On highly nonintegrable functions and homogenous polynomials

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Abstract

In this note we construct a sequence of homogenous polynomials on the unit ball $\mathbb{B}_d$ in $\mathbb{C}^d$ which are big in each point of the unit sphere $\mathbb{S}$. As an application we construct a holomorphic function on $\mathbb{B}_d$ which is not integrable in any power on any complex subspace.

1 Introduction

Let $\mathbb{S}$ denotes the unit sphere in the complex space $\mathbb{C}^d$. In the paper [5] a sequence $(p_n(z))_{n=0}^{\infty}$ of homogenous polynomials in $\mathbb{C}^d$ was constructed such that $|p_n(z)| \leq 1$ for all n and all $z \in \mathbb{S}$ and such that $\int_{\mathbb{S}} |p_n(z)|^2 \, d\sigma(z) \geq c > 0$ for all n. Such polynomials can be used to produce holomorphic functions in $\mathbb{B}_d$ (the unit ball of $\mathbb{C}^d$) with "bad behaviour" on almost all slices cf. [5] Remark 1.10. The restriction "almost all" to is caused by the fact that each $p_n(z)$ has zeros on $\mathbb{S}$ (unless $d = 1$ which is a trivial case) and to conclude something on all slices one has to control the location of the sets where

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$p_n(z)$ is small. On the other hand from the function theory point of view it is interesting to have results for all slices (see e.g. [2]). In this note we construct a sequence of homogenous polynomials which allows us to control behaviour on all slices. Our arguments in this note are modifications of some arguments from [5] and [7] and [1]. As an application we construct a holomorphic function in the unit ball $\mathbb{B}_d$ which is not integrable in any power on any slice.

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1.1 Geometric notions

In the complex d–dimensional space $\mathbb{C}^d$ we will always consider the natural scalar product $\langle .,. \rangle$. On the unit sphere $\mathbb{S}$ we will consider a unitary invariant pseudo-metric $\rho(z_1, z_2)$ defined as

$$\rho(z_1, z_2) =: \sqrt{1 - |\langle z_1, z_2 \rangle|}. \tag{1}$$

It is clear that $\rho(z_1, z_2) = 0$ if and only if $z_1 = \lambda z_2$ for some $\lambda \in \mathbb{C}$ (and clearly $|\lambda| = 1$). As usual by $B(z; r)$ we will denote the open ball with center z and radius r i.e.

$$B(z_0; r) =: \{z \in \mathbb{S} \, : \, \rho(z_0, z) < r\}.$$

There is a natural, unitary invariant (Lebesgue) measure on $\mathbb{S}$. We will normalize it so that the measure of the whole sphere $\mathbb{S}$ equals 1 and we will denote this measure by σ. Using (1.4.5) of [4] we easily compute that

$$\sigma(B(z; r)) = (2r^2 - r^4)^{d-1}. \tag{2}$$

This clearly gives

$$r^{2d-2} \leq \sigma(B(z; r)) \leq 2^{d-1} r^{2d-2}. \tag{3}$$

Clearly for small r's the constant on the right hand side can be made as close to 1 as we wish. A subset $A \subset \mathbb{S}$ is called *α–separated* if $\rho(z_1, z_2) > \alpha$ for all distinct elements z_1 and z_2 of A. It is clear that for $\alpha > 0$ each α–separated subset of $\mathbb{S}$ is finite. We will talk about maximal α–separated sets. We always mean maximal in terms of inclusion of sets.

2 Some homogenous polynomials

All homogenous polynomials of degree n constructed in this paper will have a form

$$p(z) = \sum_{j=1}^{s} \langle z, \zeta_j \rangle^n \tag{4}$$

for some some finite subset $\{\zeta_1, \dots, \zeta_s\}$ of $\mathbb{S}$. In order to be able to control values of the polynomial p we will usually assume that the set $\{\zeta_1, \dots, \zeta_s\}$ is α–separated for some α. The natural and useful degree of separation for polynomials of degree n is $\frac{1}{\sqrt{n}}$. We will start with two Lemmas on separated sets.

Lemma 1 *Suppose that $\{\zeta_1, \dots, \zeta_s\}$ is a $\frac{C}{\sqrt{N}}$–separated subset of $\mathbb{S}$. For any given $z \in \mathbb{S}$ let*

$$A_k(z) =: \Big\{ i \,:\, \frac{kC}{2\sqrt{N}} \le \rho(z, \zeta_i) \le \frac{(k+1)C}{2\sqrt{N}} \Big\}.$$

Then for $k = 1, 2, \dots$ the set $A_k(z)$ has at most $2^{d-1}(k+2)^{2d-2}$ elements. The set $A_0(z)$ has at most one element.

PROOF. The assertion about A_0 is clear. Since balls $B(\zeta_j; \frac{C}{2\sqrt{N}})$ are disjoint and

$$\bigcup_{i \in A_k(z)} B(\zeta_i; \tfrac{C}{2\sqrt{N}}) \subset B(z; \tfrac{(k+2)C}{2\sqrt{N}})$$

we get

$$\begin{aligned}
\# A_k(z) &\le \#\{i \,:\, \rho(z, \zeta_i) < \tfrac{(k+1)C}{2\sqrt{N}}\} \\
&\le \frac{\sigma(B(z, \frac{(k+2)C}{2\sqrt{N}}))}{\sigma(B(z, \frac{C}{2\sqrt{N}}))} \\
&\le \frac{2^{d-1}\left(\frac{(k+2)C}{2\sqrt{N}}\right)^{2d-2}}{\left(\frac{C}{2\sqrt{N}}\right)^{2d-2}} = 2^{d-1}(k+2)^{2d-2}.
\end{aligned}$$

•

Lemma 2 *Suppose that we have an $\frac{\alpha}{\sqrt{N}}$–separated set $A \subset \mathbb{S}$. For each $\beta > \alpha$ there exists $K = K(\alpha, \beta)$ such that the set A can be partitioned into K disjoint $\frac{\beta}{\sqrt{N}}$–separated sets.*

PROOF. Let us select from A a maximal $\frac{\beta}{\sqrt{N}}$–separated subset A_1. Next from $A \setminus A_1$ let us select a maximal $\frac{\beta}{\sqrt{N}}$–separated subset A_2. We continue in this way till we exhaust A. Let A_s be the last non–empty set in this procedure. Take $\zeta \in A_s$. Since A_{s-1} is a maximal $\frac{\beta}{\sqrt{N}}$– separated subset of $A \setminus \bigcup_{j=1}^{s-2} A_j$ we see that $\zeta \notin A_{s-1}$, so $B(\zeta; \frac{\beta}{\sqrt{N}}) \cap A_{s-1} \neq \emptyset$. Analogously we see that $B(\zeta; \frac{\beta}{\sqrt{N}}) \cap A_{s-2} \neq \emptyset$ etc. So we see that $B(\zeta; \frac{\beta}{\sqrt{N}})$ contains at least s distinct elements of A. Looking at measures of balls like in Lemma 1 we see that $B(\zeta; \frac{\beta+\alpha/2}{\sqrt{N}})$ contains s disjoint balls of radius $\frac{\alpha}{2\sqrt{N}}$. From 3 we obtain

$$s \left(\frac{\alpha}{2\sqrt{N}}\right)^{2d-2} \leq 2^{d-1} \left(\frac{\beta+\alpha/2}{\sqrt{N}}\right)^{2d-2}$$

so $s \leq 2^{3d-3}(\frac{\beta}{\alpha} + \frac{1}{2})^{2d-2}$. This gives the required decomposition. •

Now we are ready to state some estimates about polynomials 4.

Proposition 1 *There exists a constant C (rather big) such that for each $\frac{C}{\sqrt{N}}$–separated subset $\{\zeta_1, \ldots, \zeta_s\}$ of $\mathbb{S}$ and each integer k, $N \leq k \leq 2N$ the polynomial*

$$p(z) =: \sum_{j=1}^{s} \langle z, \zeta_j \rangle^k$$

satisfies

1. *$|p(z)| \leq 2$ for all $z \in \mathbb{S}$*
2. *$|p(z)| \geq 0.5$ for each $z \in \mathbb{S}$ such that $\rho(z, \zeta_j) \leq \frac{1}{4\sqrt{N}}$ for some $j = 1, 2, \ldots, s$*

provided N is big enough.

PROOF. Note that if $\rho(z, \zeta_j) \geq \frac{\alpha}{\sqrt{N}}$ and $N \leq k \leq 2N$ then we have

$$|\langle z, \zeta_j \rangle^k| \leq (1 - \tfrac{\alpha^2}{N})^k \leq e^{-\frac{\alpha^2 k}{N}} \leq e^{-\alpha^2}. \tag{5}$$

Let us consider sets $A_k(z)$ defined in Lemma 1. From Lemma 1 we obtain

$$\begin{aligned} |p(z)| &\leq \sum_{j=1}^{s} |\langle z, \zeta_j \rangle|^k \\ &\leq \sum_{k=0}^{\infty} \sum_{i \in A_k(z)} |\langle z, \zeta_i \rangle|^k \\ &\leq 1 + \sum_{k=1}^{\infty} e^{-(kC/2)^2} 2^{d-1} (k+2)^{2d-2}. \end{aligned}$$

It is clear that we can fix a $C > 0.5$ and such that

$$\sum_{k=1}^{\infty} e^{-(kC/2)^2} 2^{d-1}(k+2)^{2d-2} \leq 0.1.$$

Such choice of C clearly ensures 1.

For a fixed j and $z \in \mathbb{S}$ such that $\rho(z, \zeta_j) < \frac{1}{4\sqrt{N}}$ we have for $i \neq j$

$$\rho(z, \zeta_i) \geq \tfrac{C}{\sqrt{N}} - \tfrac{1}{4\sqrt{N}} \geq \tfrac{1}{4\sqrt{N}}. \tag{6}$$

This shows that

$$|\langle z, \zeta_j \rangle^k| \geq (1 - \tfrac{1}{16N})^k \geq (1 - \tfrac{1}{16N})^{2N}$$

so for N big enough we have

$$|\langle z, \zeta_j \rangle^k| \geq \left(\tfrac{1}{3}\right)^{1/8} \geq 0.87. \tag{7}$$

Analogously like in the argument for 1 we see from 6 that

$$\sum_{i \neq j} |\langle z, \zeta_i \rangle^k| \leq \sum_{k=1}^{\infty} \sum_{i \in A_k(z)} |\langle z, \zeta_i \rangle^k| \leq 0.1 \tag{8}$$

Since

$$|p(z)| \geq |\langle z, \zeta_j \rangle^k| - \sum_{i \neq j} |\langle z, \zeta_i \rangle^k|$$

from 7 and 8 we obtain 2. •

Now we are ready for the main technical result of this note.

Theorem 1 *There exists an integer $k = k(d)$ and a sequence $p_n(z)$ of homogenous polynomials of degree n (for n big enough) such that*

1. $|p_n(z)| \leq 2$ *for all* $z \in \mathbb{S}$
2. *for each s (big enough)* $\sum_{n=ks}^{k(s+1)-1} |p_n(z)| \geq 0.5$ *for all* $z \in \mathbb{S}$.

PROOF. Let k be an integer given by Lemma 2 for $\alpha = 0.25$ and $\beta = C$ where C is the constant given by Proposition 1. For $N = sk$ (and such that Proposition 1 holds) let us fix a maximal $\frac{1}{4\sqrt{N}}$–separated subset $A \subset \mathbb{S}$

and using Lemma 2 let us divide it into k disjoint $\frac{C}{\sqrt{N}}$–separated subsets $A_0, A_1, \ldots A_{k-1}$. For $n = sk + j$ we define

$$p_n(z) =: \sum_{\zeta \in A_j} \langle z, \zeta \rangle^n .$$

From Proposition 1 we infer that $|p_n(z)| \leq 2$ (so 1 holds) and $|p_n(z)| \geq 0.5$ for

$$z \in \bigcup_{\zeta \in A_j} B(\zeta; \frac{1}{4\sqrt{N}}).$$

Since $A = \bigcup_{l=0}^{k-1} A_l$ is a maximal $\frac{1}{4\sqrt{N}}$–separated subset of $\mathbb{S}$ we infer that

$$\bigcup_{j=0}^{k-1} \bigcup_{\zeta \in A_j} B(\zeta; \tfrac{1}{4\sqrt{N}}) = \bigcup_{\zeta \in A} B(\zeta; \tfrac{1}{4\sqrt{N}}) = \mathbb{S}$$

This gives 2. •

Remark 1. The sets A_j used in the above proof need not be maximal $\frac{C}{\sqrt{N}}$–separated subsets of $\mathbb{S}$. If we enlarge them to sets $\tilde{A}_j$, maximal $\frac{C}{\sqrt{N}}$–separated subsets of $\mathbb{S}$ then there are signs ε_ζ^n such that polynomials

$$\tilde{p}_n(z) = \sum_{\zeta \in \tilde{A}_j} \varepsilon_\zeta^n \langle z, \zeta \rangle^n$$

will satisfy

$$\int_{\mathbb{S}} |\tilde{p}_n(z)|^2 \, d\sigma(z) > c > 0$$

for all n and some C. This follows from arguments following Lemma 2.7 from [5]. Clearly those polynomials will also satisfy 1 and 2 of Theorem 1

Remark 2. The possibility of generalizing arguments from [5] to yield results like our Theorem 1 was known to A. B. Aleksandrov. In his paper [1] he formulates a result (Theorem 4) that there is a K (dependent only on dimension d) such that for each n there are homogenous polynomials of degree n, $p_n^s(z)$ where $s = 1, 2, \ldots, K$ such that for some constants $C \geq c > 0$ we have $C \geq \sum_{s=1}^K |p_n^s(z)| \geq c > 0$ for all $s \in \mathbb{S}$. It is easy to modify our proof of Theorem 1 to get this fact.

3 An application

As an easy application of Theorem 1 let us show the following fact:

The function

$$\sum_n n^{\ln n} p_n(z) =: f(z)$$

is a holomorphic function in $\mathbb{B}_d$ *such that for each hyperplane* $\Pi \subset \mathbb{C}^d$ *and any* $p > 0$

$$\int_{\Pi \cap \mathbb{B}_d} |f(z)|^p \, d\nu(z) = \infty \tag{9}$$

where $d\nu$ *is the volume measure on* $\Pi \cap \mathbb{B}_d$.

Since $|p_n(z)| \leq 2|z|^n$ and the series $\sum n^{\ln n}|z|^n$ converges for $|z| < 1$ we see that $f(z)$ is a holomorphic function in $\mathbb{B}_d$. Since f is a holomorphic function we easily see that 9 is equivalent to

$$\int_{z \in \Pi,\, 0.5 < |z| < 1} |f(z)|^p \, d\nu(z) = \infty. \tag{10}$$

Writing 10 in polar coordinates (see e.g. 1.4.3. in [4]) we see that in order to show 9 it suffices to consider only Π equal to complex lines. It is also clear that only small p's matter. Thus we must show that for each $w \in \mathbb{S}$ and each $1 > p > 0$ the function $g_w(\lambda) =: f(\lambda w)$ defined for $\lambda \in \mathbb{C}$ and $|\lambda| < 1$ satisfies

$$\int_{|\lambda|<1} |g_w(\lambda)|^p d\nu(\lambda) = \infty. \tag{11}$$

But it is known (cf. [3] or [6]) that if a function on the unit disc $g(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ satisfies

$$\int_{|\lambda|<1} |g(z)|^p d\nu(\lambda) < \infty$$

then

$$|a_n| = o(n^{\frac{2}{p}-1}). \tag{12}$$

But $g_w(\lambda)$ has the power series expansion

$$g_w(\lambda) = \sum_n n^{\ln n} p_n(w) \lambda^n$$

so we infer from Theorem 1 than 12 does not hold. This shows our claim.

This example improves a bit upon Theorem 1 from [2]

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