Non-similarity of Walsh and trigonometric systems

P. Wojtaszczyk*
Instytut Matematyki
Uniwersytet Warszawski
02–097 Warszawa, Banacha 2
e-mail wojtaszczyk@mimuw.edu.pl
and
Institute of Mathematics PAN
00-950 Warsaw, @Sniadeckich 8

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Abstract

We show that in L_p for $p \neq 2$ the constants of equivalence between finite initial segments of the Walsh and the trigonometric systems has power type growth. We also show that Riemann ideal norms connected with those systems have power type growth.

1 Introduction

There are numerous similarities between the trigonometric system and the Walsh system; both are bounded orthonormal bases in L_2 and both are characters of a compact abelian group. Many results are parallel (cf. [2]). However it is also known (cf. [3]) that as bases in $L_p[0,1]$ they are not equivalent unless $p = 2$. In this note we are interested in the "quantitative" estimates for this non-equivalence. The argument in [3] gives only a "logarithmic" difference. Unfortunately we are unable to provide exact estimates, but our results give a much stronger "power type" non-equivalence. We also provide a power type non-equivalence in the language of certain operator ideals. This complements some observations and supports some conjectures made in [1].

We consider the classical Walsh system in the following classical order called the Paley order. If an integer $n = 0, 1, 2, \dots$ can be written as $n = \sum_{k=0}^{\infty} p_k 2^k$ with $p_k = 0, 1$ (obviously this sum is finite) then we put the n-th Walsh function $w_n(t) = \prod_{k=0}^{\infty} r_k(t)^{p_k}$, where $r_0, r_1, \dots$ are Rademacher functions. The Walsh system in this order is called Walsh-Paley system cf. [2].

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We are interested in the comparison between norm convergence properties of Walsh series and trigonometric series. More precisely we are interested in best constants $K(p,N)$ and $C(p,N)$ in the following inequalities:

$$\| \sum_{n=0}^{2^N-1} a_n e^{in\alpha} \|_p \leq C(p,N) \| \sum_{n=0}^{2^N-1} a_n w_n \|_p \tag{1}$$

and

$$\| \sum_{n=0}^{2^N-1} a_n w_n \| \leq K(p,N) \| \sum_{n=0}^{2^N-1} a_n e^{in\alpha} \|_p. \tag{2}$$

The choice of dyadic sums is motivated by the the existence in this case of a simple representation for the kernels involved c.f.(8). Clearly one can easily pass from estimates (1) and (2) to analogous estimates for sums of arbitrary length. Let us consider two sequences of operators

$$T_N(f) = \sum_{n=0}^{2^N-1} \langle f, w_n \rangle e^{in\alpha} \tag{3}$$

$$S_N(f) = \sum_{n=0}^{2^N-1} \langle f, e^{in\alpha} \rangle w_n. \tag{4}$$

Then formally $T_N^* = S_N$. Let $\mathbf{W}_N = \text{span}\{w_n : n = 0,1,\ldots,2^N-1\}$ and $\mathbf{T}_N = \text{span}\{e^{in\alpha} : n = 0,1,\ldots,2^N-1\}$. The above spaces equipped with L_p norm will be denoted by $\mathbf{W}_N^p$ and $\mathbf{T}_N^p$ respectively. Then clearly $C(p,N) = \|T_N : \mathbf{W}_N^p \to \mathbf{T}_N^p\|$ and $K(p,N) = \|S_N : \mathbf{T}_N^p \to \mathbf{W}_N^p\|$. Since $\mathbf{W}_N^p$ is norm one complemented in L_p, $1 \leq p \leq \infty$ and $\mathbf{T}_N^p$ is complemented in L_p by the Riesz projection whose norms are for $1 < p < \infty$ bounded uniformly in N and in L_1 and L_∞ are bounded by N (c.f. [4] I p.67 and 266), we infer that

$$C(p,N) = \|T_N : L_p \to L_p\| \tag{5}$$

and

$$\frac{\|S_N : L_p \to L_p\|}{\beta(N,p)} \leq K(p,N) \leq \|S_N : L_p \to L_p\| \tag{6}$$

where $\beta(N,p) = C_p$ if $1 < p < \infty$ and $\beta(N,1) = \beta(N,\infty) = cN$. Also by duality we have

$$\|S_N : L_p \to L_p\| = \|T_N : L_{p'} \to L_{p'}\| = C(p',N)$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Clearly the operator T_N is an integral operator given by the kernel

$$T_N(f)(\alpha) = \int_0^1 \big(\sum_{n=0}^{2^N-1} w_n(t) e^{in\alpha} \big) f(t)\, dt.$$

We will denote

$$F_N(t,\alpha) = \sum_{n=0}^{2^N-1} w_n(t) e^{in\alpha}. \tag{7}$$

The following representation of $F_N(t,\alpha)$ will be of fundamental importance in our considerations:

$$\begin{aligned} F_N(t,\alpha) &= \sum_{n=0}^{2^{N-1}-1} w_n(t)e^{in\alpha} + r_{N-1}(t)e^{i2^{N-1}\alpha} \sum_{n=0}^{2^{N-1}-1} w_n(t)e^{in\alpha} \\ &= \big(1 + r_{N-1}(t)e^{i2^{N-1}\alpha}\big) F_{N-1}(t,\alpha) \end{aligned}$$

so by induction we get

$$F_N(t,\alpha) = \prod_{k=0}^{N-1} (1 + r_k(t)e^{i2^k\alpha}) \tag{8}$$

The following Lemma will be used several times in this paper

Lemma 1. *For any A_j and B_j we have*

$$\tfrac{1}{2\pi} \int_0^{2\pi} \prod_{j=1}^{N} (A_j + B_j \cos 2^j\alpha)\, d\alpha = \prod_{j=1}^{N} A_j$$

This Lemma is well known. It follows immediately if we write $\cos 2^j\alpha = \frac{1}{2}(e^{i2^j\alpha} + e^{-i2^j\alpha})$ and expand the product. We see that there is no cancellation in the expansion so it is a trigonometric polynomial with free term equal to $\prod_{j=1}^{N} A_j$. This gives the Lemma.

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2 Non-equivalence

All our results will follow from estimates for various mixed norms of the function $F_N(t,\alpha)$. Let us start with some Propositions which will be used in the proofs of Theorem 1 and Theorem 2.

Proposition 2. *There exists a number $b < \frac{1}{2}$ such that*

$$\big(1 + \frac{\sqrt{3}}{2}\big)^N \le \sup_\alpha \int_0^1 \big|F_N(t,\alpha)\big|\, dt \le 2 \cdot 2^{bN} \tag{9}$$

for $N = 1, 2, \ldots$.

Proof. Using independence of Rademacher functions we get

$$\begin{aligned} \sup_\alpha \int_0^1 \big|F_N(t,\alpha)\big|\, dt &= \sup_\alpha \prod_{k=0}^{N-1} \int_0^1 |1 + r_n(t)e^{i2^k\alpha}|\, dt \\ &= \sup_\alpha \prod_{k=0}^{N-1} \frac{|1 + e^{i2^k\alpha}| + |1 - e^{i2^k\alpha}|}{2}. \end{aligned}$$

Since

$$\big(|1+e^{i\alpha}|+|1-e^{i\alpha}|\big)^2 = 4+2|1-e^{2i\alpha}| = 4(1+|\sin\alpha|)$$

we infer that

$$\sup_\alpha \int_0^1 \big|F_N(t,\alpha)\big|\,dt = \sqrt{\sup_\alpha \prod_{k=1}^N (1+|\sin 2^k\alpha|)}. \tag{10}$$

Let us start with an upper estimate. Let us fix ξ, $0<\xi<\frac{1}{3}\pi$. If for some k we have

$$(n+\tfrac{1}{2})\pi-\xi \le 2^k t \le (n+\tfrac{1}{2})\pi+\xi \tag{11}$$

then

$$(2n+1)\pi-2\xi \le 2^{k+1}t \le (2n+1)\pi+2\xi. \tag{12}$$

Now let s denote the number of k's in the product (10) which satisfy (11). Clearly there exists at least $s-1$ numbers k in the product (10) for which (12) holds. In particular $s\le 1+N/2$. The factors where (11) holds are estimated by 2 and factors where (12) holds are at most $(1+\sin 2\xi)$. All the other do not satisfy (11) so are at most $1+|\sin(\frac{1}{2}\pi+\xi)|$. From this we infer that

$$\sup_\alpha \prod_{k=0}^{N-1}(1+|\sin 2^k\alpha|) \le 2^s\big[(1+\sin 2\xi)\big]^{s-1}(1+|\sin(\tfrac{1}{2}\pi+\xi)|)^{N-2s+1}.$$

Let us denote $A(\xi)=[2(1+\sin 2\xi)]^{1/2}$, $B(\xi)=1+|\sin(\frac{1}{2}\pi+\xi)|$ and $C(\xi)=\max[A(\xi),B(\xi)]$. With this notation we have

$$\sup_\alpha \prod_{k=0}^{N-1}(1+|\sin 2^k\alpha|) \le 2C(\xi)^N \tag{13}$$

for each $0<\xi<\frac{1}{3}\pi$. Looking at graphs of $A(\xi)$ and $B(\xi)$ we easily see that there exists ξ_0 such that $A(\xi_0)=B(\xi_0)$ and then $C(\xi_0)<2$. This gives $b<2$.

Now let us work out the lower estimate. We take $\gamma=\frac{1}{2}(\sum_{k=1}^\infty 4^{-k})=\frac{1}{6}$. For even $k=2s$ we have $2^k\gamma=4^s\gamma=$ integer $+\frac{1}{2}+\gamma$. For odd $k=2s+1$ we have $2^k\gamma=$ integer $+2\gamma$. Since $\frac{1}{2}+\gamma=\frac{2}{3}$ and $|\sin\frac{2}{3}\pi|=|\sin\frac{1}{3}\pi|$ we get

$$\prod_{k=1}^N(1+|\sin 2^k\pi\gamma|) = \big(1+\sin\pi/3\big)^N = (1+\frac{\sqrt{3}}{2})^N. \tag{14}$$

$\square$

Remark Estimating numerically we get $\xi_0\sim 0.4609285$ which yields $b\sim 1.89564$. Since $1+(\sqrt{3}/2)\sim 1.866025$ we see that the arguments given above are quite precise.

Proposition 3.

$$\frac{1}{2\pi}\int_0^1\int_0^{2\pi} F_N(t,\alpha)^4\,d\alpha\,dt = 6^N.$$

Proof. First note that from (8) we get

$$\begin{aligned}
F_N(t,\alpha)^4 &= \prod_{k=0}^{N-1} \left[(1+r_k(t)e^{i2^k\alpha})(1+r_k(t)e^{-i2^k\alpha})\right]^2 \\
&= \prod_{k=0}^{N-1} \left(2+2r_k(t)\cos 2^k\alpha\right)^2 \\
&= \prod_{k=0}^{N-1} \left(4+8r_k(t)\cos 2^k\alpha + 4\cos^2 2^k\alpha\right) \\
&= \prod_{k=0}^{N-1} \left(6+2\cos 2^{k+1}\alpha + 8r_k(t)\cos 2^k\alpha\right).
\end{aligned}$$

Integrating this over t and using the independence of Rademacher functions we get

$$\int_0^1 F_N(t,\alpha)^4\,dt = \prod_{k=1}^{N} \left(6+2\cos 2^k\alpha\right). \tag{15}$$

Lemma 1 completes the proof. □

Proposition 4. *For every $R > \frac{1}{4}(2+\sqrt{2}+\sqrt{6})$ there exists C_R such that*

$$\sup_t \frac{1}{2\pi}\int_0^{2\pi} |F_N(t,\alpha)|\,d\alpha \le C_R R^N$$

for $N = 1, 2, \ldots$.

Proof. Using (8) we have

$$\begin{aligned}
|F_N(t,\alpha)| &= \prod_{k=0}^{N-1} \left|1+r_k(t)e^{i2^k\alpha}\right| = \prod_{k=0}^{N-1} \sqrt{2+2r_k(t)\cos 2^k\alpha} \\
&= 2^{N/2}\prod_{k=0}^{N-1} \sqrt{1+r_k(t)\cos 2^k\alpha}.
\end{aligned} \tag{16}$$

Lemma 5. *For $|x| \le 1$ we have*

$$\sqrt{1+x} \le 1 + \tfrac{1}{2}x - \gamma x^2$$

for $\gamma = 1.5 - \sqrt{2}$.

Proof of the Lemma: We consider the function $f(x) = \sqrt{1+x} - 1 - \frac{1}{2}x + \gamma x^2$. Differentiating we see that f has a local minimum at $x = 0$ and that $f''(x) \ge 0$ on an interval $[-1,\xi]$ and $f''(x) \le 0$ on the interval $[\xi, 1]$ for certain $\xi > 0$. This implies that in order to prove the inequality it suffices to check that $f(-1) \ge 0$, $f(1) \ge 0$ and $f(0) \ge 0$ which one easily checks.

Using Lemma 5 and (16) we infer that

$$\begin{aligned}
&\sup_t \frac{1}{2\pi}\int_0^{2\pi} |F_N(t,\alpha)|\,d\alpha \\
\leq\ & 2^{N/2}\sup_t \frac{1}{2\pi}\int_0^{2\pi}\prod_{k=0}^{N-1}\left(1+\tfrac{1}{2}r_k(t)\cos 2^k\alpha-\gamma\cos^2 2^k\alpha\right)d\alpha \\
=\ & 2^{N/2}\sup_t \frac{1}{2\pi}\int_0^{2\pi}\prod_{k=0}^{N-1}\left[(1-\tfrac{\gamma}{2})+\tfrac{1}{2}r_k(t)\cos 2^k\alpha-\tfrac{\gamma}{2}\cos 2^{k+1}\alpha\right]d\alpha \qquad (17)\\
=\ & 2^{N/2}\sup_t \frac{1}{2\pi}\int_0^{2\pi}\prod_{k=0}^{N-1}\left[(1-\tfrac{\gamma}{2})+\tfrac{1}{4}r_k(t)(e^{i2^k\alpha}+e^{i2^k\alpha})-\tfrac{\gamma}{4}(e^{i2^{k+1}\alpha}+e^{i2^{k+1}\alpha})\right]d\alpha.
\end{aligned}$$

Now let us consider inductively products of the form

$$\varphi_N=\prod_{k=0}^{N-1}\left[(1-\tfrac{\gamma}{2})+\tfrac{1}{4}r_k(t)(e^{i2^k\alpha}+e^{i2^k\alpha})-\tfrac{\gamma}{4}(e^{i2^{k+1}\alpha}+e^{i2^{k+1}\alpha})\right].$$

Clearly each φ_N is a trigonometric polynomial of the form

$$\varphi_N(\alpha)=\sum_{|s|<2^{N+1}} a_s e^{is\alpha} \qquad (18)$$

Let us single out coefficients corresponding to $s=0$ and $s=\pm 2^N$ and write

$$\varphi_N(\alpha)=P_N+Q_N e^{i2^N\alpha}+Q_N e^{-i2^N\alpha}+\sum_{s\neq 0,s\neq\pm 2^N} a_s e^{is\alpha} \qquad (19)$$

Since

$$\varphi_{N+1}(\alpha)=\varphi_N(\alpha)\cdot\left[(1-\tfrac{\gamma}{2})+\tfrac{1}{4}r_N(t)(e^{i2^N\alpha}+e^{-i2^N\alpha})-\tfrac{\gamma}{4}(e^{i2^{N+1}\alpha}+e^{-i2^{N+1}\alpha})\right] \qquad (20)$$

from (20), (18) and (19) we infer that

$$P_{N+1}=P_N(1-\tfrac{\gamma}{2})+2r_N(t)\tfrac{1}{4}Q_N \qquad (21)$$

and

$$Q_{N+1}=\tfrac{1}{4}r_N(t)Q_N+\tfrac{\gamma}{4}P_N. \qquad (22)$$

To estimate P_N from above we define inductively two sequences p_n and q_n by the conditions:

$$p_0=1-\tfrac{\gamma}{2} \qquad q_0=\tfrac{1}{4} \qquad (23)$$

$$\begin{aligned}
p_{n+1} &= p_n(1-\tfrac{\gamma}{2})+\tfrac{1}{2}q_n \qquad (24)\\
q_{n+1} &= \tfrac{1}{4}q_n+\tfrac{\gamma}{4}p_n. \qquad (25)
\end{aligned}$$

We easily see that for each $t\in[0,1]$ we have $|P_N|\leq p_N$ and also $|Q_N|\leq q_N$.

Lemma 6. *Let sequences p_n and q_n be defined as in (23)–(25). Then for $n = 0, 1, 2, \ldots$*

$$p_n \leq C_a a^n \tag{26}$$

for $a > 1 - \frac{\gamma}{2} + \frac{1}{4}(\sqrt{5 - 2\sqrt{2}} - \sqrt{2})$

Since $\frac{1}{2\pi}\int_0^{2\pi} \varphi_N(\alpha)\, d\alpha = P_N$ (see Lemma 1), from (17) and Lemma 6 we infer that

$$\sup_t \frac{1}{2\pi}\int_0^{2\pi} |F_N(t, \alpha)|\, d\alpha \leq C 2^{N/2} a^N.$$

□

Proof of Lemma 6: Let us define $\Gamma_n = q_n/p_n$. Then from (23)–(25) we have $\Gamma_0 = [4(1 - \frac{\gamma}{2})]^{-1}$ and $\Gamma_{n+1} = f(\Gamma_n)$ where

$$f(x) = \frac{x + \gamma}{4 - 2\gamma + 2x}.$$

One easily checks that for $x \geq 0$ we have $0 < f'(x) < 1$. This implies that Γ_n converges to the fix point of f i.e. the solution of the equation $f(x) = x$, let call it g. A standard computation yields

$$g = \frac{\sqrt{5 - 2\sqrt{2}} - \sqrt{2}}{2}.$$

So for any $a > g$ we have $q_n \leq a p_n$ for big n's. So from (24) we infer that for big n's we have $p_{n+1} \leq [(1 - \frac{\gamma}{2}) + \frac{1}{2}a]p_n$ which gives the Proposition.

Proposition 7. *For $1 \leq p \leq 2$ we have $C(p, N) \geq (2^{-p/2} + \frac{1}{2})^{N/p}$.*

Proof. For each $t \in [0, 1]$ we have

$$\|\sum_{n=0}^{2^N-1} w_n(t) e^{in\cdot}\|_p^p \leq C(p, N)^p \|\sum_{n=0}^{2^N-1} w_n(t) w_n\|_p^p. \tag{27}$$

It is easy and well known (see e.g. [2] p.7) that $\|\sum_{n=0}^{2^N-1} w_n(t) w_n\|_p^p$ does not depend on t and equals $2^{-N} \cdot 2^{Np}$. Note that

$$\begin{aligned}
\int_0^1 \|\sum_{n=0}^{2^N-1} w_n(t) e^{in\cdot}\|_p^p\, dt &= \frac{1}{2\pi}\int_0^1\int_0^{2\pi} |F_N(t, \alpha)|^p\, dt\, d\alpha \\
&= \frac{1}{2\pi}\int_0^{2\pi} \prod_{k=0}^{N-1} \frac{|1 + e^{i2^k\alpha}|^p + |1 - e^{i2^k\alpha}|^p}{2}\, d\alpha \qquad (28)\\
&= 2^{-N} 2^{Np/2} \frac{1}{2\pi}\int_0^{2\pi} \prod_{k=0}^{N-1} \left(|1 + \cos 2^k\alpha|^{p/2} + |1 - \cos 2^k\alpha|^{p/2}\right) d\alpha.
\end{aligned}$$

To estimate it further we will need the following Lemma

Lemma 8. *For $0 \le s \le 1$ and $|x| \le 1$ the following inequality holds:*

$$|1+x|^s + |1-x|^s \ge 2 - (2-2^s)x^2. \tag{29}$$

We will postpone the proof of this Lemma for a while.
Using this Lemma we can continue (28)

$$\begin{aligned}
&\ge \; 2^{N(p/2-1)}\frac{1}{2\pi}\int_0^{2\pi}\prod_{k=0}^{N-1}\left[2-(2-2^{p/2})\cos^2 2^k\alpha\right]d\alpha \\
&= \; 2^{N(p/2-1)}\frac{1}{2\pi}\int_0^{2\pi}\prod_{k=0}^{N-1}\left[2-\tfrac{1}{2}(2-2^{p/2})-\tfrac{1}{2}(2-2^{p/2})\cos 2^{k+1}\alpha\right]d\alpha.
\end{aligned}$$

From Lemma 1 we get that this integral equals $[2-\frac{1}{2}(2-2^{p/2})]^N$. This gives that

$$\int_0^1 \|\sum_{n=0}^{2^N-1} w_n(t)e^{in\cdot}\|_p^p\,dt \ge 2^{N(p/2-1)}[2-\tfrac{1}{2}(2-2^{p/2})]^N = [\tfrac{1}{2}2^{p/2}+\tfrac{1}{4}2^p]^N. \tag{30}$$

Thus we get

$$C(p,N)^p \ge 2^N 2^{-Np}[\tfrac{1}{2}2^{p/2}+\tfrac{1}{4}2^p]^N = (\tfrac{1}{2}+2^{-p/2})^N$$

which gives the claim. □

Proof of Lemma 8: Consider the function

$$f(x) = |1+x|^s + |1-x|^s - 2 + (2-2^s)x^2.$$

Clearly f is even. Differentiating twice we see that $f(x)$ is convex on certain interval $(-\mu,\mu)$ with $\mu > 0$ and concave for $|x| \ge \mu$. From this we infer that for (29) to hold it suffices to have $f(0) \ge 0$ and $f(1) \ge 0$ which is easily seen to be true.

Before we start with our main Theorem we will need one more Lemma.

Lemma 9. *Let $\psi(\alpha) = \sum_{k=0}^{2^N-1} a_k e^{ik\alpha}$. Then for $2 \le p \le \infty$ we have*

$$\|\psi\|_\infty \le 3\cdot 2^{N/p}\|\psi\|_p$$

Proof. Let V_N be the shifted de la Valleé-Poussin kernel. It is a trigonometric polynomial of the form $V_N(\alpha) = \sum_{k=-2^{N-1}}^{3\cdot 2^{N-1}} c_k e^{ik\alpha}$ such that $0 \le c_k \le 1$ and $c_k = 1$ for $k = 0,1,\ldots,2^N-1$ and $\frac{1}{2\pi}\int_0^{2\pi}|V_N(\alpha)|\,d\alpha \le 3$. We consider an operator $Tf = f * V_N$. Then $\|T : L_\infty \to L_\infty\| \le 3$ and

$$\|T : L_2 \to L_\infty\| \le \left(\sum |c_k|^2\right)^{1/2} \le \sqrt{2}\cdot 2^{N/2}.$$

From Riesz-Thorin interpolation Theorem we infer that for $2 \le p \le \infty$ we have $\|T : L_p \to L_\infty\| \le 3\cdot 2^{N/p}$. Since for ψ as in the Lemma we have $T\psi = \psi$ we get the claim. □

Theorem 1. *Quantities $C(p,N)$ satisfy the following estimates:*

1. *for* $1 \le p \le 2$ *we have*

$$(2^{-p/2} + \tfrac{1}{2})^{N/p} \le C(p,N) \le 2^{\frac{2}{p}-1} b^{(\frac{2}{p}-1)N} \tag{31}$$

where $b < 2$ *is the constant from Proposition 2.*

2. *for* $2 \le p \le \infty$ *we have*

$$c 2^{\xi(p)N} \le C(p,N) \le C_R R^{(1-\frac{2}{p})N} \tag{32}$$

where R *is any number* $> \frac{1}{4}(2+\sqrt{2}+\sqrt{6})$ *and* $\xi(p) > 1$ *for all* $p > 2$. *The function* $\xi(p)$ *satisfies*

$$\xi(p) \ge \begin{cases} \log_2\left(\frac{2}{\sqrt{3}}\left(\frac{3}{4}\right)^{1/p}\right) & \textit{for } 2 < p \le 4, \\ 0.44 - \frac{1}{p} & \textit{for } 4 \le p < \infty. \end{cases} \tag{33}$$

Proof. The left hand side in (31) is Proposition 7. Clearly $C(2,N) = \|T_N : L_2 \to L_2\| = 1$. From (5) we infer that

$$C(1,N) = \|S_N : L_\infty \to L_\infty\| = \sup_\alpha \int_0^1 |F_N(t,\alpha)|\, dt$$

so from Proposition 2 we infer that $C(1,N) \le b^N \cdot 2^N$. Using the Riesz-Thorin interpolation theorem (c.f. [4] II.p.93) we get

$$C(p,N) \le 2^{\frac{2}{p}-1} \cdot b^{(\frac{2}{p}-1)N}.$$

The upper estimate in (32) follows directly from the Riesz-Thorin theorem and Proposition 4. To prove the lower estimate in (31) we use Lemma 9 to obtain that for any bounded f we have

$$\|T_N(f)\|_\infty \le 3{\cdot}2^{N/p}\|T_N(f)\|_p \le 3{\cdot}2^{N/p}\|T_N : L_p \to L_p\|\|f\|_p \le 3{\cdot}2^{N/p}\|T_N : L_p \to L_p\|\|f\|_\infty$$

so

$$\|T_N : L_p \to L_p\| \ge \tfrac{1}{3} 2^{-N/p} \|T_N : L_\infty \to L_\infty\|.$$

So from Proposition 2 we get

$$C(p,N) \ge \tfrac{1}{3} 2^{-N/p} 2^{aN} = \tfrac{1}{3} 2^{N(a-\frac{1}{p})}. \tag{34}$$

This is a sensible estimate for $p > 1/\alpha$ but not for all $p > 2$. From Remark after the proof of Proposition 2 we see that $1/a < 4$ so we will use (34) for $p \ge 4$. For $2 < p \le 4$ we take $a_n = e^{ins}$ in (1) and get

$$\frac{1}{2\pi} \int_0^1 \Big| \sum_{n=0}^{2^N-1} e^{ins} e^{in\alpha} \Big|^p d\alpha \le C(p,N)^p \int_0^1 \Big| \sum_{n=0}^{2^N-1} e^{ins} w_n(t) \Big|^p dt. \tag{35}$$

Classical estimates for the Dirichlet kernel (cf. [4] p.67) give $\frac{1}{2\pi}\int_0^1 |\sum_{n=0}^{2^N-1} e^{in(s+\alpha)}|^p \, d\alpha \geq c2^{N(p-1)}$. Using this and integrating (35) over s and using Hölder's inequality we get

$$
\begin{aligned}
c2^{N(p-1)} &\leq C(p,N)^p \frac{1}{2\pi}\int_0^1\int_0^1 |F_N(t,\alpha)|^p \, dt d\alpha \\
&\leq C(p,N)^p \Big(\frac{1}{2\pi}\int_0^1\int_0^1 |F_N(t,\alpha)|^4 \, dt d\alpha\Big)^{\frac{p}{2}-1} \Big(\frac{1}{2\pi}\int_0^1\int_0^1 |F_N(t,\alpha)|^2 \, dt d\alpha\Big)^{2-\frac{p}{2}}.
\end{aligned}
$$

The second integral above clearly equals 2^N so from Proposition 3 we get

$$C(p,N)^p \geq c2^{N(p-1)} 6^{N(1-\frac{p}{2})} 2^{N(\frac{p}{2}-2)}$$

which yields

$$C(p,N) \geq c\Big(\frac{2}{\sqrt{3}}\big(\frac{3}{4}\big)^{1/p}\Big)^N.$$

$\square$

Remark Note that one can select from our arguments a very simple proof of the main result of [3] that the trigonometric system and the Walsh system are equivalent in L_p only when $p = 2$. By duality it suffices to consider only $p > 2$. Then Proposition 3 and the argument starting from (35) give the non-equivalence for $2 < p \leq 4$. Riesz-Thorin interpolation Theorem shows that this implies the non-equivalence also for $p > 4$.

3 Riemann norms

In this section we want to express the "power type" difference between Walsh and trigonometric systems in the language of Riemann ideal norms (cf. [1]).

For two orthonormal systems $\mathcal{A}_n = \{\varphi_i\}_{i=1}^n$ and $\mathcal{B}_n = \{\psi_i\}_{i=1}^n$ and a Banach space X we define the constant $\rho(X|\mathcal{A}_n, \mathcal{B}_n)$ as the least constant c such that the inequality

$$\Big(\int \|\sum_{i=1}^n x_i\varphi_i(t)\|_X^2 \, dt\Big)^{1/2} \leq c\Big(\int \|\sum_{i=1}^n x_i\psi_i(t)\|_X^2 \, dt\Big)^{1/2} \tag{36}$$

holds for all sequences $(x_i)_{i=1}^n \subset X$.

This concept, its variants and ramifications are discussed in [1] 3.3. We will be interested only in $n = 2^N$ and $\mathcal{A}_n$ and $\mathcal{B}_n$ being either the beginning segment of the Walsh system or the beginning segment of the trigonometric system. We want to contribute to the problem of estimating such quantities from below for $X = L_1$ and $X = L_\infty$. This question is discussed in section 6.5.4 of [1].

Let us introduce the following notation: $\mathcal{W}_N = \{w_n\}_{n=0}^{2^N-1}$ and $\mathcal{T}_N = \{e^{in\alpha}\}_{n=0}^{2^N-1}$. We have the following

Theorem 2. *The following inequalities hold:*

$$\rho(L_\infty|\mathcal{W}_N, \mathcal{T}_N) \geq C\Big(\sqrt{\frac{6\sqrt{3}}{\sqrt{86}}}\Big)^N \sim C(1.058599)^N \tag{37}$$

$$\rho(L_\infty|\mathcal{T}_N,\mathcal{W}_N) \geq C\Big(\frac{2}{\sqrt{2+\sqrt{2}}}\Big)^N \sim C(1.082392)^N \tag{38}$$

$$\rho(L_1|\mathcal{W}_N,\mathcal{T}_N) \geq \frac{c}{N}\big(\sqrt{1+\frac{1}{\sqrt{2}}}\big)^N \sim \frac{c}{N}(1.306563)^N \tag{39}$$

$$\rho(L_1|\mathcal{T}_N,\mathcal{W}_N) \geq \big(\sqrt{1+\frac{1}{\sqrt{2}}}\big)^N \sim C(1.306563)^N \tag{40}$$

Proof. First we will reduce the estimates (37)–(40) to estimates of some mixed norms of $F_N(t,\alpha)$. Let us consider (37) and let us take $x_j = e^{ij\alpha}$. Then from (36) we infer that

$$\begin{aligned}\rho(L_\infty|\mathcal{W}_N,\mathcal{T}_N) \;&\geq\; \Big(\frac{1}{2\pi}\int_0^1 \big(\sup_\alpha |\sum_{j=0}^{2^N-1} e^{ij\alpha}e^{ij\beta}|\big)^2\, d\beta\Big)^{1/2} \\ &\times\; \Big(\int_0^1 \sup_\alpha |\sum_{j=0}^{2^N-1} e^{ij\alpha}w_n(t)|^2\, dt\Big)^{-1/2}.\end{aligned}$$

Since the sup in the first integral does not depend on β and equals 2^N we get

$$\rho(L_\infty|\mathcal{W}_N,\mathcal{T}_N) \geq 2^N\big(\int_0^1 \sup_\alpha |F_N(t,\alpha)|^2\, dt\big)^{-1/2}. \tag{41}$$

Analogously, taking $x_j = w_j$ we get

$$\rho(L_\infty|\mathcal{T}_N,\mathcal{W}_N) \geq 2^N\big(\frac{1}{2\pi}\int_0^1 \sup_t |F_N(t,\alpha)|^2\, d\alpha\big)^{-1/2}. \tag{42}$$

To prove (39) we take $x_j = e^{ij\cdot}$ and obtain from (36)

$$\begin{aligned}\rho(L_1|\mathcal{W}_N,\mathcal{T}_N) \;&\geq\; \Big(\int_0^1 \|\sum_{j=0}^{2^N-1} e^{ij\cdot}w_j(t)\|_1^2\, dt\Big)^{1/2} \\ &\times\; \Big(\frac{1}{2\pi}\int_0^1 \|\sum_{j=0}^{2^N-1} e^{ij\cdot}e^{ij\alpha}\|_1^2\, d\alpha\Big)^{-1/2}\end{aligned}$$

Since $\sum_{j=0}^{2^N-1} e^{ij\cdot}e^{ij\alpha}$ is a translate of the Dirichlet kernel we see that its L_1 norm does not depend on α and is $\geq CN$ by the classical estimates of the Dirichlet kernel (see [4] p.67), so we get

$$\rho(L_1|\mathcal{W}_N,\mathcal{T}_N) \geq \frac{c}{N}\Big(\int_0^1 \big(\frac{1}{2\pi}\int_0^1 |F_N(t,\alpha)|\, d\alpha\big)^2 dt\Big)^{1/2}. \tag{43}$$

Analogously to prove (40) we take $x_j = w_j$ and obtain from (36)

$$\begin{aligned}\rho(L_1|\mathcal{T}_N,\mathcal{W}_N) \;&\geq\; \big(\frac{1}{2\pi}\int_0^1 \big(\int_0^1 |F_N(t,\alpha)|\, dt\big)^2\, d\alpha\big)^{1/2} \\ &\times\; \big(\int_0^1 \big(\int_0^1 |\sum_{n=0}^{2^N-1} w_n(s)w_n(t)|\, ds\big)^2 dt\big)^{-1/2}.\end{aligned}$$

Since $\sum_{n=0}^{2^N-1} w_n(s)w_n(t)$ is a (dyadic) translation of the Dirichlet kernel of the Walsh system it is well known and easy (see [2] Paley's Lemma p.7) that it's L_1 norm equals 1 so we get

$$\rho(L_1|\mathcal{T}_N,\mathcal{W}_N) \geq \Big(\frac{1}{2\pi}\int_0^1 \Big(\int_0^1 |F_N(t,\alpha)|\,dt\Big)^2 d\alpha\Big)^{1/2}. \tag{44}$$

From estimates (41)–(44) we see that Theorem 2 immediately follows from the following Proposition 10. ☐

Proposition 10. *The following inequalities hold*

$$\begin{aligned}
\int_0^1 \sup_\alpha |F_N(t,\alpha)|^2\,dt &\leq 2^N\cdot\Big(\tfrac{1}{3}\sqrt{\tfrac{86}{3}}\Big)^N && (45)\\
\frac{1}{2\pi}\int_0^{2\pi} \sup_t |F_N(t,\alpha)|^2\,d\alpha &\leq 2^N(1+\tfrac{1}{\sqrt{2}})^N && (46)\\
\frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^1 |F_N(t,\alpha)|\,dt\Big)^2 d\alpha &\geq (\tfrac{1}{2}+\tfrac{1}{\sqrt{2}})^N && (47)\\
\int_0^1\Big(\frac{1}{2\pi}\int_0^{2\pi} |F_N(t,\alpha)|\,d\alpha\Big)^2 dt &\geq (\tfrac{1}{2}+\tfrac{1}{\sqrt{2}})^N && (48)
\end{aligned}$$

Proof. Let us start with the proof of (45). First observe that

$$\begin{aligned}
|F_N(t,\alpha)|^2 &= \prod_{n=0}^{N-1} |1+r_n(t)e^{i2^n\alpha}|^2\\
&= \prod_{n=0}^{N-1}(2+2r_n(t)\cos 2^{n+1}\alpha) && (49)\\
&= 2^N\prod_{n=1}^{N}(1+r_{n+1}(t)\cos 2^n\alpha).
\end{aligned}$$

Let $\varphi_s(t,\alpha) = (1+r_{s+1}(t)\cos 2^s\alpha)(1+r_{s+2}(t)\cos 2^{s+1}\alpha)$. With this notation we have

$$\sup_\alpha |F_N(t,\alpha)|^2 \leq 2^N \prod_{s=0}^{[N/2]+1}\sup_\alpha \varphi_{2s+1}(t,\alpha). \tag{50}$$

Clearly functions $\sup_\alpha \varphi_{2s+1}(t,\alpha)$ are stochastically independent as functions of t for $s=0,1,\dots,[N/2]$ and have the same distribution in t as the function $\psi(t) = \sup_\alpha(1+r_1(t)\cos\alpha)(1+r_2(t)\cos 2\alpha)$. Thus we have to find maxima in α of four functions corresponding to all possible choices of $r_i(t)=\pm 1$. After some tedious calculations we conclude that two of them have maximum equal 4 and two have the maximum $B = 2\frac{10}{27}$. This and (50) give

$$\begin{aligned}
\int_0^1 \sup_\alpha |F_N(t,\alpha)|^2\,dt &\leq 2^N\prod_{s=0}^{[N/2]+1}\int_0^1 \sup_\alpha \varphi_{2s+1}(t,\alpha)\,dt\\
&= 2^N\big(2+\tfrac{1}{2}B\big)^{N/2} = 2^N\big(\tfrac{1}{3}\sqrt{\tfrac{86}{3}}\big)^N. && (51)
\end{aligned}$$

Now let us prove (46). From (49) we get

$$\sup_t |F_N(t,\alpha)|^2 = 2^N \prod_{n=1}^{N} (1 + |\cos 2^n \alpha|). \tag{52}$$

Expanding the product we get

$$\frac{1}{2\pi}\int_0^{2\pi} \prod_{n=1}^{N} (1 + |\cos 2^n\alpha|)\, d\alpha = 1 + \sum_{1\le k_1<k_2<\cdots<k_s\le N} \frac{1}{2\pi}\int_0^{2\pi} \prod_{j=1}^{s} |\cos 2^{k_j}\alpha|\, d\alpha. \tag{53}$$

Each integral in (53) we estimate using the Cauchy inequality and Lemma 1 as follows

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \prod_{j=1}^{s} |\cos 2^{k_j}\alpha|\, d\alpha &\le \Big(\frac{1}{2\pi}\int_0^{2\pi} \prod_{j=1}^{s} \cos^2 2^{k_j}\alpha\, d\alpha\Big)^{1/2} \\ &= \Big(\frac{1}{2\pi}\int_0^{2\pi} \prod_{j=1}^{s} \tfrac{1}{2}(1+\cos 2\cdot 2^{k_j}\alpha)\, d\alpha\Big)^{1/2} = 2^{-s/2}. \end{aligned} \tag{54}$$

Thus substituting (54) into (53) we get

$$\frac{1}{2\pi}\int_0^{2\pi} \prod_{n=1}^{N} (1 + |\cos 2^k\alpha|)\, d\alpha \le \big(1 + \sum_{1\le k_1<k_2<\cdots<k_s\le N} 2^{-s/2}\big) = (1+\tfrac{1}{\sqrt{2}})^N$$

so from (52) we get (46).

Both (47) and (48) are estimated from below by $\big(\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 |F_N(t,\alpha)|\, dt\, d\alpha\big)^2$ so (30) gives the claim. □

REMARK If we split the product in (50) into triples, not in pairs like we did, we will have to analyze eight functions

$$(1\pm\cos\alpha)(1\pm\cos 2\alpha)(1\pm\cos 4\alpha).$$

It can be checked (I used a computer) that two of them will have the maximum equal to 8, two will have the maximum ≤ 4.33, two ≤ 4.186 and two ≤ 3.78. This will lead to the estimate

$$\int_0^1 \sup_\alpha |F_N(t,\alpha)|^2\, dt \le 2^N (5.074)^{N/3} \sim 2^N (1.7718371)^N \tag{55}$$

which is slightly better then (51) because numerically (51) gives $\le 2^N(1.784709)^N$. Obviously also this estimate is not optimal.

3.1 Concluding remarks.

1. Our estimates are quite sloppy and do not give the asymptotically corrects values. It would be interesting to have precise estimates.

2. The Paley order of the Walsh functions was crucial in our considerations. This order may be considered as natural but there are other as natural. We conjecture, however that in reality our results are essentially independent of order. The following question seems to be natural and very interesting: *does there exists a permutation σ of natural numbers such that $\{e^{in\alpha}\}_{n=0}^{\infty}$ is equivalent in L_p for some $p \neq 2$ to $\{w_{\sigma(n)}\}_{n=0}^{\infty}$?*

3. Both systems we consider in this paper are sets of characters of an Abelian compact group. It is an interesting question whether character sets of non-isomorphic topological groups can be equivalent systems in L_p. In particular one may investigate the equivalence of Vilenkin systems (cf. [2]) with trigonometric system or with the Walsh system or between themselves.

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