$$|\mathbf{v}_1| = |\mathbf{v}_2| \leq 6|I| \tag{5.11}$$

and such that

$$\mathbf{v}_1 \cup \mathbf{v}_2 \supset 3I \tag{5.12}$$

where $3I$ denotes the interval with the same center as I but three times longer. The interval $\mathbf{v}_1 \cup \mathbf{v}_2$ will be denoted by ω. Let us write

$$f = \sum_{\mathbf{v}\subset\omega} + \sum_{\mathbf{v}:\mathbf{v}\cap\omega=\emptyset,\ |\mathbf{v}|\leq|I|} + \sum_{\mathbf{v}:|\mathbf{v}|>|I|} a(\mathbf{v}) f_{\mathbf{v}} = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3. \tag{5.13}$$

From (5.8) we get

$$\int_I |\sum\nolimits_1|^2 \leq \int_{\mathbf{T}} |\sum\nolimits_1|^2 \leq 2C|\omega| \leq C|I| \tag{5.14}$$

From (5.8) we infer in particular that $|a(\mathbf{v})| \leq C|\mathbf{v}|^{1/2}$ so for $x \in I$ we have

$$\begin{aligned} |\sum\nolimits_2(x)| &\leq \sum_{\mathbf{v}:\mathbf{v}\cap\omega=\emptyset,\ |\mathbf{v}|\leq|I|} |a(\mathbf{v})||f_{\mathbf{v}}(x) \\ &\leq C \sum_{k:\ L(k)\leq|I|} 2^{k/2} \sum_{\mathbf{v}:\mathbf{v}\cap\omega=\emptyset,\ |\mathbf{v}|=L(k)} |f_{\mathbf{v}}(x)|. \end{aligned} \tag{5.15}$$

Using (5.12) and Lemma 5.4 we majorise the internal sum in (5.15) by

$$\begin{aligned} &C\sum_{s=1}^{\infty} |\mathbf{v}| \left(|I| + (s-\frac{1}{2})|\mathbf{v}| \right)^{-3/2} \leq |\mathbf{v}||\mathbf{v}|^{-3/2} C \sum_{s=1}^{\infty} \left(\frac{|I|}{|\mathbf{v}|} + s \right)^{-3/2} \\ &\leq C|\mathbf{v}|^{-1/2} \left(\frac{|I|}{|\mathbf{v}|} \right)^{-1/2} \leq C|I|^{-1/2}. \end{aligned} \tag{5.16}$$

From (5.15) and (5.16) we immediately get that for $x \in I$

$$\sum\nolimits_2(x) \leq C|I|^{-1/2} \sum_{k:L(k)\leq|I|} 2^{k/2} = C \tag{5.17}$$

For arbitrary points $x, x_0 \in I$ we have

$$\begin{aligned} |\sum\nolimits_3(x) - \sum\nolimits_3(x_0)| &\leq \sum_{\mathbf{v}:|\mathbf{v}|>|I|} |a(\mathbf{v})||f_{\mathbf{v}}(x) - f_{\mathbf{v}}(x_0)| \\ &\leq C \sum_{k:L(k)>|I|} 2^{-k/2} \sum_{\mathbf{v}:|\mathbf{v}|=L(k)} |f_{\mathbf{v}}(x) - f_{\mathbf{v}}(x_0)| \\ &= C \sum_{k:L(k)>|I|} 2^{-k/2} \int_x^{x_0} \sum_{\mathbf{v}:|\mathbf{v}|=L(k)} |f'_{\mathbf{v}}(\xi)| d\xi \end{aligned} \tag{5.18}$$

Since

$$\|\sum_{\mathbf{v}:|\mathbf{v}|=2^{-k}} |f'_{\mathbf{v}}(\xi)|\|_\infty = \sup\{\|(\sum_{\mathbf{v}:|\mathbf{v}|=2^{-k}} \varepsilon_{\mathbf{v}} f_{\mathbf{v}})'\|_\infty : |\varepsilon_{\mathbf{v}}|=1\}$$

and since each $f_{\mathbf{v}}$ is a trigonometric polymomial of degree at most $C\cdot 2^k$ from Bernstein's inequality and Lemma 5.4 (or Lemma 2.1) we get

$$\|\sum_{\mathbf{v}:|\mathbf{v}|=L(k)} |f'_{\mathbf{v}}(\xi)|\|_\infty \leq C\cdot 2^k \|\sum_{\mathbf{v}:|\mathbf{v}|=L(k)} |f_{\mathbf{v}}|\|_\infty \leq C\cdot 2^{3k/2}. \tag{5.19}$$

From (5.19) and (5.18) we get

$$\begin{aligned} |\textstyle\sum_3(x) - \sum_3(x_0)| &\leq C \sum_{k:L(k)>|I|} 2^{-k/2}\cdot|x-x_0|\cdot 2^{3k/2} \\ &\leq C|I|\cdot \sum_{k:L(k)>|I|} 2^k \leq C. \end{aligned} \tag{5.20}$$

From (5.14), (5.17) and (5.20) we get $\frac{1}{|I|}\int_I |f-\sum_3(x_0)|^2 \leq C$ so $f\in BMO$.

Proof of Proposition 5.6. Observe first that norms $\|f_{\mathbf{v}}\|_1$ and $\|f_{\mathbf{v}}\|_{\mathrm{Re}H_1}$ are uniformly equivalent for $\mathbf{v}\in\mathcal{J}$. This can be done exactly like in the proof of Theorem 4.6. This implies that

$$\|f_{\mathbf{v}}\|_{\mathrm{Re}H_1} \leq c|\mathbf{v}|^{1/2}. \tag{5.21}$$

For given $\mathbf{w}\in\mathcal{J}$ let us write

$$f = \sum_{\mathbf{v}\in\mathcal{J}} a(\mathbf{v})f_{\mathbf{v}} = \sum_{\mathbf{v}\subset\mathbf{w}} + \sum_{\mathbf{v}:|\mathbf{v}|\leq|\mathbf{w}| \text{ and } \mathbf{v}\cap\mathbf{w}=\emptyset} + \sum_{\mathbf{v}:|\mathbf{v}|>|\mathbf{w}|} a(\mathbf{v})f_{\mathbf{v}} = \textstyle\sum_1+\sum_2+\sum_3$$

Note that from (5.21) we get

$$|a(\mathbf{v})| = |\langle f, f_{\mathbf{v}}\rangle| \leq \|f\|_{BMO}\cdot\|f_{\mathbf{v}}\|_{\mathrm{Re}H_1} \leq c|\mathbf{v}|^{1/2}. \tag{5.22}$$

We will show that

$$|\int_{\mathbf{w}} \textstyle\sum_1| \leq C|\mathbf{w}| \tag{5.23}$$

$$\left(\frac{1}{|\mathbf{w}|}\int_{\mathbf{w}} |\textstyle\sum_2|^2\right)^{1/2} \leq C \tag{5.24}$$

$$|\textstyle\sum_3(x_0) - \sum_3(x)| \leq C \quad \text{for } x, x_0\in\mathbf{w} \tag{5.25}$$

$$\left(\int_{\mathbf{T}\setminus\mathbf{w}} |\textstyle\sum_1|^2\right)^{1/2} \leq C|\mathbf{w}|^{1/2} \tag{5.26}$$

Using (5.22) and Lemma 5.4 we get

$$\begin{aligned}
\left|\int_{\mathbf{w}} \Sigma_1\right| &\le \sum_{\mathbf{v}\subset\mathbf{w}} |a(\mathbf{v})| \left|\int_{\mathbf{w}} f_{\mathbf{v}}\right| \\
&= \sum_{\mathbf{v}\subset\mathbf{w}} |a(\mathbf{v})| \left|\int_{\mathbf{T}\setminus\mathbf{w}} f_{\mathbf{v}}\right| \\
&\le C \sum_{k: L(k)\le|\mathbf{w}|} 2^{-k/2} \sum_{\mathbf{v}\subset\mathbf{w}\ :\ |\mathbf{v}|=L(k)} \left|\int_{\mathbf{T}\setminus\mathbf{w}} f_{\mathbf{v}}\right| \\
&\le C \sum_{k\,:\,L(k)\le|\mathbf{w}|} 2^{-k/2} \sum_{s=1}^{\frac{|\mathbf{w}|}{|\mathbf{v}|}} \int_{s|\mathbf{v}|}^{\infty} 2^{-k} t^{-3/2} dt \\
&\le C \sum_{k\,:\,L(k)\le|\mathbf{w}|} 2^{-3k/2} \sum_{s=1}^{\frac{|\mathbf{w}|}{|\mathbf{v}|}} (s|\mathbf{v}|)^{-1/2} \\
&\le C \sum_{k\,:,\ L(k)\le|\mathbf{w}|} 2^{-k} \left(\frac{|\mathbf{w}|}{|\mathbf{v}|}\right)^{1/2} \\
&= C|\mathbf{w}|^{1/2} \sum_{k\,:\,L(k)\le|\mathbf{w}|} 2^{-k/2} \le C|\mathbf{w}|.
\end{aligned}$$

Thus (5.23) holds. We also have

(5.27)
$$\begin{aligned}
\left(\int_{\mathbf{w}} |\Sigma_2|^2\right)^{1/2} &\le \sum_{k\,:\,L(k)\le|\mathbf{w}|} \left(\int_{\mathbf{w}} \Big| \sum_{\mathbf{v}\ :\ \mathbf{v}\cap\mathbf{w}=\emptyset,\ |\mathbf{v}|=L(k)} a(\mathbf{v}) f_{\mathbf{v}}\Big|^2\right)^{1/2} \\
&\le \sum_{k\,:\,L(k)\le|\mathbf{w}|} 2^{-k/2} \left(\int_{\mathbf{w}} \Big(\sum_{\mathbf{v}\ :\ \mathbf{v}\cap\mathbf{w}=\emptyset,\ |\mathbf{v}|=L(k)} |f_{\mathbf{v}}|\Big)^2\right)^{1/2}.
\end{aligned}$$

Elementary computations using Lemma 5.4. show that for $x\in\mathbf{w}$ we have
(5.28)
$$\sum_{\mathbf{v}\,:\,\mathbf{v}\cap\mathbf{w}=\emptyset,\ |\mathbf{v}|=L(k)} |f_{\mathbf{v}}(x)| \le C \max_{a,b}\Big(\min(2^{k/2},(x-a)^{-1/2}),\ \min(2^{k/2},(b-x)^{-1/2})\Big)$$

where a and b denote endpoints of the interval $\mathbf{w}$, i.e. $\mathbf{w}=[a,b]$. From (5.27) and (5.28) we get

$$\left(\int_{\mathbf{w}} |\Sigma_2|^2\right)^{1/2} \le C \sum_{k\,:\,L(k)\le|\mathbf{w}|} 2^{-k/2} \left(\ln(1+2^k|\mathbf{w}|)\right)^{1/2} \le C|\mathbf{w}|^{1/2}$$

so (5.24) holds.

The proof of (5.25) is exactly the same as the proof of (5.20).

Using Lemma 5.4. we get

$$
\begin{aligned}
\left(\int_{\mathbf{T}\setminus\mathbf{w}}|\textstyle\sum_1|^2\right)^{1/2} &\leq \sum_{\mathbf{v}\subset\mathbf{w}}|a(\mathbf{v})|\left(\int_{\mathbf{T}\setminus\mathbf{w}}|f_{\mathbf{v}}|^2\right)^{1/2}\\
&\leq C\sum_{k\,:\,L(k)\leq|\mathbf{w}|}2^{-k/2}\sum_{\mathbf{v}\subset\mathbf{w}\,:\,|\mathbf{v}|=L(k)}\left(\int_{\mathbf{T}\setminus\mathbf{w}}|f_{\mathbf{v}}|^2\right)^{1/2}\\
&\leq C\sum_{k\,:\,L(k)\leq|\mathbf{w}|}2^{-k/2}\sum_{s=1}^{\frac{|\mathbf{w}|}{|\mathbf{v}|}}\left(\int_{s\cdot 2^{-k}}^{\infty}2^{-2k}x^{-3}dx\right)^{1/2}\\
&\leq C\sum_{k\,:\,L(k)\leq|\mathbf{w}|}2^{-3k/2}\sum_{s=1}^{\frac{|\mathbf{w}|}{|\mathbf{v}|}}(s\cdot 2^{-k})^{-1}\\
&\leq C\sum_{k\,:\,L(k)\leq|\mathbf{w}|}2^{-k/2}\ln(1+\frac{|\mathbf{w}|}{|\mathbf{v}|})\leq C|\mathbf{w}|^{1/2}.
\end{aligned}
$$

This shows that (5.26) holds. Now we are ready to show (5.10). From (5.23), (5.24) and (5.25) we get $|\sum_3(x_0)-\frac{1}{|\mathbf{w}|}\int_{\mathbf{w}}f|\leq C$ so from the definition of the BMO norm we get

$$\left(\frac{1}{|\mathbf{w}|}\int_{\mathbf{w}}|f-\textstyle\sum_3(x_0)|^2\right)^{1/2}\leq C. \tag{5.29}$$

From (5.24), (5.25) and (5.29) we see that

$$\int_{\mathbf{w}}|\textstyle\sum_1|^2\leq C|\mathbf{w}|. \tag{5.30}$$

Since $\sum_{\mathbf{v}\subset\mathbf{w}}|a(\mathbf{v})|^2=\int_{\mathbf{T}}|\sum_1|^2$ (5.10) follows immediately from (5.30) and (5.26).

Remark We will show (see Corollary 6.3) that in spaces A, H_1, C and L_1 we have to have a linear growth of v_n with the slope bigger then the smallest algebraically possible. So our constructions for those spaces are close to optimal. It seems to be an open problem what is the situation for unconditional bases in $L_p(\mathbf{T})$ for $1<p<\infty,\ p\neq 2$. To be more precise let us ask following questions: *Let* $\left(\Phi_k\right)_{k=0}^{\infty}$ *be an unconditional polynomial basis in* $L_p(\mathbf{T})$, $1<p<\infty$, $p\neq 2$.

(1) *Can we have* $\lim_n\frac{v_n}{n}=\frac{1}{2}$ *?*
(2) *Can we have* $v_{2n+1}=n$ *for infinitely many* n*'s ?*

It is well known that $\{e^{ikt}\}_{k=-\infty}^{\infty}$ is an unconditional basis in $L_p(\mathbf{T})$ only if $p=2$ [Wo2, Proposition II.D.9]. On the other hand it is a classical result of Marcinkiewicz (see e.g. [Wo2, Theorem II.E.9]) that span$\{e^{ikt}\}_{|k|\leq n}$ in $L_p(\mathbf{T})$ $1<p<\infty$ is uniformly isomorphic to ℓ_p^{2n+1}, so it has an unconditional basis. The problem is to glue those bases together. We suspect that the answer to our second question above is positive. It is impossible, however to have $v_{2n+1}=n$ for all n's. This was observed by A. Pełczyński. The proof is based on the following fact (Corollary 9 from [K-P])

Fact A. *If $(x_n)_{n=0}^{\infty}$ is a normalised, unconditional basis in L_p, $2 < p < \infty$, then* $\liminf \|x_n\|_2 = 0$.

Now suppose that $(\phi_n)_{n=0}^{\infty}$ is a normalised, unconditional basis in $L_p(\mathbf{T})$ for some p, $1 < p < \infty$, $p \neq 2$ such that $\operatorname{span}(\phi_n)_{n=0}^{2N} = \operatorname{span}(e^{ikt})_{|k|\le N}$ for all N's. Thus $\phi_n = \sum_{|j|\le \frac{n+1}{2}} b_j^n e^{ijt}$. The biorthogonal functionals $(\phi_n^*)_{n=0}^{\infty}$ form a normalised unconditional basis in $L_{p'}(\mathbf{T})$ and $\phi_n^* = \sum_{|j|\ge \frac{n}{2}} a_j^n e^{ijt}$. Thus we have

$$1 = \phi_n^*(\phi_n) = \sum_{\frac{n}{2}\le |j| \le \frac{n+1}{2}} a_j^n b_j^n. \tag{5.31}$$

If $1 < p < 2$ we majorise (5.31) by $\|\phi_n\|_p \cdot \|\phi_n^*\|_2$ so we infer from Fact A that $(\phi_n^*)_{n=0}^{\infty}$ is not an unconditional basis, so $(\phi_n)_{n=0}^{\infty}$ also is not an unconditional basis. If $2 < p < \infty$ we majorise (5.31) by $\|\phi_n\|_2 \cdot \|\phi_n^*\|_{p'}$ and once again Fact A gives that $(\phi_n)_{n=0}^{\infty}$ is not an unconditional basis.

Section 6. Estimates from below for v_n.

The main result of this Section is Theorem 6.1. which is an extension of a result of [Pri]. It gives immediately that if $(\Phi_k)_{k=0}^{\infty}$ is any basis in $C(\mathbf{T})$ or $L_1(\mathbf{T})$ consisting of trigonometric polynomials, then $v_n(\Phi_k) \ge (1+\varepsilon)n/2$ and for polynomial bases in $A(\mathbf{T})$ and $H_1(\mathbf{T})$ we have $v_n \ge (1+\varepsilon)n$. Our argument is a small modification and simplification of arguments in [Pri] and is given here mostly to make the paper selfcontained.

Theorem 6.1.. *Let $W_n = span\{e^{ikt}\}_{k=0}^{n}$ and let X be a subspace of W_n of codimension m. Assume that P is an operator from $A(\mathbf{T})$ (resp. from $H_1(\mathbf{T})$) into W_n such that $P|X = id$. Then $\|P\| \ge c \ln \frac{n}{m}$ for some absolute constant c.*

Proof. We will present the proof simultaneously for both spaces $A(\mathbf{T})$ and $H_1(\mathbf{T})$, so the symbol $\|.\|$ will denote one of the norms (fixed for the whole proof). In the proof we will be dealing with polynomials in two variables t and τ, so when we want to stress the variable with respect to which the norm is computed we will use the symbol $\|.\|_{(t)}$ or $\|.\|_{(\tau)}$. It suffices to consider integers m such that $m < 16^{-3}n$ and $\frac{n}{m} = k^3$ for some integer k. From known properties of the Dirichlet kernel (see [Kor]) we infer that there exists a polynomial $F(t) = \sum_{s=0}^{2k} \hat{F}(s)e^{ist}$ such that

$$\|F\| = 1 \quad \text{and} \quad \hat{F}(k) = 0 \tag{6.1}$$

$$\Big\| \sum_{s=0}^{k-1} \hat{F}(s)e^{ist} \Big\| \ge c \ln k. \tag{6.2}$$

We will write $F_1 = \sum_{s=0}^{k-1} \hat{F}(s)e^{ist}$ and $F_2 = \sum_{s=k+1}^{2k} \hat{F}(s)e^{ist}$. For $M = \frac{n}{k} - 3m = k^2m - 3m$ we put

$$f(t) = F(Mt), \quad f^1(t) = F_1(Mt), \quad f^2(t) = F_2(Mt). \tag{6.3}$$

Clearly $\|f\| = \|F\|$, $\|f^1\| = \|F_1\|$ and $\|f^2\| = \|F_2\|$. Let us fix functionals $\left(x_j^*\right)_{j=1}^{m}$ such that $X = \bigcap_{j=1}^{m} \ker x_j^* \cap W_n$. Let us consider the following system of linear equations

$$x_j^*\big(e^{iMqt} \cdot \sum_{l=0}^{2mk} \alpha_l e^{ilt}\big) = 0 \tag{6.4}$$

for $j = 1, 2, \dots, m$ and $q = 0, 1, \dots, k-1, k+1, \dots, 2k$ with $2mk+1$ unknown α_l. Since it is a system of $2mk$ equations there exists a non-zero solution which we identify with a polynomial $Q(t) = \sum_{l=0}^{2mk} \alpha_l e^{ilt}$ with $\|Q\| = 1$.

We will use the standart notation: for a function g on $\mathbf{T}$ we denote by g_τ the function $g_\tau(t) = g(t+\tau)$. Let us consider the function

$$[P(f_\tau \cdot Q)]_{-\tau} = [P(f_\tau^1 \cdot Q)]_{-\tau} + [P(f_\tau^2 \cdot Q)]_{-\tau} \tag{6.5}$$

Note that $f_\tau^1 \cdot Q$ is for every $\tau \in \mathbf{T}$ a polynomial in t of degree at most $2mk + M(k-1) < 2mk + n - 3mk = n - mk < n$. Thus we infer from (6.4) that for each $\tau \in \mathbf{T}$ the function $f_\tau^1 \cdot Q \in X$, so

$$P(f_\tau^1 \cdot Q) = f_\tau^1 \cdot Q. \tag{6.6}$$

Let us consider $[P(f_\tau^2 \cdot Q)]_{-\tau}$ as a function of τ (for a fixed $t \in \mathbf{T}$). One checks that it is a polynomial and the lowest non-zero coefficient has number $\geq (k+1)M - n$. From our restrictions on m we see that $(k+1)M - n \geq 4mk$. Since $[f_\tau^1 \cdot Q]_{-\tau} = f^1 \cdot Q_{-\tau}$ we see that (fot every $t \in \mathbf{T}$) it is a polynomial in τ of degree at most $2mk$. Using (6.5) and properties of the de la Vallée Poussin kernel in the variable τ we get

$$\begin{aligned} \Big\| \| [P(f_\tau \cdot Q)]_{-\tau} \|_{(t)} \Big\|_{(\tau)} &= \Big\| \| [P(f_\tau \cdot Q)]_{-\tau} \|_{(\tau)} \Big\|_{(t)} \\ &= \Big\| \| f^1 \cdot Q_{-\tau} + [P(f_\tau^2 \cdot Q)]_{-\tau} \|_{(\tau)} \Big\|_{(t)} \\ &\geq c \Big\| \| f^1 \cdot Q_{-\tau} \|_{(\tau)} \Big\|_{(t)} = c \|f^1\| \cdot \|Q\| \geq c \ln k \end{aligned} \tag{6.7}$$

On the other hand

$$\begin{aligned} \Big\| \| [P(f_\tau \cdot Q)]_{-\tau} \|_{(t)} \Big\|_{(\tau)} &= \Big\| \| [P(f_\tau \cdot Q)] \|_{(t)} \Big\|_{(\tau)} \\ &\leq \|P\| \Big\| \| f_\tau \cdot Q \|_{(t)} \Big\|_{(\tau)} = \|P\| \end{aligned}$$

so comparing with (6.7) we get $\|P\| \geq c \ln k \geq c \ln \frac{n}{m}$.

Corollary 6.2. *Suppose that P is an operator from the space $C(\mathbf{T})$ or from the space $L_1(\mathbf{T})$ into span$\{e^{ikt}\}_{|k| \leq n}$. Assume that $P|X = id$ for some $X \subset$ span$\{e^{ikt}\}_{|k| \leq n}$ and codim $X = m$. Then $\|P\| \geq c \ln \frac{2n+1}{m}$.*

Proof. Since the multiplication by e^{int} is an isometry of $C(\mathbf{T})$ and $L_1(\mathbf{T})$ we see that we can assume as well that $X \subset \operatorname{span}\{e^{ikt}\}_{k=0}^{2n}$. From Theorem 6.1 we see that $\|P|A(\mathbf{T})\|$ or $\|P|H_1(\mathbf{T})\|$ satisfies the desired estimate. This clearly implies the estimate for $\|P\|$.

Corollary 6.3.

(1) *Let $(\Psi_k)_{k=0}^\infty$ be a polynomial basis in the space $C(\mathbf{T})$ or $L_1(\mathbf{T})$. Then there exists an $\varepsilon > 0$ such that $v_n \geq (\frac{1}{2} + \varepsilon)n$.*
(2) *Let $(\Psi_k)_{k=0}^\infty$ be a polynomial basis in the space $A(\mathbf{T})$ or $H_1(\mathbf{T})$. Then there exists an $\varepsilon > 0$ such that $v_n \geq (1 + \varepsilon)n$.*

Proof. We will prove only the case (1). The proof of (2) is the same with Theorem 6.1 replacing Corollary 6.2. To prove (1) we apply Corollary 6.2 with X being $\operatorname{span}\{\Psi_k\}_{k=0}^n \subset \operatorname{span}\{e^{ikt}\}_{|k| \leq v_n}$ and P being the natural partial sum projection with respact to the basis (Ψ_k). We get $\operatorname{bc}(\Psi_k) \geq \|P\| \geq c \ln \frac{2v_n+1}{2v_n+1-n}$ so $v_n \geq \frac{C}{2(C-1)} n - 1$ for some $C > 1$.

Let us point out that results of Privalov [Pri] and also our Theorem 6.1 generalise the main results of [Sh1] and [Sh2] and solves problems asked in those papers. We have

Corollary 6.4. *Let X be either the space $C(\mathbf{T})$ or the space $L_1(\mathbf{T})$ and assume that $T : X \longrightarrow span(e^{ikt})_{|k|\leq n} \subset X$ is such an operator that $T^*|Y = id_Y$ for some subspace $Y \subset X^*$. Then*

$$\|T\| \geq c \ln \tfrac{2n+1}{2n+2-\dim Y}.$$

Proof. Clearly we have $\dim Y \leq 2n+1$. The assumption means that 1 is an eigenvalue of T^* with the multiplicity at most $\dim Y$. But elementary spectral theory, or linear algebra, shows that 1 is an eigenvalue of T^{**} with multiplicity at least $\dim Y$. Since $T^{**}(X^{**}) \subset \mathrm{span}(e^{ikt})_{|k|\leq n}$ and $T^{**}|X = T$ we see that there exists a subspace $Y_* \subset \mathrm{span}(e^{ikt})_{|k|\leq n}$ such that $\dim Y_* = \dim Y$ and $T|Y_* = id_{Y_*}$. Thus Theorem 6.1 gives the claim.

Theorem 2 of [Sh1] is Corollary 6.4 for $X = C(\mathbf{T})$ and $Y = \mathrm{span}(\nu_j)_{j=1}^m$ where ν_j's are measures with disjoint supports. The main result of [Sh2] is Corollary 6.4. for $X = L_1(\mathbf{T})$ and $Y = \mathrm{span}(f_j)$ where f_j's are disjointly supported functions in $L_\infty(\mathbf{T})$. The problems asked in those papers deal with removing the assumption of disjointness of support. Our Corollary 6.4 clearly showes that the only assumption needed is linear independence.

An analog of Corollary 6.4 for spaces H_1 and A clearly also holds with the same proof.

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