UNIQUENESS OF UNCONDITIONAL BASES IN QUASI-BANACH SPACES WITH APPLICATIONS TO HARDY SPACES, II

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Abstract. We prove that a wide class of quasi-Banach spaces has a unique up to a permutation unconditional basis. This applies in particular to Hardy spaces H_p for $p < 1$. We also investigate the structure of complemented subspaces of $H_p(D)$. The proofs use in essential way matching theory.

Instroduction

In this paper we study the problem of uniqueness up to permutations of unconditional bases in quasi-Banach spaces. Suppose that X is a quasi-Banach space (in particular a Banach space) with a quasi-norm $\|.\|$ and an unconditional basis $(x_n)_{n\in N}$. We always assume that the basis is normalised i.e. $\|x_n\| = 1$ for all $n \in N$. Let $(y_m)_{m\in M}$ be an unconditional basis in another quasi-Banach space Y. We say that those bases are equivalent (and write it as $(x_n)_{n\in N} \sim (y_m)_{m\in M}$) if there exists a 1–1 and onto map $\Phi : N \longrightarrow M$ such that the map $x_n \mapsto y_{\Phi(n)}$ extends by linearity to an isomorphism between X and Y. The terms "permutatively equivalent" or "equivalent up to a permutation" are also used in the literature. We say that a quasi-Banach space X has a unique unconditional basis if it has an unconditional basis and all (normalised) unconditional bases in X are equivalent.

In the context of Banach spaces it is quite exceptional for a space to have a unique unconditional basis. What is known can be found in [BCLT] and the references quoted there and also in the forthcoming paper [CK]. The general introduction to the problem can be found in [M].

In the context of quasi-Banach spaces which are not Banach spaces the uniqueness of unconditional basis seems to be a norm rather then exception. It was shown in [K] that a wide class of non-locally convex Orlicz sequences spaces, including ℓ_p-spaces for $0 < p < 1$ have a unique unconditional basis. The case of Lorentz sequence spaces was studied by Nawrocki and Ortyński [NO]. The uniqueness of unconditional basis in non- locally convex Lorentz sequence spaces was established in [KLW] Theorem 2.6. Actually in [KLW] it was shown that (under some assumptions to be explained later), given two unconditional bases $(x_n)_{n\in N}$ and $(y_m)_{m\in M}$ in a quasi-Banach space X we can partition N into a finite number of disjoint sets

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$N_1, N_2, \ldots, N_k$ in such a way that each basic sequence $(x_n)_{n\in N_s}$ is equivalent to a subbasis of the basis $(y_m)_{m\in M}$, and naturally the same holds with roles of the bases reversed. This allowed to treat the above mentioned case of Lorentz sequence spaces and also to obtain non-isomorphism of Hardy spaces H_p for $p < 1$ in different number of variables. We were unable to decide if H_p has a unique unconditional basis. This problem was a driving force of the present investigation and we solve it in the affirmative. Our main technical result is the following

Theorem B.9. *Suppose X is a natural, quasi- Banach space with strictly absolute unconditional basis $(e_m, e_m^*)_{m\in M}$ and suppose that $(u_n, u_n^*)_{n\in N}$ is some other unconditional basis in X. Then we can partition each index set into four disjoint subsets, N_1, N_2, N_3, N_4 and M_1, M_2, M_3, M_4 in such a way that*

$$(u_n)_{n\in N_1} \sim (e_n)_{m\in M_1}$$

$$(u_n)_{n\in N_4} \sim (e_m)_{m\in M_4} \sim (e_m)_{m\in M_3}$$

$$(u_n)_{n\in N_2} \sim (u_n)_{n\in N_3} \sim (e_m)_{m\in M_2}$$

As a corollary we obtain

Theorem B.12. *Let X be a natural quasi-Banach space with strongly absolute unconditional basis $(e_m)_{m\in M}$. Assume also that X is isomorphic to some of its cartesian powers X^s, $s = 2, 3, \ldots$. Then all normalised, unconditional bases in X are permutatively equivalent.*

Generally we follow the ideas of [KLW]. The essential new ingredient is combinatorial. We use the classical Hall-König Lemma (Marriage Theorem) and some of its refinements. This is explained in Section A. The second Section B deals with general quasi-Banach spaces. Here we prove our main results. The final Section C is devoted to Hardy spaces. We give a simple proof of uniqueness of unconditional basis in H_p and investigate the structure of complemented subspaces of H_p for $p < 1$. Our results in this section are in the nature of examples, but in our opinion they show that the structure of complemented subspaces of H_p is extremally complicated.

Our notation is rather standard. In combinatorics we follow the expository article [B] and in the theory of quasi-Banach spaces we follow [KPR]. Let me only point out that $|.|$ may denote (depending on the context) one of the following: absolute value of the number, cardinality of a finite set or the Lebesque measure of a subset of interval $[0, 1]$.

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Section A : Matching in bipartite graphs

A bipartite graph G is a triple (N, Δ, M) with N and M disjoint sets and Δ a set of unordered pairs, one element from N and one element from M. The elements of the set $N \cup M$ are called **vertices** of the graph G and the set $N \cup M$ of all vertices is sometimes denoted $V(G)$. The elements of Δ are called **edges** and Δ is sometimes denoted by $E(G)$ — the edge set of G. We say that two vertices $a, b \in V(G)$ are

joined if the pair $(a, b) \in \Delta$. We call a subset $A \subset \Delta$ **one– sided** if it is contained either in N or in M. For a one–sided set of vertices A we denote

$$\phi(A) = \{v \in V(G) \ : \ (a, v) \in \Delta \ \text{ for some } a \in A\}.$$

It is clear that $\phi(A)$ is also one–sided and belongs to the different set (N or M) then A. If A is a one element set $A = \{a\}$ then we will use the notation $\phi(a)$ to denote $\phi(\{a\})$. With this convention

$$\phi(A) = \bigcup_{a \in A} \phi(a).$$

The degree of a vertex $a \in V(G)$ is $|\phi(A)|$, the cardinality of $\phi(A)$, i.e. the number of vertices joined with a.

Let $A \subset V(G)$ be a one–sided set. A matching of A is a $1-1$ map $\psi : A \longrightarrow V(G)$ such that $(a, \psi(a)) \in \Delta$ for every $a \in A$. Clearly $\psi(A)$ is also a one–sided set. A one–sided set which has a matching is called **matchable**.

A necessary and sufficient condition for the existence of a matching of A is given by the following classical result, called usually the Hall-König Lemma.

Theorem A.1. *Suppose G is a bipartite graph such that the degree of each vertex of G is finite. For a one– sided set $A \subset V(G)$ there exists a matching of A if and only if for every finite subset $B \subset A$ we have*

$$|\phi(B)| \geq |B|.$$

It follows easily from Theorem A.1. that the increasing union of matchable sets is matchable, so every matchable subset $A \subset V(G)$ is contained in a maximal matchable set A_1.

The following concept will play a very important role in our considerations:

Definition A.2. Let $G = (N, \Delta, M)$ be a bipartite graph such that every vertex has finite degree and let c be a positive number. We say that N satisfies c-**Hall-König** condition if for every finite subset $A \subset N$ we have

$$|\phi(A)| \geq c|A|.$$

In this paper we are only interested in the case $0 < c < 1$.

The following consequences of Theorem A.1. will be crucial for further arguments.

Corollary A.3. *Suppose $G = (N, \Delta, M)$ is a bipartite graph in which every vertex has finite degree. Assume also that N satisfies c-Hall-König condition with $c = 1/K$ for some integer $K = 2, 3, \dots$. Then*

a) there exists a decomposition $N = N_1 \cup \dots \cup N_K$ of N into K disjoint, matchable subsets,

b) if we write $N = \bigcup_{\alpha \in A} N_\alpha$ with N_α disjoint sets with $|N_\alpha| = K$ for every $\alpha \in A$, then there exists a matchable subset $\bar{N} \subset N$ such that $N_\alpha \cap \bar{N} \neq \emptyset$ for every $\alpha \in A$.

Proof. a) Let us replace the graph G by the graph $G_1 = (N, \Delta_1, M_1)$ where

$$M_1 = M \times \{1, 2, \dots, K\}$$

and

$$\Delta_1 = \{(n, (m, s)) \in N \times M_1 \ : \ (n, m) \in \Delta\}.$$

It is clear that the set N in G_1 satisfies 1-Hall- König condition, so there exists a matching $\psi : N \longrightarrow M_1$. We define $N_s = \psi^{-1}(M \times \{s\})$ for $s = 1, 2, \dots, K$. The sets N_s are clearly matchable in G, the matching of N_s is given by $\psi_s(n) = m$ where $\psi(n) = (m, s)$.

b) Let us replace the graph G by the graph $G_1 = (A, \Delta_1, M)$ (A is the set of indices α) where $(\alpha, m) \in \Delta_1$ if $(n, m) \in \Delta$ for some $n \in N_\alpha$

The graph G_1 satisfies 1-Hall König condition. To see it let us take any finite subset $B \subset A$. Then

$$\phi(B) = \bigcup_{\alpha \in B} \phi(\alpha) = \bigcup_{\alpha \in B} \bigcup_{n \in N_\alpha} \phi(n)$$

so

$$|\phi(B)| \geq K^{-1} |\bigcup_{\alpha \in B} \bigcup_{n \in N_\alpha} \phi(n)| = K^{-1} |B| K = |B|.$$

So there exists a matching $\psi : A \longrightarrow M$. This means that $(\alpha, \psi(\alpha)) \in \Delta_1$ for every $\alpha \in A$, so for every $\alpha \in A$ there exists $n_\alpha \in N_\alpha$ such that $(n_\alpha, \psi(\alpha)) \in \Delta$. The set $\{n_\alpha\}_{\alpha \in A}$ is the desired set $\bar{N}$ $\square$

Let us recall one more classical result of graph theory. Really it is the classical Schroeder-Bernstein theorem of set theory but with an improvement which has been observed by Banach in [Ba]

Theorem A.4. *Suppose (N, Δ, M) is a bipartite graph such that both N and M are matchable. Then there exists a matching $\psi : N \longrightarrow M$ such that $\psi(N) = M$.*

Let $G = (N, \Delta, M)$ be a bipartite graph. A path of the length k in the graph G is a sequence $(a_o, a_1, a_2, \dots, a_k)$ of elements of $V(G)$ such that for each $j = 0, 1, \dots, k-1$ the vertex a_j is joined with the vertex a_{j+1} i.e. $(a_j, a_{j+1}) \in \Delta$. We will say that such a path joins a_o with a_k. Given the graph G we can consider graphs $G^k = (N, \Delta_k, M)$ for $k = 0, 1, \dots$ where $(n, m) \in \Delta_k$ if and only if there exists a path in G of length $2k+1$ joining n with m. Clearly $G^o = G$. If $(a_o, a_1, \dots, a_{2k+1})$ is a path of length $2k+1$ joining a_o with a_{2k+1} then $(a_o, a_1, a_o, a_1, a_2 \dots, a_{2k+1})$ is a path of length $2(k+1)+1$ joining a_o with a_{2k+1}. This means that $\Delta \subset \Delta_1 \subset \Delta_2 \subset \dots$.

Lemma A.5. *Suppose that the degree of each vertex in $G^o = (N, \Delta, M)$ is at most C. Suppose that $B \subset N$ is matchable in G^k with the matching $\psi : B \longrightarrow M$. Then there exists a partition of B into at most C^{2k} disjoint sets $B_1, \dots, B_s$ such that for each $j = 1, 2, \dots, s$ there exist 1-1 maps ψ_l^j for $l = 2, 3, \dots, 2k$ on B_j such that for every $b \in B_j$ the sequence $(b, \psi_1^j(b), \psi_2^j(b), \dots, \psi_{2k}^j(b), \psi(b))$ is a path in G^o of length $2k+1$.*

Proof. Let $\psi : B \longrightarrow M$ be a matching in G^k. This means that there are maps $\psi_2, \dots, \psi_{2k}$ defined on B such that for each $b \in B$ the sequence

$$(b, \psi_2(b), \dots, \psi_{2k}(b), \psi(b))$$

is a path of length $2k+1$ in G^o. Since b is joined with at most C vertices, we can partition B into at most C sets $B^1, B^2, \dots, B^s$ such that $\psi_2|B^j$ is $1-1$ for each $j = 1, 2, \dots, s$. Now each vertex $\psi_2(b)$ can be joined with at most C vertices so each set $B^1, \dots, B^s$ can be partitioned into at most C sets on which $\psi_3 \circ \psi_2$ is 1-1. Thus we have partitioned B into at most C^2 sets on each of which both ψ_2 and ψ_3 are 1-1. Continuing in this manner we get the claim. □

Remark.. Let us observe that it is possible to have both conclusions a) and b) of Corollary A.3. satisfied. We have

Proposition A.6. *Suppose $G = (N, \Delta, M)$ is a bipartite graph such that every vertex has finite degree. Assume also that N satisfies the 1/2-Hall-König condition. Let us have $N = \bigcup_{\alpha \in A} N_\alpha$ with N_α's disjoint and $|N_\alpha| = 2$ for all $\alpha \in A$. Then there exists a partition $N = N_1 \cup N_2$ into two disjoint, matchable subset such that $N_\alpha \cap N_1 \neq \emptyset$ for all $\alpha \in A$.*

Proof. We use Corollary A.3.a) to partition N as $\bar{N}_1 \cup \bar{N}_2$ into two matchable subsets. Let us fix the matchings $\psi_i : \bar{N}_1 \longrightarrow M$ for $i = 1, 2$. Let us consider the graph $G_1 = (N, \Delta_1, M_1)$ where $M_1 = \psi_1(\bar{N}_1) \cup \psi_2(\bar{N}_2)$ and $\Delta_1 = \{(n, \psi_1(n))\}_{n \in \bar{N}_1} \cup \{(n, \psi_2(n))\}_{n \in \bar{N}_2}$.

It is clear from the definition that each vertex from N belongs to exactly one edge in $E(G_1)$ and each vertex from M_1 belongs to either one or two edges. It follows that N in G_1 satisfies 1/2-Hall-König condition and that M_1 is matchable in G_1. Now let us apply Corollary A.3.b) to obtain $\bar{N} \subset N$ matchable in G_1 and such that $\bar{N} \cap N_\alpha \neq \emptyset$ for all $\alpha \in A$. Take N_1 to be a maximal matchable in G_1 subset on N containing $\bar{N}$ and let $N_2 = N \setminus N_1$. From 4.9(1) of [B] we infer that there exists a matching $\psi : N_1 \xrightarrow{\text{onto}} M_1$. This implies that $\Delta_1 \setminus \{(n, \psi(n)\}_{n \in N_1}$ gives a matching of N_2. So both N_1 and N_2 are matchable in G_1 so also in G. □

Remark. There is a real need for some kind of argument for Proposition A.6. In the situation of Corollary A.3. with $K = 2$ we can have $\bar{N} \subset N$ satisfying the conclusion of b) with $N \setminus \bar{N}$ unmatchable. Here is an example: $G = (N, \Delta, M)$ with $N = \{1, 2, 3, 4\}$ and $M = \{a, b\}$ and $\Delta = \{(1, a), (2, b), (3, a), (3, b), (4, b)\}$. The partition of N is given by $\{1, 2\}$ and $\{3, 4\}$. If we take $\bar{N} = \{1, 3\}$ we see that it satisfies the conclusion of Corollary A.3.b) but $N \setminus \bar{N} = \{2, 4\}$ is unmatchable.

Section B: General situation

Let $(e_m, e_m^*)_{m \in M}$ be a biorthogonal system in a quasi-Banach space X i.e. we have

$$e_m^*(e_s) = \begin{cases} 1, & \text{if } m = s \\ 0, & \text{if } m \neq s \end{cases}$$

The system $(e_m, e_m^*)_{m \in M}$ is an unconditional basis in X if for every $x \in X$ the series $\sum_{m \in M} e_m^*(x) e_m$ converges unconditionally to x. This implies that there exists a constant K such that

$$\| \sum_{m \in M} \beta_m e_m^*(x) e_m \| \leq K \sup_{m \in M} |\beta_m| \cdot \|x\|$$

for all $x \in X$. The smallest such constant K will be called an **unconditional basis constant** ot the basis $(e_m, e_m^*)_{m \in M}$. We will *always* additionally assume that $\|e_m\| = 1$ for all $m \in M$. Since actually the elements $(e_m)_{m\in M}$ determine the functionals $(e_m^*)_{m\in M}$ it is customary to speak about $(e_m)_{m\in M}$ as being an unconditional basis. We will use this convention sometimes, but very often we will actually need the biorthogonal functionals. Let $(e_m, e_m^*)_{m\in M}$ be an unconditional basis in X and let $(y_n, y_n^*)_{n\in N}$ be an unconditional basis in Y. We say that those bases are equivalent (and write $(e_m) \sim (y_n)$ if there exists a 1–1 and onto map $\Phi : M \longrightarrow N$ such that the map $e_m \mapsto y_{\Phi(m)}$ extend by linearity to an isomorphism between X and Y. We say that $(e_m, e_m^*)_{m\in M}$ is equivalent to a subbasis of $(y_n)_{n\in N}$ if there exists a 1–1 (not necessarily onto map $\Phi : M \longrightarrow N$ such that the map $e_m \mapsto y_{\Phi(m)}$ extends by linearity to an isomorphism between X and the closed linear span of $(y_s)_{s\in\Phi(M)}$.

The unconditional basis $(e_m, e_m^*)_{m\in M}$ in X is **strongly absolute**, if for every $\varepsilon > 0$ there is a constant C_ε such that for any scalars $(a_m)_{m\in M}$, only finitely many of them non- zero, we have

$$\sum_{m\in M} |a_m| \le C_\varepsilon \sup_{m\in M} |a_m| + \varepsilon \Big\| \sum_{m\in M} a_m e_m \Big\|.$$

This definition was introduced in [KLW]. Its intuitive meaning is that the space X is far from being a Banach space.

The other notion we will need is that of a **natural** quasi-Banach space. We will state this definition very briefly and refer the reader to [K1] or [KLW] for more details. A quasi-Banach lattice X is L-convex if there exists an $\varepsilon > 0$ so that if $u \in X$, $u \ge 0$ and $\|u\| = 1$ then for any $x_i \in X$, $1 \le i \le n$ with $0 \le x_i \le u$ and such that $\frac{1}{n}(x_1 + x_2 + \cdots + x_n) \ge (1-\varepsilon)u$ we have $\max_{1\le i\le n} \|x_i\| \ge \varepsilon$. A quasi-Banach space Y is called **natural** if it is isomorphic to a subspace of an L-convex quasi-Banach lattice.

Let us simply mention that all function spaces occurring naturally in analysis and their subspaces are natural. In particular Hardy spaces are natural. In our proofs this notion enters only once (but in a crucial way) in the proof of Proposition B.4. Since I decided to refer the reader for this proof to [KLW] instead of repeating two pages of arguments, this brevity should not cause any problems.

If X is a quasi-Banach space, by $\hat{X}$ we will denote its Banach envelope (cf. [KPR] p. 27).

In this Section we will always consider a natural quasi- Banach space X with a normalised, strongly absolute unconditional basis $(e_m, e_m^*)_{m\in M}$. We will also consider Y, a complemented subspace of X with a normalised, unconditional basis $(u_n, u_n^*)_{n\in N}$. We will assume that unconditional basis constants of both bases are at most K. Since $(u_n, u_n^*)_{n\in N}$ is assumed to be an unconditional basis in Y we have u_n^*'s defined only on Y. But Y is assumed to be complemented, so we can extend u_n^*'s to the whole of X. This gives that the projection $P : X \overset{\text{onto}}{\longrightarrow} Y$ is given by the formula

$$P(x) = \sum_{n\in N} u_n^*(x) u_n.$$

We will treat u_n and u_n^* as sequences indexed by M. More precisely since $(e_m)_{m\in M}$ is an unconditional basis in X we have $u_n = \sum_{m\in M} u_n(m) e_m$ and $u_n^* = \sum_{m\in M} u_n^*(m) e_m^*$ where naturally $u_n(m) = e_m^*(u_m)$ and $u_n^*(m) = u_n^*(e_m)$. The reader should note that the first of the above sums is unconditionally convergent in norm while the second is only w^*-convergent. We can consider $(e_m, e_m^*)_{m\in M}$ as an unconditional basis in $\hat{X}$. Since in X it is strongly absolute, in $\hat{X}$ it is equivalent to the unit vector basis in $\ell_1(M)$. This implies in particular that there exists a constant C such that $|u_n^*(m)| \leq C$ for all $n \in N$ and $m \in M$.

The numbers $U(n,m) = u_n^*(m) \cdot u_n(m)$ will be of fundamental importance in our considerations, so for future reference let us summarise their properties

Lemma B.1. *There exists a constant C such that*
a) for every $n \in N$ we have $\sum_{m\in M} U(n,m) = 1$ and $\sum_{m\in M} |U(n,m)| \leq C$
b) for every $m \in M$ we have $\sum_{n\in N} |U(n,m)| \leq C\|P\|$
c) for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $n \in N$ we have $\sum_{\{m:|U(n,m)<\delta\}} |U(n,m)| \leq \varepsilon$
d) for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $m \in M$ we have $\sum_{\{n:|U(n,m)|<\delta\}} |U(n,m)| \leq \varepsilon$
d) if $Y = X$ i.e. $(u_n)_{n\in N}$ is another unconditional basis in X then $\sum_{n\in N} U(n,m) = 1$ for all $m \in M$

For each $\delta > 0$ we define the bipartite graph $B(\delta) = (N, \Delta_\delta, M)$ by the condition that $(n,m) \in \Delta_\delta$ if and only if $|U(n,m)| \geq \delta$.

Lemma B.2. *For every $\delta > 0$ there exists a constant $C(\delta)$ such that the degree of each vertex in $V(B(\delta))$ is at most $C(\delta)$.*

Proof. This follows immediately from Lemma B.1. a) and b). □

Lemma B.3. *There exists a $\delta_o > 0$ such that for all δ, $0 < \delta < \delta_o$ we have*
a) the degree of each vertex $n \in N$ is at least 1
b) there exists a $c > 0$ such that N in $B(\delta)$ satisfies c-Hall-König condition.

Proof. Using the unconditionality of the basis $(e_m, e_m^*)_{m\in M}$ and the fact that it is strongly absolute we have

$$\begin{aligned}
1 = u_n^*(u_n) = \sum_m U(n,m) &\leq \sum_m |U(n,m)| \\
&\leq C_\varepsilon \sup_m |U(n,m)| + \varepsilon \|\sum_m U(n,m) e_m\| \\
&\leq C_\varepsilon \sup_m |U(n,m)| + \varepsilon CK \|\sum_m u_n(m) e_m\| \\
&= C_\varepsilon \sup_m |U(n,m)| + \varepsilon CK
\end{aligned}$$

If we take ε such that $\varepsilon CK < \frac{1}{2}$ then we obtain that for each n

$$\sup_m |U(n,m)| > \frac{1}{2C_\varepsilon} = \delta_o.$$

This proves a). To prove b) observe that (because the basis $(e_m, e_m^*)_{m\in M}$ is strongly absolute) for each $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon)$ such that for every $x \in X$ we have

$$\sum_{m\,:\,|e_m^*(x)|<\delta} |e_m^*(x)| \leq \varepsilon \|x\|.$$

Take a finite set $A \subset N$. We have

$$\begin{aligned}|A| &= |\sum_{n\in A} u_n^*(u_n)| = |\sum_{n\in A}\sum_{m\in M} U(n,m)| \\ &\le \sum_{n\in A}\sum_{m\in\phi(A)} |U(n,m)| + \sum_{n\in A}\sum_{m\notin\phi(A)} |U(n,m)|\end{aligned}$$

By Lemma B.1.c we can find a δ_o such that for $\delta < \delta_o$ the second sum above will be at most $\frac{1}{2}|A|$. Thus using Lemma B.1.b we have

$$\frac{1}{2}|A| \le \sum_{m\in\phi(A)}\sum_{n\in A} |U(n,m)| \le C|\phi(A)|$$

This proves b). □

The importance of all this for the problem of equivalence of unconditional bases rests on the following

Proposition B.4. *Suppose a subset $N_1 \subset N$ admits a matching $\psi : N_1 \longrightarrow M$ in the graph $B(\delta)$. Then the basic sequences $(u_n)_{n\in N_1}$ and $(e_{\psi(n)})_{n\in N_1}$ are equivalent.*

The proof of this Proposition is the argument given at the end of the proof of Theorem 2.3 of [KLW].

Now we can appeal to Corollary A.3.a) to get Theorem 2.3. of [KLW]. Actually all we did was to rewrite the proof of this Theorem using a bit of the language of graph theory.

Corollary B.5. *(see* [KLW] *Theorem 2.3) Suppose X is a natural quasi-Banach space with a strongly absolute, normalised unconditional basis $(e_m, e_m^*)_{m\in M}$. Suppose also that Y is a complemented subspace of X with a normalised unconditional basis $(u_n, u_n^*)_{m\in M}$. Then there is a decomposition of N into finite union of disjoint sets $N_1, N_2, \dots, N_s$ such that each basic sequence $(u_n)_{n\in N_j}$, $j = 1, 2, \dots, s$ is equivalent to a subbasis of the basis $(e_m, e_m^*)_{m\in M}$. In particular $(u_m, u_m^*)_{m\in M}$ is strongly absolute.*

Proof. Combine Lemma B.3.b), Proposition B.4 and Corollary A.3.a). □

From now on we will assume that $Y = X$ i.e. that $(u_n)_{n\in N}$ *is another unconditional basis in* X. We would like to show that $(u_n)_{n\in N}$ is equivalent to a subbasis of $(e_m)_{m\in M}$. This we are unable to do in general. We can however show that it is possible to partition N into only **two** subsets N_1 and N_2 so that each $(u_n)_{n\in N_i}$ is equivalent to a subbasis of $(e_m)_{m\in M}$. To do this we will consider paths in a graph.

To our graph $B(\delta)$ let us apply the procedure described before Lemma A.5. We get the following

Corollary B.7. *Suppose $B \subset N$ is matchable in $B^k(\delta)$ with the matching ψ. Then the basic sequences $(u_n)_{n\in B}$ and $(e_{\psi(n)})_{n\in B}$ are equivalent.*

Proof. We use Lemma A.5. We get a partition of B into $B_1, \dots, B_s$ and we get maps as described in this Lemma. From Proposition B.4. we infer that $(u_n)_{n\in B_j}$ is equivalent to $(e_{\psi_2^j(n)})_{n\in B_j}$. But by the same Proposition $(e_{\psi_2^j(n)})_{n\in B_j}$ is equivalent to $(u_{\psi_3^j(n)})_{n\in B_j}$. Continuing in this manner $2k$–times we infer that $(u_n)_{n\in B_j}$ is equivalent to $(e_{\psi(n)})_{n\in B_j}$. Since B_j's form a partition of B and ψ is 1-1 we infer that $(u_n)_{n\in B}$ and $(e_{\psi(n)})_{n\in B}$ are equivalent. □

Remark. . This Corollary is valid also (with the same proof) when $\mathrm{span}(u_n)_{n\in N}$ is a complemented subspace of X.

Now we will show that for two bases in X graphs $B^k(\delta)$ can satisfy the c–Hall–König condition with the constant c as closed to 1 as one wants.

Proposition B.8. *Assume that $(u_n)_{n\in N}$ and $(e_m)_{m\in M}$ are two unconditional and strongly absolute bases in a natural space X. Let us fix α, $0<\alpha<1$. Then there exists a $\delta>0$ and k such that both N and M in the graph $B^k(\delta)$ satisfy α–Hall–König condition.*

Proof. Let us start with the argument for N. First we need to visualise the edges of the graph $B^k(\delta)$ in terms of numbers $U(n,m)$ analysed in Lemma B.1. From the definition of $B^k(\delta)$ we see that $(n,m)\in E(B^k(\delta))$ if there exists a sequence $n_o,m_1,n_1,\dots,m_k,n_k,m_{k+1}$ where $n_o=n$ and $m_{k+1}=m$ such that for $i=0,1,\dots,k$ we have $|U(n_i,m_{i+1})|>\delta$ and $|U(n_i,m_i)|>\delta$. Given our α fix $\varepsilon=\frac{1-\alpha}{6}$ and fix δ so that conditions c) and d) of Lemma B.1. hold. We will show that $k>\frac{12}{1-\alpha}C^2+1$ will work.

Let us fix an arbitrary finite set $A_1\subset N$ and call $|A_1|=N_1$. Let

$$B_1=\phi(A_1)=\{m\in M\ :\ |U(n,m)|>\delta \text{ for some } n\in A_1\}.$$

Now we form inductively sets

$$A_{s+1}=\{n\ :\ |U(n,m)|>\delta\ \text{ for some } m\in B_s\}$$

and

$$B_{s+1}=\{m\ :\ |U(n,m)|>\delta\ \text{ for some } n\in A_{s+1}\}$$

It is clear that $A_1\subset A_2\subset\dots$ and $B_1\subset B_2\subset\dots$. Let us call $|A_s|=N_s$ and $|B_s|=K_s$. Sequences N_s and K_s are increasing. It should be clear from what we said at the beginning of this proof that

$$B_s=\{m\in M\ :\ (n,m)\in E(B^s(\delta))\ \text{ for some }\ n\in A_1\}.$$

This means that our goal is to show that $K_k\ge\alpha N_1$. From Lemma B.1.a) we have

$$\begin{aligned}\sum_{n\in A_{s+1}}\sum_{m\in B_s}U(n,m)&=\sum_{m\in B_s}\sum_{n\in N}U(n,m)-\sum_{m\in B_s}\sum_{n\notin A_{s+1}}U(n,m)\\&=K_s-\sum_{m\in B_s}\sum_{n\notin A_{s+1}}U(n,m)\end{aligned}$$

Since for $m\in B_s$ and $n\notin A_{s+1}$ we have $|U(n,m)|<\delta$, our choice of δ and Lemma B.1.d. yields

$$(1-\varepsilon)K_s\le\sum_{n\in A_{s+1}}\sum_{m\in B_s}U(n,m)\le(1+\varepsilon)K_s \tag{1}$$

Analogously

$$(1-\varepsilon)N_s\le\sum_{n\in A_s}\sum_{m\in B_s}U(n,m)\le(1+\varepsilon)N_s \tag{2}$$

Observe however that Lemma B.1.b) implies

$$\sum_{n\in A_s}\sum_{m\in B_s} U(n,m) \le C|B_s| = CK_s \tag{3}$$

so from (2) and (3) we get

$$(1-\varepsilon)N_s \le CK_s. \tag{4}$$

We have

$$\begin{aligned}&\sum_{n\in A_{s+1}\setminus A_s}\ \sum_{m\in B_{s+1}\setminus B_s} U(n,m)\\ &= (\sum_{n\in A_{s+1}}\ \sum_{m\in B_{s+1}} - \sum_{n\in A_{s+1}}\ \sum_{m\in B_s} - \sum_{n\in A_s}\ \sum_{m\in B_{s+1}\setminus B_s})U(n,m)\end{aligned}$$

Since for each $n \in A_s$ and $m \in B_{s+1}\setminus B_s$ we have $|U(n,m)| < \delta$ we infer from Lemma B.1.c) and (1) and (2) that

$$\begin{aligned}(1-\varepsilon)N_{s+1} - (1+\varepsilon)K_s - \varepsilon N_s &\le \sum_{n\in A_{s+1}\setminus A_s}\ \sum_{m\in B_{s+1}\setminus B_s} U(n,m)\\ &\le (1+\varepsilon)N_{s+1} - (1-\varepsilon)K_s + \varepsilon N_s\end{aligned} \tag{5}$$

Since Lemma B.1.a) clearly implies

$$\sum_{n\in A_{s+1}\setminus A_s}\ \sum_{m\in B_{s+1}\setminus B_s} U(n,m) \le C|A_{s+1}\setminus A_s| = C(N_{s+1}-N_s)$$

from (5) and the monotonicity of N_s we get

$$(1-2\varepsilon)N_{s+1} - (1+\varepsilon)K_s \le C(N_{s+1}-N_s) \tag{6}$$

Suppose now that $K_k \le \alpha N_1$. Thus for all $s = 1,2,\ldots,k$ we have

$$K_s \le \alpha N_1. \tag{7}$$

From (4) we now infer that for $s = 1,2,\ldots,k$

$$N_s \le \frac{\alpha C}{1-\varepsilon}N_1 \tag{8}$$

Summing (6) for $s = 1,2,\ldots,k-1$ and using (8) we get

$$(1-2\varepsilon)\sum_{s=1}^{k-1} N_{s+1} - (1+\varepsilon)\sum_{s=1}^{k-1} K_s \le C(N_{k+1}-N_1) \le C(\frac{\alpha C}{1-\varepsilon}-1)N_1 \tag{9}$$

Using our supposition (7) and the monotonicity of N_s the left hand side of (9) can be minorised by

$$(1-2\varepsilon)(k-1)N_1 - (1+\varepsilon)\alpha N_1(k-1) \tag{10}$$

From (9) and (10) and the definition of ε we see that

$$C(\frac{\alpha C}{1-\varepsilon}-1) \ge [(1-2\varepsilon)-(1+\varepsilon)\alpha](k-1) \ge \frac{1-\alpha}{2}(k-1).$$

This contradicts our choice of ε and k and so completes the proof for N. Since our assumptions and constructions work as well for M as for N we obtain the Proposition. □

Theorem B.9. *Suppose X is a natural, quasi- Banach space with strictly absolute unconditional basis $(e_m, e_m^*)_{m\in M}$ and suppose that $(u_n, u_n^*)_{n\in N}$ is some other unconditional basis in X. Then we can partition each index set into four disjoint subsets, N_1, N_2, N_3, N_4 and M_1, M_2, M_3, M_4 in such a way that*

$$(u_n)_{n\in N_1} \sim (e_n)_{m\in M_1}$$

$$(u_n)_{n\in N_4} \sim (e_m)_{m\in M_4} \sim (e_m)_{m\in M_3}$$

$$(u_n)_{n\in N_2} \sim (u_n)_{n\in N_3} \sim (e_m)_{m\in M_2}$$

Proof. We use Proposition B.8 to obtain the graph $B^k(\delta)$ where both N and M satisfy $\frac{1}{2}$- Hall-König condition. It follows from Corollary A.3. that there are maximal matchable subsets $S \subset N$ and $V \subset M$ such that $N \setminus S$ and $M \setminus V$ are also matchable. It follows from [B] 4.9.1. that there exists a matching $\psi : S \xrightarrow{\text{onto}} V$. Let us fix matchings $\psi_1 : N \setminus S \longrightarrow M$ and $\psi_2 : M \setminus V \longrightarrow N$. We have

$$\psi_2(M \setminus V) \cap \psi^{-1}(\psi_1(N \setminus S)) = \emptyset.$$

To show this assume to the contrary that for some $m \in M \setminus V$ and $n \in N \setminus S$ we have $\psi\psi_2(m) = \psi_1(n)$. Then we can define matching Ψ on the set $S \cup \{n\}$ by the formulas $\Psi = \psi$ on the set $S \setminus \{\psi_2(m)\}$, $\Psi(\psi_2(m)) = m$ and $\Psi(n) = \psi_1(n)$. One checks that it is really the matching, so S was not maximal; a contradiction. This also implies that $\psi_1(N \setminus S) \cap \psi\psi_2(M \setminus V) = \emptyset$.

Now we define

$$N_1 = S \setminus \{\psi_2(M \setminus V) \cup \psi^{-1}\psi_1(N \setminus S)\}$$

$$M_1 = V \setminus \{\psi_1(N \setminus S) \cup \psi\psi_2(M \setminus V)\}$$

$$N_3 = N \setminus S$$

$$M_4 = M \setminus V$$

$$N_4 = \psi_2(M \setminus V)$$

$$M_2 = \psi_1(N \setminus S)$$

$$N_2 = \psi^{-1}\psi_1(N \setminus S)$$

$$M_3 = \psi\psi_2(M \setminus V).$$

One easily checks that we get the desired partitionings and that the matchings we have chosen establish the desired equivalences (use Corollary B.7). □

Proposition B.10. *Suppose that X is a natural quasi-Banach space with a strongly absolute unconditional basis $(e_m, e_m^*)_{m\in M}$. Suppose that $(u_n, u_n^*)_{n\in N}$ is another unconditional basis in X. Let $s = 2, 3, \dots$ be given. Assume that $N = \bigcup_{\alpha\in A} N_\alpha$ where N_α's are disjoint subset each of cardinality s. Then there exists a subset $V \subset N$ such that*
a) $(u_n)_{n\in V}$ is equivalent to a subbasis of $(e_m)_{m\in M}$
b) $V \cap N_\alpha \neq \emptyset$ for all $\alpha \in A$.

Proof. It follows directly from Proposition B.8. and Corollary A.3. and Corollary B.7. □

Proposition B.11. *Suppose that X and Y are quasi- Banach spaces with normalised unconditional bases $(x_n)_{n\in N}$ and $(y_m)_{m\in M}$ respectively. Assume that $(x_n)_{n\in N}$ is equivalent to a subbasis $(y_{\sigma(n)})_{n\in N}$ of $(y_n)_{n\in N}$ and that $(y_m)_{m\in M}$ is equivalent to a subbasis $(x_{\gamma(m)})_{m\in M}$ of $(x_n)_{n\in N}$. Then the bases $(x_n)_{n\in N}$ and $(y_m)_{m\in M}$ are permutatively equivalent; in particular X is isomorphic to Y.*

Proof. Let us consider a bipartite graph G with vertex set $N\cup M$ (assumed to be disjoint sets) and edge set $\{(n,\sigma(n))\}_{n\in N}\cup\{(m,\gamma(m))\}_{m\in M}$. This implies that in G both n and M are matchable. From Theorem A.4. we see that there is a matching $\psi : N \overset{\text{onto}}{\longrightarrow} M$. From the definition of the edge set we see that we can split N into N_1 and N_2 in such a way that $\psi|N_1=\sigma$ and $\psi|N_2=\gamma^{-1}$. This means that $(x_n)_{n\in N_1}$ is equivalent to $(y_m)_{m\in\psi(N_2)}$ and $(x_n)_{n\in N_2}$ is equivalent to $(y_m)_{m\in\psi(N_2)}$. Since $N=N_1\cup N_2$ and $N_1\cap N_2=\emptyset$ and $M=\psi(N_1)\cup\psi(N_2)$ we conclude that $(x_n)_{n\in N}$ and $(y_m)_{m\in M}$ are permutatively equivalent. $\square$

Remark.. This argument is very general. It applies not only to quasi-Banach spaces (in particular Banach spaces), but practically to any decent kind of space. Surprisingly enough Proposition B.11. seems to be unknown to the specialists. Usually such conclusions were obtained using the decomposition method, which requires some additional properties of bases (like being isomorphic to its square or something similar) (cf. [BCLT] Prop. 7.7).

Theorem B.12. *Let X be a natural quasi-Banach space with strongly absolute unconditional basis $(e_m)_{m\in M}$. Assume also that X is isomorphic to some of its cartesian powers X^s, $s=2,3,\ldots$. Then all normalised, unconditional bases in X are permutatively equivalent.*

Proof. Since X^s is isomorphic to X we get also that X^{s^2} is isomorphic to X. Thus in X there is a normalised, unconditional basis equivalent to the direct sum of s^2 copies of the basis $(e_m)_{m\in M}$. We call this basis $(u_n)_{n\in N}$ where naturally $N=M\times S\times S$ where S is a set of cardinality s. We partition N into sets of cardinality s as $N=\bigcup_{m\in M,s\in S}\{m\}\times\{s\}\times S$. Applying Proposition B.10 (with the integer s) we realise that the subbasis of $(e_m)_{m\in M}$ given by this Proposition has a subbasis equivalent to the direct sum of s copies of the basis $(e_m)_{m\in M}$. From Proposition B.11. we infer that the basis $(e_m)_{m\in M}$ is permutatively equivalent to the direct sum of s copies of itself. Now let $(u_n)_{n\in N}$ denote any other normalised unconditional basis in X. It follows from Theorem B.9. that $(u_n)_{n\in N}$ can be split into two pieces, each equivalent to a subbasis of $(e_m)_{m\in M}$. But as we know, $(e_m)_{m\in M}$ is equivalent to the direct sum of s copies of itself $s\geq 2$, so actually we see that $(u_n)_{n\in N}$ is equivalent to a subbasis of $(e_m)_{m\in M}$.

Reversing the role of bases in the above argument we conclude that $(e_m)_{m\in M}$ is equivalent to a subbasis of $(u_n)_{n\in N}$, so by Proposition B.11. all bases in X are permutatively equivalent. $\square$

Corollary B.13. *Suppose that X is a natural quasi-Banach space with the strongly absolute unconditional basis. If for some $s=2,3,\ldots$ X is isomorphic to X^s, then X is isomorphic to $X\oplus X$, so X is isomorphic to X^k for every $k=1,2,3,\ldots$.*

Proof. Let us consider X with the basis $(x_n)_{n\in N}$ and $Y=X\oplus X$ with the basis $(y_m)_{m\in M}=(x_n)_{n\in N}\oplus(x_n)_{n\in N}$. Clearly $(x_n)_{n\in N}$ is equivalent to a subbasis of $(y_m)_{m\in M}$. We know from Theorem B.12. that $(x_n)_{n\in N}$ is equivalent to the direct sum of s copies of itself. Since $s\geq 2$, it means that $(y_m)_{m\in M}$ is equivalent to a subbasis of $(x_n)_{n\in N}$. Proposition B.11. shows that bases $(x_n)_{n\in N}$ and $(y_m)_{m\in M}$ are equivalent, so $X \sim X \oplus X$. $\square$

Remarks.. One may be tempted to believe that we do not need Proposition B.8., in other words that N satisfies the 1–Hall–König condition in $B(\delta)$ for sufficiently small δ. This however is not the cases as the following example shows.

Example B.14. Let us fix an integer $n \geq 2$. In ℓ_p^{n+1} consider the following biorthogonal system

$$
\begin{aligned}
u_1 &= (1,1,0,\dots,0) & u_1^* &= (1,0,-1,-1,\dots,-1)\\
u_2 &= (1,0,1,0,\dots,0) & u_2^* &= (1,-1,0,-1,\cdots-1)\\
\\
u_n &= (1,0,\dots,0,1) & u_n^* &= (1,-1,\dots,-1,0)\\
u_{n+1} &= (1,\frac{1}{n-1},\dots,\frac{1}{n-1}) & u_{n+1}^* &= (-(n-1),n-1,\dots,n-1)
\end{aligned}
$$

If we treat ℓ_p as a direct sum of countably many copies of ℓ_p^{n+1}, the above system yields an unconditional basis in ℓ_p. The other basis is the unit vector basis. It is easy to see that, for any $\delta > 0$, the best constant c in the c–Hall– König condition we can get in this situation is $\frac{1}{n}$.

Actually this example also shows that Proposition B.8 does not work when $(u_n)_{n\in N}$ is an unconditional basis in a complemented subspace. Simply consider the basis which is the infinite direct sum of $u_1, u_2, \dots, u_n$. In this case taking paths does not add a single edge to the graph.

It may be true that for two unconditional bases in X there is a graph $B^k(\delta)$ which satisfies 1–Hall- -König condition. This would lead to Theorem B.12. without the assumption that X is isomorphic to X^s. It is however relatively easy to see that in order to prove it one would have to use something more about the matrix $[U(n,m)]_{n\in N,m\in M}$ then is summarised in Lemma B.1. I also believe that one really has to use quite big k's in order to get the conclusion of Proposition B.8 (for $\alpha = \frac{1}{2}$ say). Here is a finite dimensional example which shows that $B^2(\delta)$ does not work for any $\delta > 0$. We look at R^5 and take vectors

$$(1,1,0,0,0),\ (1,0,1,0,0),\ (1,0,0,1,0),\ (1,0,0,0,1),\ (0,1,1,1,-2).$$

One checks that biorthogonal functionals are:

$$
\begin{aligned}
&(1,0,-1,-1,-1),\ (1,-1,0,-1,-1),\ (1,-1,-1,0,-1),\\
&(-2,2,2,2,3),\ (-1,1,1,1,1).
\end{aligned}
$$

This gives the matrix $[U(n,m)]$ as

$$
\begin{bmatrix}
1,0,0,0,0\\
1,0,0,0,0\\
1,0,0,0,0\\
-2,0,0,0,3\\
0,1,1,1,-2
\end{bmatrix}
$$

Let us now return to the situation considered at the beginning of this Section, i.e. when $(u_n, u_n^*)_{n\in N}$ is an unconditional basis in a complemented subspace of a natural space X with a strongly absolute unconditional basis $(e_m, e_m^*)_{m\in M}$. It is very tempting to conjecture that $(u_n)_{n\in N}$ is equivalent to a subbasis of $(e_n)_{n\in N}$. Unfortunately in general we can not say anything beyond Corollary B.5. However when we compare Corollary B.5. and Theorem B.12 we get the following

Corollary B.15. *Suppose that X is a natural quasi- Banach space with a strongly absolute unconditional basis $(e_m)_{m\in M}$ and that $(u_n)_{n\in N}$ is an unconditional basis in a complemented subspace Y of X. Assume additionally that X is isomorphic to X^s for some $s = 2, 3, \dots$. Then $(u_n)_{n\in N}$ is equivalent to a subbasis of $(e_m)_{m\in M}$*

Section C: Hardy spaces

Let us now turn our attention to Hardy spaces H_p. We will work exclusively in the framework of dyadic Hardy spaces.

Let $\mathcal{J}$ denote the family of all dyadic subintervals of the interval $[0,1]$ that is the family of intervals of the form $[\frac{k}{2^n}, \frac{k+1}{2^n}]$ where $n = 0, 1, 2\dots$ and $k = 0, 1, \dots, 2^n - 1$. The set $\mathcal{J}$, when ordered by inclusion forms a canonical dyadic tree. For each interval $I = [\frac{k}{2^n}, \frac{k+1}{2^n}] \in \mathcal{J}$ we define the corresponding Haar function h_I normalised in H_p by the formula

$$h_I(t) = \begin{cases} 2^{n/p}, & \text{if } \frac{2k}{2^{n+1}} \le t < \frac{2k+1}{2^{n+1}} \\ -2^{n/p}, & \text{if } \frac{2k+1}{2^{n+1}} \le t < \frac{2(k+1)}{2^{n+1}} \\ 0, & \text{otherwise} \end{cases}$$

We define H_p space as the space of distributions f of the form $f = \sum_{I\in\mathcal{J}} a_I h_I$ for which the following expression is finite

$$\|f\| = \left[\int_0^1 (\sum_{I\in\mathcal{J}} |a_I h_I(t)|^2)^{p/2}\right]^{1/p}. \tag{1}$$

This definition makes perfect sense for $0 < p < \infty$. For $1 \le p < \infty$ $\|.\|$ is really a norm and H_p is a Banach space. When $0 < p < 1$ (1) defines a p norm and the resulting Hardy space H_p is a quasi- Banach space. In the rest of this section we will **always** assume that $0 < p < 1$.

It is clear from the definition that the Haar system $(h_I)_{I\in\mathcal{J}}$ is a normalised unconditional basis in H_p. This basis is strongly absolute (cf. [KLW]).

We will not use much more about those spaces besides the definition. Let us note however that their origin lies in martingale theory where they play important role (cf. [G]). Their importance stems also from the fact (cf. [W]) that they are isomorphic to the classical Hardy spaces of analytic functions on the unit disc. For a general theory of those Hardy spaces see [D].

The following properties of the basis $(h_I)_{I\in\mathcal{J}}$ in H_p are easy and well known.

Proposition C.1. *a) If $I_o \in \mathcal{J}$ then the system $(h_I)_{I\subset I_o,\ I\in\mathcal{J}}$ is equivalent to the Haar system.*

b) The Haar system $(h_I)_{I\in\mathcal{J}}$ is equivalent to its infinite direct sum in the ℓ_p-sense.

It follows from Corollary B.5 and the above Proposition that if X is a complemented subspace of H_p, $p<1$, with an unconditional basis $(e_j)_{j\in J}$, then the basis $(e_j)_{j\in J}$ is equivalent to the subbasis of the Haar basis.

Our aim in this section (motivated by the above remark) is to study subbases of the Haar basis. It turns out that they exhibit very complicated structure. Before we proceed let us point out the following

Fact C.2. *Every normalised unconditional basis in H_p, $p<1$ is equivalent to the Haar basis.*

This is clearly a corollary of Theorem B.12. and the above Proposition C.1.b), but in this particular case the proof can be obtained more easily as follows:

We know from Corollary B.5 that each unconditional basis in H_p is equivalent to a subbasis of the Haar basis. In the other direction we proceed like in the proof of Corollary B.5, but instead of Corollary A.3.a we use Corollary A.3.b. The easiest way to see the argument is to assume that $K=2^s$ and take

$$N_1=\{h_I\ :\ |I|<2^s\}$$

and the remaining N_α's as

$$\{h_{I_o},h_{I_1},\dots,h_{I_{2^s-1}}\}$$

where $I_o\in\mathcal{J}$ and $I_o\subset[0,2^{-s}]$ and for $j=1,2,\dots,2^s-1$ we have $I_j=I_o+j2^{-s}$. It is now quite clear that taking one out of each N_α we get a basis equivalent to the Haar basis. The appeal to Theorem A.4 finishes the proof of the Fact.

Let us introduce now a notation which will be used in the rest of this section. If $A\subset\mathcal{J}$ then the closed linear span of $(h_I)_{I\in A}$ in H_p will be denoted by $H_p(A)$. The set of all dyadic intervals of length 2^{-n} will be denoted by $\mathcal{J}_n$. Given $A\subset\mathcal{J}$ by A_n we will mean $A\cap\mathcal{J}_n$ and by $\phi_n(A)$ the number $|A_n|2^{-n}$. Thus A_n is the portion of A in the n–th level of the dyadic tree and $\phi_n(A)$ is the relative density of A in the n–th level.

Lemma C.3. *Suppose we have sets $A_k\subset\mathcal{J}_{n_k}$ for some increasing sequence of integers (n_k) and let*

$$x_k=\|\sum_{I\in A_k}h_I\|^{-1}\sum_{I\in A_k}h_I$$

and let us denote $|A_k|2^{-n_k}=\sum_{I\in A_k}|I|=|supp\,x_k|$ by a_k. Then if $\inf a_k>0$ then (x_k) is in H_p equivalent to the unit vector basis in ℓ_2,

Proof. Let χ_k denotes the characteristic function of $\operatorname{supp}x_k$. We easily see that $|x_k|=a_k^{-1/p}\chi_k$. From the definition of the norm we get that for any sequence of scalars α_k we have

$$\|\sum_k\alpha_kx_k\|=\Big[\int_0^1(\sum_k\alpha_k^2a_k^{-2/p}\chi_k)^{p/2}\Big]^{1/p}.$$

This expression clearly increases when we replace each χ_k by the constant function 1, so

$$\|\sum_k\alpha_kx_k\|\le(\inf a_k)^{-1/p}(\sum_k|\alpha_k|^2)^{1/2}.$$

On the other hand, since the norm of the integral is smaller then the integral of the norm (use the norm in $\ell_{2/p}$), we have

$$\begin{aligned}\Big(\sum_k |\alpha_k|^2\Big)^{p/2} &= \Big[\sum_k \Big|\int_0^1 \alpha_k^p a_k^{-1}\chi_k(t)\,dt\Big|^{2/p}\Big]^{p/2}\\ &\le \int_0^1 \Big(\sum_k |\alpha_k| a_k^{-2/p}\chi_k(t)\Big)^{2/p} dt\end{aligned}$$

so we have the claim. □

Lemma C.4. *Let $x_k = \sum_{I\in L^k} b_I h_I$ be a sequence of vectors in H_p such that $\|x_k\| = 1$ for all k and L^k's are disjoint subsets of $\mathcal{J}$. If $\liminf_k |\bigcup_{I\in L^k} I| = 0$ then (x_k) has a subsequence equivalent to the unit vector basis in ℓ_p, so it is* **not** *equivalent to the unit vector basis in ℓ_2.*

Proof. We can pass to a subsequence and assume without loss of generality that vectors x_k are almost disjoint. Then one easily checks that they span ℓ_p.

Our first aim now is to discuss for what sets $A \subset \mathcal{J}$ the space $H_p(A)$ is isomorphic to H_p. By Fact C.2 it is the same as when the system $(h_I)_{I\in A}$ is equivalent to the whole Haar system. □

Proposition C.5. *Let the set $A \subset \mathcal{J}$ be given. Suppose that*

(*) *for every $\delta > 0$ and for every natural number s there exists a natural number $k = k(\delta, s)$ such that for all $n \in [k, k+s]$ we have $\psi_n(A) < \delta$.*

Then $H_p(A)$ is not isomorphic to H_p.

Proof. Assume to the contrary that $H_p(A)$ is isomorphic to H_p. Then by Fact C.2. there exists a 1–1 and onto map $\Phi : \mathcal{J} \longrightarrow A$ which gives the equivalence of the bases. Now for $n = 1, 2, \dots$ take $\delta_n = \frac{1}{n}$ and $s_n = n$ and denote the corresponding $k(\frac{1}{n}, n)$ by k_n. Observe that (unless A is empty, but then there is nothing to do) $\lim_n k_n = \infty$.

For $l \in [k_n + \frac{n}{3}, k_n + \frac{2n}{3}] = K_n$ let

$$x_l = \Big\|\sum_{I\in V_l} h_I\Big\|^{-1} \sum_{I\in V_l} h_I$$

where $V_l = \{I \in \mathcal{J}_l \ : \ |\Phi(I)| > 2^{-k_n}\}$. Since Φ is 1–1, there is at most 2^{k_n+1} elements in $\mathcal{J}_l$ which are not in V_l. From Lemma C.3 we infer that the sequence (x_l) where $l \in \bigcup_{n=1}^\infty K_n$ is equivalent in H_p to the unit vector basis in ℓ_2. Since Φ establishes the equivalence of bases we see that

$$\Big\|\sum_{I\in V_l} h_I\Big\|^{-1} \sum_{I\in V_l} h_{\Phi(I)}$$

is also equivalent to the unit vector basis in ℓ_2. This implies by Lemma C.4 that there exists a $\Delta > 0$ such that $\sum_{I\in V_l} |\Phi(I)| > \Delta$. So for e fixed n and $l \in K_n$ we have

$$\sum_{l\in K_n}\sum_{I\in V_l} |\Phi(I)| \ge \frac{n}{3}\Delta.$$

On the other hand, since each $\Phi(I)$ in the above sum has $|\Phi(I)| > 2^{-k_n}$ and there are very few intervals in A_l for $l \in [k_n, k_n+n]$ we have

$$\sum_{l\in K_n}\sum_{I\in V_l}|\Phi(I)| \le n\cdot\frac{1}{n} + 2^{-k_n-n}\cdot\sum_{l\in K_n}|V_l|$$
$$\le 1 + 2^{-k_n-n}2^{k_n+\frac{2n}{3}+1} = 1 + 2^{-\frac{n}{3}+1}$$

Thus we get $\Delta \le \frac{3}{n} + \frac{1}{n}2^{-\frac{n}{3}+1}$ which is a contradiction. □

I suspect that condition (*) of Proposition C.5 is both necessary and sufficient for the basis $(h_I)_{I\in A}$ to be equivalent to the whole Haar basis. This however I cannot prove. The following Theorem is only a partial result.

Theorem C.6. *Let $B = (k_n)$ be a strictly increasing sequence of natural numbers and let $A \subset \mathcal{J}$ be defined as $A = \bigcup_{n=1}^{\infty}\mathcal{J}_{k_n}$. Then $H_p(A)$ is isomorphic to H_p if and only if $\sup_n(k_{n+1}-k_n) < \infty$.*

In the proof of this Theorem we will need the following

Lemma C.7. *Suppose that the set A is as in the above Theorem and given a natural number r let us define the set A^o as $\bigcup_{n=1}^{\infty}\mathcal{J}_{k_n+r}$. Then the basis $(h_I)_{I\in A}$ is equivalent to a subbasis of $(h_I)_{I\in A^o}$.*

Proof of the Lemma. The map is given as

$$[\frac{s}{2^{k_n}},\frac{s+1}{2^{k_n}}] \mapsto [\frac{s}{2^{k_n+r}},\frac{s+1}{2^{k_n+r}}].$$

Since this map is actually a linear change of variables, one easily checks that it gives an isometric embedding. □

Proof of the Theorem. If $\sup k_{n+1}-k_n = \infty$ then Proposition C.5 clearly implies that $H_p(A)$ is not isomorphic to H_p. To prove the converse implication, it suffices to show that the Haar basis is equivalent to a subbasis of $(h_I)_{I\in A}$. Let $\sup_n k_{n+1}-k_n = K$ Let us fix an integer r such that $0 \le r < K$. For each $s = 0,1,2\dots$ there is at least one n such that $sK + r \le k_n < (s+1)K + r$. Splitting the integers into at most K parts and aplying Lemma C.7 we infer that the basis $(h_I)_{I\in A}$ is (for every r) equivalent to a subbasis of a basis $(h_I)_{I\in A^r}$ where $A^r = \bigcup_{s=1}^{\infty}\mathcal{J}_{sK+r}$. But the same argument shows that the basis $(h_I)_{I\in A^r}$ is equivalent to a subbasis of $(h_I)_{I\in A}$. Thus by Proposition B.11 bases $(h_I)_{I\in A}$ and $(h_I)_{I\in A^r}$ are all equivalent. It is easily seen that the basis $(h_I)_{I\in A^o}$ is equivalent to any of its finite direct sums, so

$$(h_I)_{I\in\mathcal{J}} \sim \sum_{r=0}^{K}(h_I)_{I\in A^r} \sim \sum_{r=0}^{K}(h_I)_{I\in A^o} \sim (h_I)_{I\in A^o} \sim (h_I)_{I\in A}.$$

This proves the Theorem. □

Remark. This Theorem can be generalised a little. Assume that $(k_n)_{n=1}^{\infty}$ is an increasing sequence of integers such that $\sup_n k_{n+1}-k_n < \infty$ and that $A = \bigcup A_{k_n}$ where $A_{k_n} \subset \mathcal{J}_{k_n}$. If $\bigcup_{I\in A_{k_n}} I = S_n$ is an interval of length $> \delta > 0$, then it is easy to see that $(h_I)_{I\in A}$ is equivalent to the Haar system. Simply take intervals from $\mathcal{J}_r$ with $2^{-r} < \delta/2$ and observe that for each S_n there is at least one $I \in \mathcal{J}_r$ contained in S_n. This implies that $(h_I)_{I\in A}$ contains a subbasis equivalent to the Haar system, thus is equivalent to the Haar system.

Also if a set $A \subset \mathcal{J}$ contains a set A_1 such that $(h_I)_{I\in A_1}$ is equivalent to the Haar system, then $(h_I)_{I\in A}$ is equivalent to the Haar system.

The following Proposition allows one to tinker a bit with the set A but seems to be insufficient to decide the general problem.

Proposition C.8. *a) Suppose that the set A satisfies $\phi_n(A) \to 0$. Then the basis $(h_I)_{I\in A}$ is equivalent to a subbasis of the natural basis in the space $(\sum_n H_p^n)_p$.*

b) If $A = B \cup C$, $A \subset \mathcal{J}$, and $H_p(A)$ is isomorphic to H_p and $\phi_n(B) \to 0$ as $n \to \infty$, then $H_p(C)$ is isomorphic to H_p.

Proof of a). This is once more a disjointness argument. Take an increasing sequence of integers such that $\phi_s(A) < 2^{-k_n-2}$ for all $s > k_{n+1}$ and write $A = A_{\text{even}} \cup A_{\text{odd}}$ where

$$A_{\text{even}} = \bigcup_{n=1}^{\infty} \{A_s \ : \ k_{2n} \le s < k_{2n+1}\}$$

$$A_{\text{odd}} = \bigcup_{n=0}^{\infty} \{A_s \ : \ k_{2n+1} \le s < k_{2n+2}.$$

It follows from our choice of k_n that

$$H_p(A_{\text{even}}) \sim \left(\sum_{n=1}^{\infty} H_p(\{A_s \ : \ k_{2n} \le s < k_{2n+1}\})\right)_p$$

and

$$H_p(A_{\text{odd}}) \sim \left(\sum_{n=0}^{\infty} H_p(\{A_s \ : \ k_{2n+1} < s \le k_{2n+2}\})\right)_p ,$$

so we get

$$H_p(A) \sim \left(\sum_{n} H_p(\{A_s \ : \ k_n \le s < k_{n+1}\})\right)_p .$$

This clearly gives the claim. $\square$

In order to prove b) we will need the following Lemma.

Lemma C.9. *For any $B \subset \mathcal{J}$ such that $\phi_n(B) \to 0$ there exists a subset $B_1 \subset \mathcal{J}\backslash B$ such that*

1. *if $I \in B_1$ then $I \subset [0, 1/2]$*
2. *$H_p(B_1)$ is isomorphic to $(\sum_n H_p^n)_p$*
3. *$B_1 \cap B = \emptyset$*

Proof. Since $\phi_s(B) \to 0$ for every integer n we can find an integer k such that

$$\phi_k(B) \cdot 2^{k-1} + \phi_{k+1}(B) \cdot 2^{(k+1)-1} + \cdots + \phi_{k+n}(B) \cdot 2^{(k+n)-1} < 2^{k-1}.$$

Thus we can find at least one s, $0 \le s < 2^{k-1}$ such that no interval from

$$B_k, B_{k+1}, \ldots, B_{k+n}$$

is contained in $[\frac{s}{2^k}, \frac{s+1}{2^k}]$. This means that

$$B_1^n = \{I \;:\; I \subset [\frac{s}{2^k}, \frac{s+1}{2^k}] \text{ and } I \in \bigcup_{r=k}^{k+n} \mathcal{J}_r\} \cap B = \emptyset.$$

Since $H_p(B_1^n)$ is naturally isometric to H_p^n we can continue in this way (taking $k = k(n)$ rapidly increasing) to obtain $B_1 = \bigcup_n B_1^n$ satisfying conditions of the Lemma. $\square$

Proof of b). Since $H_p(A)$ is isomorphic to H_p there exists a 1–1 map $\Phi : A \xrightarrow{\text{onto}} \mathcal{J}$ establishing equivalence of bases. Since $H_p(B)$ does not contain a subspace isomorphic to ℓ_2 from Lemma C.3 we infer that $\phi_n(\Phi(b)) \to 0$. Aplying Lemma C.9 we get a subset $B_1 \subset \mathcal{J}$, $B_1 \cap \Phi(B) = \emptyset$, such that bases $(h_I)_{I\in B_1}$ and $(h_I)_{I\in\Phi(B)}$ are equivalent. Let us say that $\Phi_1 : B_1 \longrightarrow \Phi(B)$ establishes the equivalence. This implies that the map $\Phi_2 : \mathcal{J} \longrightarrow \mathcal{J}$ defined as

$$\Phi_2(I) = \begin{cases} \Phi_1(I), & \text{if } I \in B_1 \\ \Phi_1^{-1}(I), & \text{if } I \in \Phi(B) \\ I, & \text{otherwise} \end{cases}$$

is well defined and establishes an isomorphism of H_p. Thus the map $\Phi_2 \circ \Phi : A \longrightarrow \mathcal{J}$ also establishes the equivalence of bases $(h_I)_{I\in A}$ and $(h_I)_{I\in\mathcal{J}}$. But $\Phi_2 \circ \Phi(C) \supset \{I \in \mathcal{J} \;:\; I \subset [1/2, 1]\}$. This shows that C contains a subbasis equivalent to the Haar basis so is equivalent to Haar basis. $\square$The rest of this section deals with some examples of subbases of the Haar basis. We will deal only with subbases consisting of full levels of the dyadic tree $\mathcal{J}$. For any number $\alpha > 1$ we consider the set

$$A_\alpha = \bigcup_{n=1}^{\infty} \mathcal{J}_{[n^\alpha]}$$

and the corresponding spaces $H_p(A_\alpha)$. We have the following

Theorem C.10. *For every $\alpha > 1$ the space $H_p(A_\alpha)$ ($0 < \alpha < 1$) is not isomorphic to any of its powers. If $\alpha \neq \beta$ the spaces $H_p(A_\alpha)$ and $H_p(A_\beta)$ are not isomorphic.*

Proof. If $H_p(A_\alpha)$ is isomorphic to some of its powers then by Corollary B.13 it is isomorphic to its square and by Theorem B.12 $H_p(A_\alpha)$ has a unique unconditional basis. Thus there exists a map $\Phi \;:\; A_\alpha \oplus A_\alpha \longrightarrow A_\alpha$ establishing the isomorphism. Let us consider all intervals $I \in \mathcal{J}_{[n^\alpha]} \oplus \mathcal{J}_{[n^\alpha]}$ such that $|\Phi(I)| \leq 2^{-[n^\alpha]}$. Since the number of intervals in A_α whose length is at most $2^{-[n^\alpha]}$ does not exceed

$$\sum_{k=1}^{n} 2^{[k^\alpha]} \leq 2^{[n^\alpha]} + n2^{[(n-1)^\alpha]}$$

for each n there is a subset $B_n \subset \mathcal{J}_{[n^\alpha]} \oplus \mathcal{J}_{[n^\alpha]}$ of cardinality at least

$$2^{[n^\alpha]} - n2^{[(n-1)^\alpha]}$$

such that $|\Phi(I)| < 2^{-[n^\alpha]}$ for all $I \in B_n$. Since

$$2^{-[n^\alpha]} \cdot |B_n| \geq 1 - n2^{[(n-1)^\alpha]-[n^\alpha]}$$

and the right hand side tends to 1 as $n \to \infty$, we infer from Lemma C.3 that $\|\sum_{I\in B_n} h_I\|^{-1}\sum_{I\in B_n} h_I$ is (at least for n greater then some integer n_o) equivalent in $H_p(A_\alpha)\oplus H_p(A_\alpha)$ to the unit vector basis in ℓ_2. On the other hand

$$\begin{aligned}\Big|\mathrm{supp}\sum_{I\in B_n} h_{\Phi(I)}\Big| &\leq \sum_{I\in B_n}|\Phi(I)|\\ &\leq 2^{-[(n+1)^\alpha]}\cdot|B+_n| \leq 2\cdot 2^{[n^\alpha]}\cdot 2^{-[(n+1)^\alpha]} \to 0 \text{ as } n \text{ tends to } \infty\end{aligned}$$

By Lemma C.4 this means that $\|\sum_{I\in B_n} h_I\|^{-1}\sum_{I\in B_n} h_{\Phi(I)}$ is not equivalent to the unit vector basis in ℓ_2. This contradicts the assumption that Φ establishes the equivalence of bases and shows that $H_p(A_\alpha)$ is not isomorphic to any of its powers.

Now let us take two real numbers $\beta > \alpha > 1$ and suppose that $H_p(A_\beta)$ is isomorphic to $H_p(A_\alpha)$. It follows from Theorem B.9. that the natural basis in $H_p(A_\alpha)$ is equivalent to a subbasis of the natural basis in $H_p(A_\beta)\oplus H_p(A_\beta)$. Let $\Phi : A_\alpha \xrightarrow{1-1} A_\beta\oplus A_\beta$ establishes the equivalence. It follows from Lemmas C.3 and C.4 that there exists a $\delta > 0$ such that for each natural number N

$$(*) \qquad \sum_{n=1}^{N}\sum_{I\in\mathcal{J}_{[n^\alpha]}}|\Phi(I)| \geq \delta N.$$

But the left hand side sum in (*) does not exceed the sum of lengths of $2\cdot 2^{[N^\alpha]}$ longest intervals in $A_\beta\oplus A_\beta$. This in turn equals at most twice the number of levels in A_β before the cardinality of the level reaches $2\cdot 2^{[N^\alpha]}$, which is at most $CN^{\alpha/\beta}$. Since $\alpha/\beta < 1$ we reach the contradiction. $\square$

Remark. 1. Similar arguments can be applied to other increasing sequences of natural numbers, not only to $[n^\alpha]$. This choice was simply the easiest way to provide a collectiion of continuum non-isomorphic complemented subspaces of H_p.

2. The alternative (although related) proof of the non- isomorphism of $H_p(A_\alpha)$ for different α's can be obtained using the invariant $b_N(X)$ defined in Theorem 3.2. of [KLW]. One gets that $b_N(H_p(A_\alpha)) \approx (\log N)^{\frac{1}{\alpha}(\frac{1}{2}-\frac{1}{p})}$

3. The spaces $H_p(A_\alpha)$ are in a sense big. It is easily seen that they contain a subspace isomorphic to the whole H_p. Such a subspace can not be complemented. We can form a "local" version of spaces $H_p(A_\alpha)$ that is spaces $X_\alpha = (\sum H_p(A^n_\alpha))_p$ where $A^n_\alpha = \bigcup_{k=1}^n \mathcal{J}_{[k^\alpha]}$. Those spaces can also be represented as spaces spanned by a subbasis of the Haar basis. It follows from Proposition B.11 that each X_α is isomorphic to its square. Since $b_N(H_p(A_\alpha)) = b_N(X_\alpha)$ we infer that X_α's are mutually non-isomorphic.

Given an increasing sequence of natural numbers $(k_n)_{n=1}^\infty$ let us form two subsets of $\mathcal{J}$ namely

$$A_1 = \bigcup_n \bigcup_{s=k_{2n+1}+1}^{k_{2n+2}} \mathcal{J}_s$$

and

$$A_2 = \bigcup_n \bigcup_{s=k_{2n}+1}^{k_{2n+1}} \mathcal{J}_s$$

and let us consider corresponding spaces $H_p(A_1)$ and $H_p(A_2)$. Clearly $H_p = H_p(A_1) \oplus H_p(A_2)$. If we assume additionally that $k_{n+1} - k_n \to \infty$ we infer from Proposition C.5 that neither $H_p(A_1)$ nor $H_p(A_2)$ are not isomorphic to H_p. Let us concentrate our attention on the family of sequences $k_n^{(\alpha)} = [n^\alpha]$ for $n = 1, 2, \ldots$, and denote the corresponding sets by A_1^α and A_2^α. Using arguments analogous to the arguments used to prove Theorem C.10 above we can show that spaces $H_p(A_i^\alpha)$ are pairwise non- isomorphic for $i = 1, 2$ and $\alpha > 1$. Thus H_p admits a continuum of different splitings into a direct sum of two subspaces. In the above examples those subspaces are non-isomorphic. It is unknown if there exists a space X non-isomorphic to H_p such that $X \oplus X$ is isomorphic to H_p.

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