ON UNCONDITIONAL POLYNOMIAL BASES IN L_p AND BERGMAN SPACES

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Abstract. In this paper we consider unconditional bases in $L_p(\mathbf{T})$, $1 < p < \infty$, $p \neq 2$ consisting of trigonometric polynomials. We give a lower bound for the degree of polynomials in such a basis (Theorem 3.4.) and show that this estimate is best possible. This is applied to the Littlewood-Paley type decompositions. We show that such a decomposition has to contain exponential gaps. We also consider unconditional polynomial bases in H_p as bases in Bergman type spaces and show that they provide explicit isomorphisms between Bergman type spaces and natural sequences spaces.

Section 1. Introduction

In this paper we consider unconditional polynomial (trigonometric) bases in spaces $L_p(\mathbf{T})$, $1 < p < \infty$, $p \neq 2$. It is well known (see e.g. [W1] II.D.9.) that the system $(e^{int})_{n=-\infty}^{\infty}$ is not an unconditional basis in $L_p(\mathbf{T})$ when $p \neq 2$. On the other hand it is known (cf. e.g. [W1] II.D.13.) that $L_p(\mathbf{T})$ has an unconditional basis when $1 < p < \infty$, so by well known stability results (cf. e.g. [W1] II.B.15.) there exists an unconditional basis consisting of trigonometric polynomials. One would like to have some control over the degree of those polynomials. Let $(\psi_k)_{k=0}^{\infty}$ be any sequence of trigonometric polynomials. Denote

$$v_n = v_n(\psi_k) = \sup\{\deg \psi_k : 0 \leq k \leq 2n\}.$$

If $(\psi_k)_{k=0}^{\infty}$ is a basis in $L_p(\mathbf{T})$ (or is merely linearly independent) we easily see that $v_n(\psi_k) \geq n$. Since the system $(e^{ikt})_{k=-\infty}^{\infty}$ in the natural order i.e. 1, e^{it}, e^{-it}, e^{2it}, $e^{-2it}, \ldots,$ is a basis in $L_p(\mathbf{T})$ for $1 < p < \infty$ the lower estimate $v_n \geq n$ is clearly the best possible for bases in $L_p(\mathbf{T})$. The problem of constructing unconditional polynomial (even orthogonal) bases in $L_p(\mathbf{T})$ for $1 < p < \infty$, $p \neq 2$ with good upper estimates for v_n was intensively studied by Z.A. Chanturija and G.E. Tkebuchava (see [Ch] for references) in the seventies when estimates $v_n \leq n^{1+\epsilon}$ were proved. The linear upper estimate for v_n was obtained by K. Woźniakowski and the present author (see [Woz] and [W-W]). In [Woz] K. Woźniakowski has constructed for each $\epsilon > 0$ an unconditional orthogonal polynomial basis in $L_p(\mathbf{T})$ with $v_n < (1+\epsilon)n$. As far as I know the question of lower estimates for v_n in the case of unconditional polynomial basis in $L_p(\mathbf{T})$ was first raised in the literature in [W-W] where an argument of A. Pełczyński is presented showing that $v_n > n$ infinitely often. The

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main result of this note is Theorem 3.4. saying that $\limsup_{n\to\infty}\frac{v_n}{n} > 1$. We also give a natural example which shows that on can have $v_n = n$ infinitely often.

Those results answer questions asked at the end of Section 5 of [W-W]. (The reader who wishes to consult [W-W] should be cautioned that the definition of v_n given in [W-W] differs slightly from the one used in the present paper.) The above mentioned results are presented in Section 3. In Section 2 we collected some definitions and known facts for future use.

In Section 4 we examine the behaviour of polynomial bases in Bergman spaces. The fact is that the polynomial bases constructed in [W-W] and [Woz] give rise to natural unconditional, polynomial bases in $H_p(\mathbf{D})$. This is discussed in detail in those papers. In this paper we prove a general result which shows that such bases are also unconditional in appropriate Bergman spaces and actually provide explicit isomorphisms between Bergman spaces and ℓ_p spaces. Actually the argument is presented in greater generality so, in particular, we solve the problem asked by M. Mateljevič and M. Pavlovič in [M-P]. We also remedy a somewhat paradoxical situation which existed in the literature. For simplicity let us discuss here only spaces

$$B_p(\mathbf{D}) = \{f(z) \text{ analytic for } |z|<1 \text{ and such that } \int_{\mathbf{D}} |f(z)|^p < \infty\}.$$

Those spaces are known to be isomorphic to ℓ_p for $0 < p < \infty$ (see [L-P] for $p \geq 1$ and [Roch] for $p < 1$) but the isomorphisms given by those authors were not constructive. In volume 77 of Studia Mathematica two papers appeared. In [W2] an explicit isomorphisms were given for $p \leq 1$ but it was unclear if the same isomorphisms work for $p > 1$. On the other hand in [M-P] explicit isomorphisms were given in the case $p > 1$ but they clearly do not work for $p \leq 1$. Now we are able to give isomorphisms which work for all $p > p_o$ where p_o can be taken as close to 0 as one wishes.

For the sake of historical accuracy I would like to acknowledge that in November of 1991 the paper [Pr] appeared where A.A. Privalov construct polynomial bases in $C(\mathbf{T})$ similar to ones considered in [W-W]. He does not however, consider the unconditional convergence in $L_p(\mathbf{T})$ nor in H_p.

The research reported in this paper was done while I visited Department of Mathematics of University of Missouri at Columbia. I would like to thank Nigel Kalton for arranging this visit and the whole Modern Analysis Group of this Department for their warm hospitality which made my stay in Columbia extremely pleasant. I would also thank Bill Johnson for helpful e-mail conversations about Lemma 3.1.

Section 2

Everywhere in this paper $\mathbf{T}$ will denote the unit circle in the complex plain and $\int_{\mathbf{T}} f(t)\,dt$ will always denote the integral with respect to the Lebesgue measure normalised so that the measure of the whole circle is 1. We will often identify $\mathbf{T}$ with the interval $[0, 2\pi)$. The symbol $L_p(\mathbf{T})$ will (as usual) denote the space of p-integrable functions with the norm

$$\|f\|_p = \big(\int_{\mathbf{T}} |f(t)|^p\,dt\big)^{1/p}.$$

By a polynomial we will mean a finite sum $\psi(t) = \sum_{n\in A} a_n e^{int}$ and the degree of such polynomial will be $\deg\psi = \max\{|n| \, : \, a_n \neq 0\}$. The space of all polynomials of degree $\leq n$ will be denoted by W_n. This notation will be used sometimes even when n is not an integer. It is obvious that $\dim W_n = 2[n] + 1$.

We will also consider the space H_p for $0 < p < \infty$ which can be defined as the closed linear span of the sequence $(e^{int})_{n=0}^{\infty}$. It is well known (cf. [Bo] or [W1] II.B.Ex 9) that the map $e^{int} \mapsto e^{2int}$ when $n \geq 0$ and $e^{int} \mapsto e^{-i(2n+1)t}$ when $n < 0$ extends by linearity to an isomorphism between $L_p(\mathbf{T})$ and H_p when $1 < p < \infty$.

We will also use the notion of de la Vallée Poussin kernel. Let us recall that the n-th de la Vallée Poussin kernel $V_n(t)$ is defined as $V_n(t) = \sum_{-2^{n+1}}^{2^{n+1}} a_k e^{ikt}$ where

$$a_k = \begin{cases} 1, & \text{for } |k| \leq 2^n \\ 1 - 2^{-n}(|k| - 2^n), & \text{for } 2^n \leq |k| \leq 2^{n+1}. \end{cases}$$

Since the de la Vallée Poussin kernels can be easily expressed as linear combinations of more classical Fejer kernels we have $\int_{\mathbf{T}} |V_n(t)|\, dt \leq 3$ and $|V_n(t)| \leq C \min(2^n, 2^{-n}t^{-2})$. This implies that the convolution with V_n define an operator on $L_p(\mathbf{T})$ for $1 \leq p \leq \infty$ with norm at most 3. It is also clear that for $f \in W_{2^m}$ and $n \geq m$ we have $f * V_n = f$. All this can be found in many textbooks, in particular in [Z], [T] or [K].

We will also use some notions from the theory of Banach spaces. If X and Y are two isomorphic Banach spaces (in particular two finite dimensional Banach spaces of the same dimension) we define the Banach-Mazur distance d(X,Y) by

$$d(X, Y) = \inf\{\|T\| \cdot \|T^{-1}\| \, : \, T : X \longrightarrow Y \text{ is an isomorphism } \}.$$

A sequence of vectors $(x_j)_{j=0}$ in a Banach space X is called a basis if every element $x \in X$ has a unique representation as $x = \sum_{j=0} a_j x_j$ where a_j's are scalars. *The sign = in the above means that the series converges in the norm of the space.* The basis is called *unconditional* if every convergent series of the form $\sum_{j=0} a_j x_j$ actually converges unconditionally. A basis is called *normalised* if $\|x_j\| = 1$ for all j's. It is known that with each basis one can associate partial sum projections defined as $P_n(\sum_{j=0} a_j x_j) = \sum_{j=0}^{n} a_j x_j$. The basis constant of the basis, denoted as $\mathrm{bc}(x_j)$, is defined as $\mathrm{bc}(x_j) = \sup\{\|P_n\| \, : \, n = 0, 1, \ldots\}$. It is known that this quantity is finite. Analogously with an unconditional basis one can associate projections as follows: for any subset A of the set of indices of the basis we define $P_A(\sum_{j=0} a_j x_j) = \sum_{j\in A} a_j x_j$. The unconditional basis constant, denoted $\mathrm{ubc}(x_j)$, is defined as $\mathrm{ubc}(x_j) = \sup_A \|P_A\|$. If (x_j) is an unconditional basis then $\mathrm{ubc}(x_j) < \infty$.

All this is explain in some detail in [W1] and very carefully in [NTJ] and in many other books.

We will use the standard convention that C denotes a constant which may vary from one occurrence to the other, even within the same calculation. Also, for a number p, $1 < p < \infty$, p' will always denote its conjugate exponent defined by $\frac{1}{p} + \frac{1}{p'} = 1$.

To conclude this preliminary section let us remark that the terms orthonormal or orthogonal will always refer to orthogonality in $L_2(\mathbf{T})$.

Section 3.

The following Lemma (at least for subspaces of ℓ_p^n) is basically known to the specialists. I was unable, however, to find any explicit reference, so I decided to give the proof.

Lemma 3.1. *There exists a constant C such that for any α with $0 < \alpha \leq 1$ and any subspace X, $X \subset W_n \subset L_p(\mathbf{T})$ where $2 < p < \infty$ with* $\dim X = k \geq \alpha(2n+1)$ *we have $d(X, \ell_2^k) \geq C\alpha(2n+1)^{\frac{1}{2}-\frac{1}{p}}$.*

Proof. It is well known (cf. [M-Z] Theorem 1 and Theorem 7 and remarks after its proof) that there exists a constant C such that for every p, $1 \leq p \leq \infty$ and any $f \in W_n$ we have
(3.1)
$$C^{-1}\Big(\frac{1}{4n}\sum_{j=0}^{4n-1}|f(\frac{2\pi j}{4n})|^p\Big)^{1/p} \leq \Big(\int_{\mathbf{T}}|f(t)|^p\,dt\Big)^{1/p} \leq C\Big(\frac{1}{4n}\sum_{j=0}^{4n-1}|f(\frac{2\pi j}{4n})|^p\Big)^{1/p}.$$

Let E denotes the set $\{f(\frac{2\pi j}{4n}) \ : \ f \in X\} \subset \mathbf{C}^{4n}$. Clearly this is a linear subspace of $\mathbf{C}^{4n}$. When we equip E with the norm $\big(\frac{1}{4n}\sum_{j=0}^{4n-1}|\beta_j|^p\big)^{1/p}$ for $1 \leq p \leq \infty$ we denote it by E_p. Corollary 1.3 of [F-J] gives that $\mathrm{d}(E_\infty, \ell_2^k) \geq \frac{\alpha}{2}\sqrt{n}$. Now we have

$$\begin{aligned}
d(E_\infty, \ell_2^k) &\leq d(E_\infty, E_p)\cdot d(E_p, \ell_2^k) \\
&\leq \|id : E_\infty \to E_p\| \cdot \|id : E_p \to E_\infty\| \cdot d(E_p, \ell_2^k) \\
&\leq \|id : \ell_\infty^{4n} \to \ell_p^{4n}\| \cdot \|id : \ell_p^{4n} \to \ell_\infty^{4n}\| \cdot d(E_p, \ell_2^k) \\
&\leq (4n)^{1/p} d(E_p, \ell_2^k)
\end{aligned} \tag{3.2}$$

so comparing (3.1) and (3.2) we get

$$d(E_p, \ell_2^k) \geq \frac{\alpha}{2}4^{-1/p}n^{\frac{1}{2}-\frac{1}{p}} \geq \alpha n^{\frac{1}{2}-\frac{1}{p}}. \tag{3.3}$$

Since $d(X, E_p) \leq C^2$ we have $d(X, \ell_2^k) \geq C^{-2}\alpha n^{\frac{1}{2}-\frac{1}{p}}$. This proves the Lemma.

□

Lemma 3.2. *There exists a constant $C = C(K, p, \alpha)$ such that if $(\psi_j)_{j=1}^n \subset W_m \subset L_p(\mathbf{T})$ with $2 < p < \infty$ is a normalised sequence with $ubc(\psi_j)_{j=1}^n \leq K$ and $\frac{n}{m} \geq \alpha$, then there exists a set $\mathcal{J} \subset \{1, 2, \ldots, n\}$ with $|\mathcal{J}| \geq \frac{n}{2}$ such that*

$$\|\psi_j\|_2 \leq Cn^{\frac{1}{p}-\frac{1}{2}} \text{ for } j \in \mathcal{J}.$$

Proof. Let $r = [\frac{n}{2}]$ and let $(\phi_j)_{j=1}^r$ denote arbitrary subsequence of the sequence $(\psi_j)_{j=1}^n$. It suffices to show that $\min_j \|\phi_j\|_2 \leq Cn^{\frac{1}{p}-\frac{1}{2}}$. On $X = \mathrm{span}(\phi_j)_{j=1}^r$ we introduce a new euclidean norm by $|||\sum_{j=1}^r \alpha_j\phi_j||| = \big(\sum_{j=1}^r |\alpha_j|^2\|\phi_j\|_2^2\big)^{1/2}$. We have

$$|||\sum_{j=1}^{r}\alpha_j\phi_j||| = \Big(\int_{\mathbf{T}}\sum_{j=1}^{r}|\alpha_j|^2|\phi_j(t)|^2dt\Big)^{1/2} \tag{3.4}$$
$$= \Big(\int_{\mathbf{T}}\int_0^1 \big|\sum_{j=0}^{r} r_j(s)\alpha_j\phi_j(t)\big|^2\,ds\,dt\Big)^{1/2}$$
$$\leq \Big(\int_{\mathbf{T}}\int_0^1 \big|\sum_{j=0}^{r} r_j(s)\alpha_j\phi_j(t)\big|^p\,ds\,dt\Big)^{1/p}$$
$$\leq \mathrm{ubc}(\phi_j)\cdot\|\sum_{j=0}^{r}\alpha_j\phi_j\|_p$$

From Lemma 1, the definition of the Banach-Mazur distance and (3.4) we infer that there exists $\sum_{j=0}^{r}\bar{\alpha}_j\phi_j \in X$ such that $\|\sum_{j=0}^{r}\bar{\alpha}_j\phi_j\|_p = 1$ but

$$|||\sum_{j=0}^{r}\bar{\alpha}_j\phi_j||| \leq KC2\alpha(2m+1)^{\frac{1}{p}-\frac{1}{2}}. \tag{3.5}$$

Since $L_p(\mathbf{T})$ has type 2 (cf. e.g. [W1] III.A.23.) we infer that

$$1 \leq \|\sum_{j=0}^{r}\bar{\alpha}_j\phi_j\|_p \leq C_p\Big(\sum_{j=0}^{r}|\bar{\alpha}_j|^2\Big)^{1/2} \tag{3.6}$$

so from (3.6) and (3.5) we get

$$\min_j\|\phi_j\|_2 \leq C_p\min_j\|\phi_j\|_2\Big(\sum_j|\bar{\alpha}_j|^2\Big)^{1/2} \leq C_p\Big(\sum_j|\bar{\alpha}_j|^2\|\phi_j\|_2^2\Big)^{1/2}$$
$$= C_p|||\sum_j\bar{\alpha}_j\phi_j||| \leq KC2\alpha(2m+1)^{\frac{1}{p}-\frac{1}{2}} \leq C(K,p,\alpha)n^{\frac{1}{p}-\frac{1}{2}}.$$

□

The above Lemma is also not surprising to the specialists. A more precise result of the similar nature is proved in [BKT] Theorem 2.5. It is possible to derive our Lemma 3.2 from the above mentioned result of J. Bourgain, N. Kalton and L. Tzafriri. Their result is much deeper so, although I was motivated by it during my work on Theorem 3.4, I decided to give a more direct proof of Lemma 3.2.

Lemma 3.3. *If $p > 2$ and $f = \sum_{j\in A} a_j e^{ijt}$ then*

$$\|f\|_p \leq |A|^{\frac{1}{2}-\frac{1}{p}}\|f\|_2.$$

Proof. From the Hausdorff-Young inequality (cf. [W] III.G.22. or [Z] XII.2.3. or [T] VI.1.5.) we get $\|f\|_p \leq \left(\sum_{j\in A}|a_j|^{p'}\right)^{\frac{1}{p'}}$, so applying Hölder's inequality for $\beta = \frac{2}{p'}$, we get

$$\|f\|_p \leq |A|^{\frac{1}{\beta' p'}}\left(\sum_{j\in A}|a_j|^2\right)^{\frac{1}{2}} = |A|^{\frac{1}{2}-\frac{1}{p}}\|f\|_2.$$

□

The above Lemmas will be used in the proof of the following Theorem.

Theorem 3.4. *Let $(\varphi_j)_{j=0}^{\infty}$ be a normalised, unconditional, polynomial basis in $L_p(\mathbf{T})$ for $1<p<\infty$, $p\neq 2$. Then* $\limsup_{n\to\infty}\frac{v_n}{n}>1$.

Proof. It is an easy and well known consequence of Riesz theorem (cf. [K] or [Z]) that for each p, $1<p<\infty$ there exists a constant R_p such that

$$\|\sum_{j<k}a_je^{ijt}+\sum_{j>s}a_je^{ijt}\|_p\leq R_p\|\sum_{j=-\infty}^{\infty}a_je^{ijt}\|_p$$

for all integers $k<s$ and all (finite) sequences of scalars $(a_j)_{j=-\infty}^{\infty}$ Let us denote $K=\operatorname{ubc}(\varphi_j)_{j=0}^{\infty}$, $X_n=\operatorname{span}(\varphi_j)_{j\leq n}$ and let $(\varphi_j^*)_{j=0}^{\infty}$ denote the sequence of biorthogonal functionals. Let K_1 denote $\sup\|\varphi_j^*\|$.

Now we will consider two cases.

Case 1), $p>2$. Let D denotes the constant $C(K,p,1/2)$ from Lemma 3.2. Let us fix an integer k, $k>8KR_p$ and $\varepsilon>0$ such that $D(4(1+\varepsilon)k\varepsilon)^{\frac12-\frac1p}<\frac{1}{2K_1}$. Note that this implies $2k\varepsilon<1$. With those choices we will show that $\limsup_{n\to\infty}\frac{v_n}{n}\geq(1+\frac{\varepsilon}{2})$. If this is not the case we fix N such that for all $2N\leq n\leq 4N$ we have $v_n<(1+\varepsilon)n$. Let us consider $\varphi_{2N+1},\varphi_{2N+2},\dots,\varphi_{4N}$. From Lemma 3.2 applied to the sequence $\varphi_{2N+1},\dots,\varphi_{4N}$ we infer that there exists a set $\mathcal{J}\subset\{2N+1,\dots,4N\}$ such that $|\mathcal{J}|>N$ and such that $\|\varphi_j\|_2\leq DN^{\frac1p-\frac12}$ for $j\in\mathcal{J}$. We can find natural numbers A and B such that $2N<A<B\leq 4N$ and such that

$$B-A\leq 2k\varepsilon N<N \tag{3.7}$$

$$|\mathcal{J}\cap(A,B]|\geq k\varepsilon N. \tag{3.8}$$

Let Q denotes the orthonormal projection onto $\operatorname{span}\{e^{ijt}\;:\;|j|>\frac{(1+\varepsilon)A}{2}\}$. For $j\in\mathcal{J}\cap(A,B]$ we have $Q\varphi_j=\sum_{\frac{(1+\varepsilon)A}{2}\leq|s|\leq\frac{(1+\varepsilon)B}{2}}\hat{\varphi}_j(s)e^{ijt}$ and obviously $\|Q\varphi_j\|_2\leq\|\varphi_j\|_2\leq DN^{\frac1p-\frac12}$. So from Lemma 3.3 we infer that for $j\in\mathcal{J}\cap(A,B]$ we have

$$\|Q\varphi_j\|_p\leq(4(1+\varepsilon)k\varepsilon N)^{\frac12-\frac1p}\|Q\varphi_j\|_2\leq D(4(1+\varepsilon)k\varepsilon)^{\frac12-\frac1p}<\frac{1}{2K_1}. \tag{3.9}$$

Now we define an operator $\Phi:W_{\frac{(1+\varepsilon)A}{2}}\longrightarrow W_{\frac{(1+\varepsilon)A}{2}}$ by the formula

$$\Phi(f)=\sum_{j\in\mathcal{J}\cap(A,B]}\varphi_j^*(f)(I-Q)\varphi_j. \tag{3.10}$$

Observe that $\Phi\mid X_A=0$ and $\|\Phi\|\leq K\cdot\|I-Q\|\leq K\cdot R_p$. This shows that Φ induces an operator $\tilde\Phi:Z\longrightarrow Z$ where $Z=W_{\frac{(1+\varepsilon)A}{2}}/X_A$, with $\|\tilde\Phi\|\leq\|\Phi\|\leq R_pK$. Since $\dim Z\leq(1+\varepsilon)A+1-(A+1)=\varepsilon A$ we have

$$\operatorname{tr}\tilde\Phi\leq\dim Z\cdot\|\tilde\Phi\|\leq\varepsilon AR_pK. \tag{3.11}$$

On the other hand (3.10), (3.9) and (3.8) give

$$
\begin{aligned}
\text{tr } \tilde{\Phi} &= \sum_{j\in\mathcal{J}\cap(A,B]} \varphi_j^*((I-Q)\varphi_j) \\
&= \sum_{j\in\mathcal{J}\cap(A,B]} \varphi_j^*(\varphi_j) - \sum_{j\in\mathcal{J}\cap(A,B]} \varphi_j^*(Q\varphi_j) \\
&\geq |\mathcal{J}\cap(A,B]| - \sum_{j\in\mathcal{J}\cap(A,B]} \|\varphi_j^*\|_{p'} \cdot \|Q\varphi_j\|_p \\
&\geq |\mathcal{J}\cap(A,B]| - \frac{1}{2}|\mathcal{J}\cap(A,B]| = \frac{1}{2}|\mathcal{J}\cap(A,B]| \\
&\geq \frac{1}{2}k\varepsilon N
\end{aligned}
\tag{3.12}
$$

Comparing (3.12) and (3.11) we get $AR_pK \geq \frac{1}{2}kN$, so $k \leq \frac{2KR_pA}{N} \leq 8KR_p$. This is a contradiction.

Case 2), $1 < p < 2$. Let D denotes now the constant $C(K, p', 1/2)$ from Lemma 3.2. We fix k and ε as in the previous case. Let S denote the orthogonal projection onto $W_{2N(1+\varepsilon)}$. The system $(\varphi_j, S\varphi_j^*)_{j=0}^{4N}$ is a biorthogonal system. Moreover ubc $(S\varphi_j^*)_{j=0}^{4N} \leq \|S\| \cdot K \cdot \text{bc } (\varphi_j)$. This can be seen as follows:

$$
\begin{aligned}
\Big\|\sum_{j=0}^{4N} \alpha_j S\varphi_j^*\Big\| &\leq \|S\| \Big\|\sum_{j=0}^{4N} \alpha_j\varphi_j^*\Big\| \leq K\|S\| \Big\|\sum_{j=0}^{4N} \pm\alpha_j\varphi_j^*\Big\| \\
&= K\|S\| \sup_{\|f\|\leq 1} \Big|\langle\sum_{j=0}^{4N} \pm\alpha_j\varphi_j^*, f\rangle\Big| \\
&= K\|S\| \sup_{\|f\|\leq 1} \Big|\langle\sum_{j=0}^{4N} \pm\alpha_j\varphi_j^*, \sum_{j=0}^{4N} \varphi_j^*(f)\varphi_j\rangle\Big| \\
&= K\|S\| \sup_{\|f\|\leq 1} \Big|\langle\sum_{j=0}^{4N} \pm\alpha_j\varphi_j^*, S(\sum_{j=0}^{4N} \varphi_j^*(f)\varphi_j)\rangle\Big| \\
&= K\|S\| \sup_{\|f\|\leq 1} \Big|\langle\sum_{j=0}^{4N} \pm\alpha_j S(\varphi_j^*), \sum_{j=0}^{4N} \varphi_j^*(f)\varphi_j\rangle\Big| \\
&\leq K\|S\| \sup_{\|f\|\leq \text{bc } (\varphi_j)} \Big|\langle\sum_{j=0}^{4N} \pm\alpha_j S(\varphi_j^*), f\rangle\Big| \\
&= K\|S\| \text{bc } (\varphi_j) \Big\|\sum_{j=0}^{4N} \pm\alpha_j S\varphi_j^*\Big\|.
\end{aligned}
\tag{3.13}
$$

Now we apply Lemma 3.2 to the sequence $\left(\|S\varphi_j^*\|_{p'}^{-1} S\varphi_j^*\right)_{j=2N+1}^{4N}$ in $L_{p'}(\mathbf{T})$ and we get a set $\mathcal{J} \subset \{2N+1, \ldots, 4N\}$ with $|\mathcal{J}| > N$ and

$$
\|S\varphi_j^*\|_2 \leq K_1 D N^{\frac{1}{p'}-\frac{1}{2}} \text{ for } j \in \mathcal{J}. \tag{3.14}
$$

Observe that this imply that $\|\varphi_j\|_2 \geq CN^{\frac{1}{p}-\frac{1}{2}}$ for $j \in \mathcal{J}$.

Now, like in the case $p > 2$, we fix integers A and B, $2N < A < B \leq 4N$ such that

$$(B-A) \leq 2k\varepsilon N \tag{3.15}$$

$$|\mathcal{J} \cap (A, B]| > k\varepsilon N. \tag{3.15}$$

Like before we define an operator Q to be the orthogonal projection onto the space span $\{e^{ijt} \ : \ |j| > \frac{(1+\varepsilon)A}{2}\}$. We also define S_1 to be the orthogonal projection onto $W_{\frac{(1+\varepsilon)B}{2}}$. From Lemma 3.3, (3.14) and (3.15) we have

$$\begin{aligned} \|S_1 Q\varphi_j^*\|_{p'} &\leq [2(1+\varepsilon)(B-A)]^{\frac{1}{2}-\frac{1}{p'}} \|S_1 Q\varphi_j^*\|_2 \\ &\leq [2(1+\varepsilon)(B-A)]^{\frac{1}{2}-\frac{1}{p'}} \|SQ\varphi_j^*\|_2 \\ &\leq [2(1+\varepsilon)(B-A)]^{\frac{1}{2}-\frac{1}{p'}} K_1 D N^{\frac{1}{p'}-\frac{1}{2}} \\ &\leq K_1 D\, [2(1+\varepsilon)2k\varepsilon]^{\frac{1}{2}-\frac{1}{p'}} \leq \frac{1}{2} \end{aligned} \tag{3.17}$$

We define $\Phi \ : \ W_{\frac{(1+\varepsilon)A}{2}} \longrightarrow W_{\frac{(1+\varepsilon)A}{2}}$ by

$$\Phi(f) = \sum_{j \in \mathcal{J}\cap(A,B]} \varphi_j^*(f)(I-Q)\varphi_j = \sum_{j \in \mathcal{J}\cap(A,B]} S_1\varphi_j^*(f)(I-Q)\varphi_j. \tag{3.17}$$

Since $\Phi(\varphi_j) = 0$ for $j \leq A$ we see that Φ induces an operator $\tilde{\Phi} : Z \longrightarrow Z$ where $Z = W_{\frac{(1+\varepsilon)A}{2}}/X_A$. Clearly $\dim Z \leq \varepsilon A$ and $\|\tilde{\Phi}\| \leq \|\Phi\| \leq \|I-Q\| \cdot \text{ubc}\ (\varphi_j) \leq R_p K$. From (3.17) and the choice of ε and k we have

$$\begin{aligned} \text{tr}\ \tilde{\Phi} &= \sum_{j \in \mathcal{J}\cap(A,B]} S_1\varphi_j^*((I-Q)\varphi_j) \\ &= \sum_{j \in \mathcal{J}\cap(A,B]} \left[\varphi_j^*(\varphi_j) - S_1 Q\varphi_j^*(\varphi_j)\right] \\ &\geq |\mathcal{J} \cap (A,B]| - \sum_{j \in \mathcal{J}\cap(A,B]} \|S_1 Q\varphi_j^*\|_{p'} \\ &\geq \frac{1}{2}|\mathcal{J} \cap (A,B]| \geq \frac{1}{2}k\varepsilon N \end{aligned}$$

This leads to a contradiction with the choice of k since

$$\frac{1}{2}k\varepsilon N \leq \text{tr}\ \tilde{\Phi} \leq \|\tilde{\Phi}\| \cdot \dim Z \leq R_p K\varepsilon A \leq R_p K\varepsilon 4N$$

so we obtain $k \leq 8R_p K$.

□

Remark. Actually in the proof of Theorem 3.4 we used only selected facts about the trigonometric system. It is thus possible to discuss not only trigonometric polynomials but also polynomials with respect to other systems. More precisely let us consider any system $(g_n)_{n=1}^{\infty}$ in $L_p(\mu)$ satisfying the following conditions:

a) $(g_n)_{n=1}^{\infty}$ is an orthonormal system in $L_2(\mu)$,
b) $\sup_n \|g_n\|_\infty < \infty$,
c) $(g_n)_{n=1}^{\infty}$ is a basis in $L_p(\mu)$,
d) spaces $\operatorname{span}(g_n)_{n=1}^{N}$ are uniformly isomorphic to subspaces of ℓ_p^k of proportional dimension i. e. there exist constants C and α such that for any $N = 1, 2, \dots$ there exists a subspace $E_N \subset \ell_P^{[\alpha N]}$ such that $\mathrm{d}(E_N, \operatorname{span}(g_n)_{n=1}^{N}) \leq C$.

Then for any unconditional basis of $L_p(\mu)$ of the form $G_k = \sum_{n=1}^{v_k} \beta_n g_n$ with $\beta_{v_k} \neq 0$ we have $\limsup_k \frac{v_k}{k} > 1$.

Among the classical systems satisfying a)–d) are the Walsh system in the natural order and the double trigonometric system $e^{int}e^{ims}$ ordered along the squares or rectangles.

Now we want to give an example showing the need for "$\limsup$" in the statement of the above Theorem.

Example 3.5. *There exists an orthogonal system of polynomials $(\psi_k)_{k=0}^{\infty}$ such that*

(1) *The system $(\psi_k)_{k=0}^{\infty}$ is an unconditional basis in $L_p(\mathbf{T})$ for $1 < p < \infty$*
(2) *deg $\psi_k \leq k$ for $k = 0, 1, 2, \dots$*
(3) *span $(\psi_k)_{k=0}^{2^n}$ = span $(e^{ikt})_{k=-2^{n-1}}^{2^{n-1}}$ for $n = 1, 2, \dots$.*

The existence of such a system follows immediately from the isomorphism between H_p and $L_p(\mathbf{T})$ described in Section 2 and the following

Proposition 3.6. *There exists an orthogonal system consisting of analytic polynomials $(\Psi_k)_{k=0}^{\infty}$ such that*

(1) $\Psi_0 = 1$
(2) *$(\Psi_k)_{k=0}^{\infty}$ is an unconditional basis in H_p for $1 < p < \infty$*
(3) *deg $\Psi_k \leq 2k$ for $k = 0, 1, 2, \dots$*
(4) *span $(\Psi_{k=0}^{2^n-1})$ = span $(e^{ikt})_{k=0}^{2^n-1}$ for $n = 1, 2, \dots$*

Proof. We define $f_n = 2^{-\frac{n}{2}} \sum_{k=0}^{2^n-1} e^{ikt}$ for $n = 0, 1, 2, \dots$ and $f_n^k = e^{2^n it} f_n(t - \frac{2\pi k}{2^n})$ for $k = 0, 1, \dots, 2^n - 1$. We put $\Psi_0 = 1$ and for $s = 2^n + k$ with $n = 0, 1, 2, \dots$ and $k = 0, 1, 2, \dots, 2^n - 1$ we put $\Psi_s = f_n^k$. With those definitions properties (1), (3) and (4) are clearly satisfied. It is also easily verified (and actually well known, cf. [Z] X.1 and X.2, that $(\Psi_k)_{k=0}^{\infty}$ is an orthogonal system. Let $(h_n)_{n=0}^{\infty}$ be the classical orthonormal Haar system. It is well known that $(h_n)_{n=0}^{\infty}$ is an unconditional basis in $L_p(\mathbf{T})$ for $1 < p < \infty$ (cf. e.g. [W1] II.D.13.). In order to establish (2) we will show that the system $(\Psi_k)_{k=0}^{\infty}$ is equivalent to $(h_n)_{n=0}^{\infty}$, i.e. that there exist constant C_p for $1 < p < \infty$ such that for all sequences of scalars $(\alpha_k)_{k=0}^{\infty}$ we have

$$c_p^{-1} \Big\| \sum_{k=0}^{\infty} \alpha_k h_k \Big\|_p \leq \Big\| \sum_{k=0}^{\infty} \alpha_k \Psi_k \Big\|_p \leq C_p \Big\| \sum_{k=0}^{\infty} \alpha_k h_k \Big\|_p \tag{3.18}$$

By duality it is enough to establish only one inequality in (3.18). We have from the

Littlewood-Paley theorem (cf. [Z] XV.4.24.)

$$\Big\|\sum_{k=0}^{\infty}\alpha_k\Psi_k\Big\|_p=\Big\|\alpha_o\Psi_o+\sum_{s=0}^{\infty}\sum_{k=2^s}^{2^{s+1}-1}\alpha_k\Psi_k\Big\|_p \tag{3.19}$$
$$\geq\left(\int_{\mathbf{T}}(|\alpha_o\Psi_o|^2+\sum_{s=0}^{\infty}|\sum_{k=2^s}^{2^{s+1}-1}\alpha_k\Psi_k|^2)^{\frac{p}{2}}\right)^{\frac{1}{p}}.$$

Let M denotes the Hardy-Littlewood maximal function (see [T] or [K] for definition and properties of M). From the Fefferman-Stein result (see e.g. [T] XII.1. Th 1.1) we get that (3.19) is minorised by (a constant times)

$$\left(\int_{\mathbf{T}}\left(|\alpha_o|^2+\sum_{s=0}^{\infty}\Big[M(\sum_{k=2^s}^{2^{s+1}-1}\alpha_k\Psi_k)\Big]^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}. \tag{3.20}$$

To minorise this we will need the following Lemma whose proof is momentarily postponed.

Lemma 3.7. *If f is a polynomial of degree n then*

$$Mf(t)\geq C\sup_{|s-t|\leq\frac{\pi}{n}}|f(s)|.$$

Let us denote $g_s(t)=e^{-i2^st}\sum_{k=2^s}^{2^{s+1}-1}\alpha_k\Psi_k(t)$. We can write

$$g_s(t)=\sum_{k=2^s}^{2^{s+1}-1}\alpha_kf_s(t-\tfrac{2\pi(k-2^s)}{2^s})=\sum_{k=0}^{2^s-1}g_s(\tfrac{2\pi k}{2^s})2^{-\frac{n}{2}}f_s(t-\tfrac{2\pi k}{2^s}).$$

Since $M(\sum_{k=2^s}^{2^{s+1}-1}\alpha_k\Psi_k)=Mg_s$ we infer from Lemma 3.7. that

$$M(\sum_{k=2^s}^{2^{s+1}-1}\alpha_k\Psi_k)\geq C\sum_{k=2^s}^{2^{s+1}-1}|\alpha_k||h_k|. \tag{3.21}$$

Putting together (3.19), (3.20) and (3.21) we get

$$\Big\|\sum_{k=0}^{\infty}\alpha_k\Psi_k\Big\|_p\geq C\Big(\int_{\mathbf{T}}\big(|\alpha_o|^2+\sum_{s=0}^{\infty}(\sum_{k=2^s}^{2^{s+1}-1}|\alpha_k||h_k|)^2\big)^{\frac{p}{2}}\Big)^{\frac{1}{p}}.$$

Since the Haar system is an unconditional basis in $L_p(\mathbf{T})$ the last expression is minorised by $C\|\sum_{s=0}^{\infty}\alpha_sh_s\|_p$. So (3.18) is proved.

□

Proof of the Lemma. Let V_n be the n-th de la Vallée Poussin kernel. Then $f(t)=\int f(\theta)V_n(t-\theta)\,d\theta$, so

$$\sup_{|s-t|<\frac{\pi}{n}}|f(s)|=\sup_{|s-t|<\frac{\pi}{n}}\Big|\int_{\mathbf{T}}f(\theta)V_n(s-\theta)\,d\theta\Big|$$
$$\leq\int_{\mathbf{T}}|f(\theta)|\sup_{|s-t|<\frac{\pi}{n}}|V_n(s-\theta)|\,d\theta$$

Using well known estimates for V_n stated in Section 2 we infer that there exists a constant C (independent of n) such that

$$\sup_{|s-t|<\frac{\pi}{n}} |V_n(s-\theta)| \leq F_n(t-\theta)$$

where $F_n(t)$ is a positive, even function on $(-\pi, \pi)$, decreasing on the interval $(0, \pi)$ and such that $\int_{-\pi}^{\pi} F_n(t)\, dt \leq C$. Thus

$$\sup_{|s-t|<\frac{\pi}{n}} |f(s)| \leq \int_{\mathbf{T}} |f(\theta)| F_n(t-\theta)\, d\theta.$$

From [T] IV.2.3. we infer that $\int |f(\theta)|F_n(t-\theta)\, d\theta \leq CMf(t)$. This proves the Lemma.

□

Remark. Suppose that $(n_k)_{k=0}^{\infty}$ is an increasing sequence of natural numbers with $n_o = 0$ such that for some constants A and B and for all $f \in H_p$ we have

$$A\|f\|_p \leq \Big[\int_{\mathbf{T}} \big(\sum_{k=0}^{\infty} \big|\sum_{s=n_k}^{n_{k+1}-1} \hat{f}(s) e^{ist}\big|^2\big)^{p/2} dt\Big]^{1/p} \leq B\|f\|_p. \tag{3.22}$$

Let us denote $b_k = n_{k+1} - n_k + 1$ and $f_k = 2^{-b_k/2} \sum_{k=0}^{b_k-1} e^{ikt}$. For $s = n_k + l$ with $0 \leq l \leq b_k$ we define

$$\Psi_s(t) = e^{in_k t} f_k(t - \tfrac{2\pi l}{b_k}). \tag{3.23}$$

It is easy to check that $(\Psi_s)_{s=0}^{\infty}$ is a complete orthonormal system in H_p. Moreover one can basically reproduce the above argument to conclude that $(\Psi_s)_{s=0}^{\infty}$ is an unconditional basis in H_p. Only two observations have to be made

1) If (3.22) holds for p then it also holds for p', so we can apply duality as before.
2) We have to replace the Haar system by an orthonormal, unconditional basic sequence χ_s in $L_p(\mathbf{T})$ such that for $s = n_k + l$ we have $|\chi_s| = 2^{b_k/2}\mathbf{1}_s$ where $\mathbf{1}_s$ is an indicator function of the interval $(\frac{2\pi l}{b_k} - \frac{\pi l}{b_k}, \frac{2\pi l}{b_k} + \frac{\pi l}{b_k})$.

To construct such a sequence is routine. Either we can construct a small perturbation of it as a block basic sequence of the Haar system or we can construct it exactly as a martingale difference sequence for suitably chosen filtration.

This construction gives a whole family of unconditional polynomial bases in H_p, and so in $L_p(\mathbf{T})$. It is also of some interest in the theory of Fourier series. Namely, when we compare the degrees of polynomials in the above bases with Theorem 3.4 we obtain

Corollary 3.8. *Suppose $1 < p < \infty$, $p \neq 2$ and suppose that increasing sequence of integers (n_k) satisfies (3.22) for some constants A, B and all $f \in H_p$. Then* $\limsup_{k\to\infty} \frac{n_{k+1}}{n_k} > 1$.

Remark. Actually Theorem 3.4 is not really needed for the proof of the above Corollary 3.8. Since the L_p and L_2 norms of functions given by (3.23) are easy to compute, we can directly apply Lemma 3.2 to get Corollary 3.8

Comment. It is well known (cf. e.g. [Z]) that if $n_{k+1} > (1+\delta)n_k$ for all k and some positive δ then (3.22) holds for $1 < p < \infty$. On the other hand it is easy to construct sequences (n_k) such that (3.22) holds and $\liminf_{k\to\infty} \frac{n_{k+1}}{n_k} = 1$, e.g. $n_{2k} = 2^k$ and $n_{2k+1} = 2^k + 1$. Corollary 3.8 shows that despite such examples some exponential gaps have to remain. I suspect that a result like Corollary 3.8 exists somewhere in the literature, but I was unable to find it. The sequences which satisfy (3.22) for all p, $1 < p < \infty$ were considered in the literature under the name *Littlewood-Paley sets* cf. [H-K]. Actually Theorem 3.7 of [H-K] implies our Corollary 3.8. for Littlewood-Paley sets.

Section 4

In this Section we will consider the Bergman-type spaces. This whole Section is strongly influenced by the paper [M-P], so we refer the reader to this paper for additional background information. We will also quote several Lemmas and estimates from [M-P] without repeating them here, so the reader interested in this Section should have [M-P] handy.

Let us recall some definitions. Let $\varphi(r)$ $0 \le r < 1$ be a non-negative, increasing function, such that

$$\varphi(tr) \le Ct^\alpha \varphi(r) \quad 0 < t < 1 \tag{4.1}$$

and

$$\varphi(tr) \ge C^{-1} t^\beta \varphi(r) \quad 0 < t < 1 \tag{4.2}$$

where C, α, β are positive numbers.

Let $0 < p, q < \infty$. For a function f defined on the unit disc $\mathbf{D}$ in the complex plain we define the norm

$$\|f\|_{p,q,\varphi} = \left(\int_0^1 (1-r)^{-1} \varphi^q(1-r) \Big(\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \Big)^{q/p} dr \right)^{1/q}. \tag{4.3}$$

Using this norm we define the space $H(p,q,\varphi)$ as the space of all functions $f(z)$ analytic in $\mathbf{D}$ and such that $\|f\|_{p,q.\varphi} < \infty$. Note that when $p = q$ and $\varphi(r) = r^{1/p}$ then $H(p,q,\varphi)$ equals the space $B_p(\mathbf{D})$ discussed in the Introduction.

We will consider systems $(\Phi_n^k)_{n=0,k=1}^{\infty,2^n}$ such that

(4.4) the system is an unconditional basis in $H_p(\mathbf{T})$ for some p, $0 < p < \infty$

(4.5) uniformly in n we have $\| \sum_{k=1}^{2^n} \alpha_k \Phi_n^k \|_{H_p} \sim \big(\sum_{k=1}^{2^n} |\alpha_k|^p \big)^{1/p}$

(4.6) there exist positive integer a such that for all n's and k's we have $\Phi_n^k(t) = \sum_{s=2^{n-a}}^{2^{n+a+1}} \beta_s^{k,n} e^{ist}$.

One example of such a system is the system considered in Proposition 3.6. Other examples are constructed in [W-W] (cf. Corollary 4.7. and 5.3.) and in [Woz1] and [Woz2].

The main result of this Section is the following

Theorem 4.1. *Suppose that* $(\Phi_n^k)_{n=0,k=0}^{\infty,2^n}$ *is a system satisfying (4.4)-(4.6) for some* p, $0<p<\infty$. *Then for every* $0<q<\infty$ *and every* $\varphi(t)$ *satisfying (4.1) and (4.2) there exists a constant* C *such that*

$$C^{-1}\Big(\sum_{n=0}^{\infty}\varphi(2^{-n})^q\Big(\sum_{k=1}^{2^n}|\alpha_n^k|^p\Big)^{q/p}\Big)^{1/q} \tag{4.7}$$
$$\leq \Big\|\sum_{n=0}^{\infty}\sum_{k=1}^{2^n}\alpha_n^k\Phi_n^k\Big\|_{p,q,\varphi} \leq C\Big(\sum_{n=0}^{\infty}\varphi(2^{-n})^q\Big(\sum_{k=1}^{2^n}|\alpha_n^k|^p\Big)^{q/p}\Big)^{1/q}$$

This Theorem implies that systems satisfying (4.4), (4.5) and (4.6) are unconditional bases in $H(p,q,\varphi)$. Those bases establish isomorphisms between $H(p,q,\varphi)$ and natural sequence spaces. In particular, when applied to the system considered in Proposition 3.6, this Theorem gives an alternative proof of [M-P] Theorem 2.6. and when applied to the system constructed in [W-W] it solves in the positive the problem asked at the end of [M-P].

In the proof of the Theorem we will use many ideas and Lemmas from [M-P].

Proof of the Theorem. Let us denote $\Delta_n=\sum_{k=1}^{2^n}\alpha_n^k\Phi_n^k$. We have

$$\begin{aligned}
&\Big\|\sum_{n=0}^{\infty}\sum_{k=1}^{2^n}\alpha_n^k\Phi_n^k\Big\|_{p,q,\varphi}^q\\
&=\int_0^1(1-r)^{-1}\varphi^q(1-r)\Big(\int_0^{2\pi}\Big|\sum_{n=0}^{\infty}\Delta_n(re^{i\theta})\Big|^p\,d\theta\Big)^{q/p}dr\\
&=\sum_{s=0}^{\infty}\int_{1-2^{-s}}^{1-2^{-s-1}}(1-r)^{-1}\varphi^q(1-r)\Big(\int_0^{2\pi}\Big|\sum_{n=0}^{\infty}\Delta_n(re^{i\theta})\Big|^p\,d\theta\Big)^{q/p}dr
\end{aligned}$$

using unconditionality we get

$$\begin{aligned}
&\geq C\sum_{s=0}^{\infty}\int_{1-2^{-s}}^{1-2^{-s-1}}(1-r)^{-1}\varphi^q(1-r)\Big(\int_0^{2\pi}\Big(\sum_{n=0}^{\infty}|\Delta_n(re^{i\theta})|^2\Big)^{p/2}\,d\theta\Big)^{q/p}dr\\
&\geq C\sum_{s=0}^{\infty}\int_{1-2^{-s}}^{1-2^{-s-1}}(1-r)^{-1}\varphi^q(1-r)\Big(\int_0^{2\pi}|\Delta_s(re^{i\theta})|^p\,d\theta\Big)^{q/p}dr
\end{aligned}$$

so from (4.6) and Lemma 3.1. of [M-P] we obtain

$$\begin{aligned}
&\geq C\sum_{s=0}^{\infty}\int_{1-2^{-s}}^{1-2^{-s-1}}(1-r)^{-1}\varphi^q(1-r)r^{q2^{s+a+1}}\|\Delta_s\|_p^q\,dr\\
&\geq C\sum_{s=0}^{\infty}\|\Delta_s\|_p^q\varphi^q(2^{-s})\int_{1-2^{-s}}^{1-2^{-s-1}}(1-r)r^{q2^{s+a+1}}dr\\
&\geq C\sum_{s=0}^{\infty}\varphi^q(2^{-s})\|\Delta_s\|_p^q.
\end{aligned}$$

Now (4.5) gives the right hand side inequality in (4.7). To prove the left hand side inequality let us observe that for $t=\min(1,p)$ we have

$$
\begin{aligned}
&\Big(\int_0^{2\pi}|\sum_{s=0}^{\infty}\Delta_s(re^{i\theta})|^p\,d\theta\Big)^{q/p}\\
&\le\Big(\sum_{s=0}^{\infty}\Big(\int_0^{2\pi}|\Delta_s(re^{i\theta})|^p\,d\theta\Big)^{t/p}\Big)^{q/t}\\
&\le\Big(\sum_{s=0}^{\infty}r^{t2^{s-a}}\|\Delta_s\|_p^t\Big)^{q/t}
\end{aligned}
$$

where the last inequality follows from Lemma 3.1 of [M-P].

We now introduce a number δ and write

$$
\begin{aligned}
&\Big(\sum_{s=0}^{\infty}r^{t2^{s-a}}\|\Delta_s\|_p^t\Big)^{q/t}=\Big(\sum_{s=0}^{\infty}2^{-s\delta t}r^{t2^{s-a-1}}\|\Delta_s\|_p^t r^{t2^{s-a-1}}2^{s\delta t}\Big)^{q/t}\\
&\le\Big[\sum_{s=0}^{\infty}2^{-s\delta q}r^{q2^{s-a-1}}\|\Delta_s\|_p^q\Big]\cdot\Big[2^{(a+1)\delta t}\sum_{s=0}^{\infty}2^{(s-a-1)\delta t}(r^t)^{2^{s-a-1}}\Big]^{q/t}
\end{aligned}
$$

and using estimate (4.1) of [M-P] we obtain

$$
\begin{aligned}
&\le C\Big[\sum_{s=0}^{\infty}2^{-s\delta q}r^{q2^{s-a-1}}\|\Delta_s\|_p^q\Big]\cdot\Big[r^t(1-r)^{-\delta t}\Big]^{q/t}\\
&\le C\sum_{s=0}^{\infty}2^{-s\delta q}r^{q2^{s-a-1}}r^q(1-r)^{-\delta q}\|\Delta_s\|_p^q.
\end{aligned}
$$

Thus we get

$$
\|\sum_{s=0}^{\infty}\Delta_s\|_{p,q,\varphi}^q\le C\sum_{s=0}^{\infty}\|\Delta_s\|_p^q2^{-s\delta q}\int_0^1 r^{q(2^{s-a-1}+1)}(1-r)^{-1-\delta q}\varphi^q(1-r)\,dr.
$$

Now we apply Lemma 4.1 of [M-P] for $\varepsilon=1+\delta q$ and $\delta=\frac{\alpha}{2}$ (where α appears in (4.1)) and we get

$$
\begin{aligned}
&\|\sum_{s=0}^{\infty}\Delta_s\|_{p,q,\varphi}^q\\
&\le C\sum_{s=0}^{\infty}\|\Delta_s\|_p^q\cdot2^{-s\delta q}\frac{1}{q(2^{s-a-1}+1)}\varphi^q\Big(\frac{1}{q(2^{s-a-1}+1)}\Big)\big[q(2^{s-a-1}+1)\big]^{1+\delta q}\\
&\le C\sum_{s=0}^{\infty}\varphi^q(2^{-s})\|\Delta_s\|_p^q
\end{aligned}
$$

This together with (4.5) give the right hand side inequality in (4.7).

□

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