ON NONATOMIC BANACH LATTICES AND HARDY SPACES

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Abstract. We are interested in the question when a Banach space X with an unconditional basis is isomorphic (as a Banach space) to an order-continuous nonatomic Banach lattice. We show that this is the case if and only if X is isomorphic as a Banach space with $X(\ell_2)$. This and results of J. Bourgain are used to show that spaces $H_1(\mathbf{T}^n)$ are not isomorphic to nonatomic Banach lattices. We also show that tent spaces introduced in [4] are isomorphic to Rad H_1.

1. Introduction

There is a natural distinction between sequence spaces and function spaces (or, between atomic and nonatomic Banach lattices) in functional analysis. As an example, let us point out the subtitles of two volumes of [15] and [16]. However, many classical function spaces (e.g. the spaces $L_p 0,1$ for $1 < p < \infty$ [22] or [16]) have unconditional bases and hence are isomorphic as Banach spaces to sequence spaces (atomic Banach lattices). On the other hand, $L_1 0,1$ and has no unconditional basis ([22] or [16]) and in the other direction the sequence spaces ℓ_p for $p \neq 2$ are not isomorphic to any nonatomic Banach lattice [1]. In this note we discuss a general criterion for deciding whether a Banach space with an unconditional basis (i.e. a sequence space) can be isomorphic to a nonatomic Banach lattice (i.e. a function space). Our main result (Theorem 2.4) gives a simple necessary and sufficent condition for an atomic Banach lattice X to be isomorphic to an order-continuous nonatomic Banach lattice; of course if X contains no copy of c_0 every Banach lattice structure on X is order-continuous.

Our main motivation is to study the Hardy space $H_1(\mathbf{T})$. After the discovery that the space $H_1(\mathbf{T})$ has an unconditional basis [17] it become natural to investigate if $H_1(\mathbf{T})$ is isomorphic to a nonatomic Banach lattice. Applying Theorem 2.4 to H_1 and using some previous results of Bourgain [2] and [3] we show that H_1 is not isomorphic to any nonatomic Banach lattice, and further more that $H_1(\mathbf{T}^n)$ is not isomorphic to a nonatomic Banach lattice for any natural number n.

We conclude by showing that the space Rad H_1 or $H_1(\ell_2)$ is isomorphic to the tent spaces T^1 introduced by Coifman, Meyer and Stein [4].

2. Lattices with unconditional bases

Our terminology about Banach lattices will agree with [16]; we also refer the reader to [9] and [10] for the isomorphic theory of nonatomic Banach lattices.

A (real) Banach lattice X is called *order continuous* if every order-bounded increasing sequence of positive elements is norm convergent. Any Banach lattice not containing c_0 is automatically order-continuous.

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For any order-continuous Banach lattice X we can define an associated Banach lattice $X(\ell_2)$ (using the Krivine calculus [16] pp. 40-42) as the space of sequences $(x_n)_{n=1}^{\infty}$ in X such that $(\sum_{k=1}^{n} !x_k!^2)^{1/2}$ is order-bounded (and hence is a convergent sequence) in X. $X(\ell_2)$ becomes an order-continuous Banach lattice when normed by $(x_n) = (\sum_{n=1}^{\infty} !x_n!^2)^{1/2}$.

If X has nontrivial cotype then $X(\ell_2)$ is naturally isomorphic to the space Rad X which is the subspace of $L_2(0,1;X)$ of functions of the form $\sum_{n=1}^{\infty} x_n r_n$ where (r_n) is the sequence of Rademacher functions. The space Rad X is clearly an isomorphic invariant of X, and so if two Banach lattices X and Y with nontrivial cotype are isomorphic it follows easily that $X(\ell_2)$ and $Y(\ell_2)$ are isomorphic. However, this result holds in general by a result of Krivine [13] or [16] Theorem 1.f.14.

Theorem 2.1. *If X, Y are order-continuous Banach lattices and $T : X \longrightarrow Y$ is a bounded linear operator, then if $(x_n) \in X(\ell_2)$ we have $(Tx_n) \in Y(\ell_2)$ and*

$$(T(x_n))_{Y(\ell_2)} \leq K_G T (x_n)_{X(\ell_2)}.$$

Here, as usual, K_G denotes the Grothendieck constant.

Proof. Essentially this is Krivine's theorem, but we do need to show that if $(x_n) \in X(\ell_2)$ then $(Tx_n) \in Y(\ell_2)$. To see this we show that $(\sum_{k=1}^{n} !Tx_k!^2)^{1/2}$ is norm-Cauchy. In fact if $m > n$ then

$$\begin{aligned}
(\sum_{k=1}^{m} !Tx_k!^2)^{1/2} - (\sum_{k=1}^{n} !Tx_k!^2)^{1/2}{}_Y &\leq (\sum_{k=n+1}^{m} !Tx_k!^2)^{1/2}{}_Y \\
&\leq K_G T (\sum_{k=n+1}^{m} !x_k!^2)^{1/2}{}_X \\
&\leq K_G T (\sum_{k=n+1}^{\infty} !x_k!^2)^{1/2}{}_X
\end{aligned}$$

which converges to zero as $n \to \infty$ by the order-continuity of X. $\square$

Corollary 2.2. *If two order-continuous Banach lattices X and Y are isomorphic as Banach spaces, then $X(\ell_2)$ and $Y(\ell_2)$ are isomorphic as Banach spaces.*

If X is a separable order-continuous nonatomic Banach lattice then X can be represented as (i.e. is linearily and order isomorphic with) a Köthe function space on $0,1$ in such a way that $L_\infty 0,1 \subset X \subset L_1 0,1$ and inclusions are continuous. It will then follow that L_∞ is dense in X, and the dual of X can be represented as a space of functions, namely $X^* = \{f \in L_1 \ : \ \int !fg!\, dt < \infty \ \text{ for every } \ g \in X\}$.

Now we are ready to state our main result. Let us observe that for re-arrangement invariant function spaces on $0,1$ this result was proved in [9] (cf. also [16] 2.d) by a quite different technique.

Theorem 2.3. *Let X be an order continuous, nonatomic Banach lattice with an unconditional basis. Then X is isomorphic as a Banach space to $X(\ell_2)$.*

Proof. We will represent X as a Köthe function space on $0,1$ as described above. Suppose $(\phi_n)_{n=1}^{\infty}$ is a normalized unconditional basis of X. Then there is an order-continuous atomic Banach lattice Y which we identify as a sequence space and operators $U : X \longrightarrow Y$ and $V : Y \longrightarrow X$ such that $UV = I_Y$, $VU = I_X$ and $U(\phi_n) = e_n$ for $n = 1, 2, \ldots$, where e_n denotes the canonical basis vectors in Y. We can regard Y^* as a space of sequences and further suppose that $e_{nY^*} = e_{nY} = 1$. We will identify $Y(\ell_2)$ as a space of double sequences with canonical basis $(e_{mn})_{m,n=1}^{\infty}$; thus for any finitely nonzero sequence we have $\sum a_{mn} e_{mnY(\ell_2)} = \sum_m (\sum_n |a_{mn}|^2)^{1/2} e_{mY}$.

Let r_n denote the Rademacher functions and for each fixed $f \in X$ note that $(r_n f)$ converges weakly to zero, since for $g \in X^*$ we have $\lim_{n\to\infty} \int r_n f g \, dt = 0$. In particular we have for each $m \in \mathbf{N}$ that $(r_n \phi_m)$ converges weakly to zero. It follows by a standard gliding hump technique that if $\eta = (2UV)^{-1}$ then we can find for each $(m,n) \in \mathbf{N}^2$ an integer $k(m,n)$ and disjoint subsets (A_{mn}) of $\mathbf{N}$ so that $U(\phi_m r_{k(m,n)})\chi_{A_{mn}} - U(\phi_m r_{k(m,n)})_Y \le \eta$.

Identifying Y^* as a sequence space, we let $\psi_m = U^*(e_m)$ and then define $v_{m,n} = \chi_{A_{mn}} U(\phi_m r_{k(m,n)}) \in Y$ and $v^*_{m,n} = \chi_{A_{mn}} V^*(\psi_m r_{k(m,n)}) \in Y^*$. Now suppose (a_{mn}) is a finitely nonzero double sequence. Then

$$\begin{aligned}
\sum_{m,n} a_{mn} v_{mnY} &\le (\sum_{m,n} |a_{mn}|^2 |U(\phi_m r_{k(m,n)})|^2)^{1/2}{}_Y \\
&\le K_G U (\sum_{m,n} |a_{mn}|^2 |\phi_m r_{k(m,n}|^2)^{1/2}{}_X \\
&= K_G U (\sum_m (\sum_n |a_{mn}|^2) |\phi_m|^2)^{1/2}{}_X \\
&= K_G U (\sum_m (\sum_n |a_{mn}|^2) |V e_m|^2)^{1/2}{}_X \\
&\le K_G^2 UV (\sum_m (\sum_n |a_{mn}|^2)^{1/2} |e_m|^2)^{1/2}{}_Y \\
&= K_G^2 UV \sum_{m,n} a_{mn} e_{mnY(\ell_2)}.
\end{aligned}$$

Here we have used Krivine's theorem twice. It follows that we can define a linear operator $S : Y(\ell_2) \to Y$ by $Se_{mn} = v_{mn}$ and then $S \le K_G^2 UV$.

Similar calculations yield that for any finitely nonzero double sequence (b_{mn}) we have:

$$\sum_{m,n} b_{mn} v^*_{mnY^*} \le K_G^2 UV \sum_m (\sum_n |b_{mn}|^2)^{1/2} e_{mY^*}.$$

Suppose then $y \in Y$ and set $a_{mn} = \langle y, v^*_{mn} \rangle$. Let F be a finite subset of $\mathbf{N}^2$. Let $\alpha_m = (\sum_n \chi_F(m,n)|a_{mn}|^2)^{1/2}$ and suppose the finitely nonzero sequence (β_m) is chosen so that $\sum \beta_m e_{mY^*} = 1$ and $\sum \beta_m \alpha_m = \sum \alpha_m e_{mY}$. Then, with the convention that $0/0 = 0$,

$$
\begin{aligned}
\sum_{(m,n)\in F} a_{mn}e_{mnY(\ell_2)} &= \sum_m \beta_m \alpha_m \\
&= \sum_{(m,n)\in F} \beta_m \alpha_m^{-1} !a_{mn}!^2 \\
&= \langle y, \sum_{(m,n)\in F} \beta_m \alpha_m^{-1} a_{mn} v_{mn}^* \rangle \\
&\le y_Y \sum_{(m,n)\in F} \beta_m \alpha_m^{-1} a_{mn} v_{mnY^*} \\
&\le K_G^2 UVy_Y .
\end{aligned}
$$

Thus for each F the map $T_F : Y \to Y(\ell_2)$ given by $T_F y = \sum_{(m,n)\in F} \langle y, v_{mn}^* \rangle e_{mn}$ has norm at most $K_G^2 UV$. More generally, we have $T_F y \le K_G^2 UV \chi_{A_F} y$ where $A_F = \cup_{(m,n)\in F} A_{mn}$.

It follows that for each $y \in Y$ the series $\sum_{m,n} \langle y, v_{mn}^* \rangle e_{mn}$ converges (unconditionally) in $Y(\ell_2)$. We can thus define an operator $T : Y \to Y(\ell_2)$ by $Ty = \sum_{m,n} \langle y, v_{mn}^* \rangle e_{mn}$ and $T \le K_G^2 UV$.

Now notice that $TS(e_{mn}) = c_{mn}e_{mn}$ where $c_{mn} = \langle v_{mn}, v_{mn}^* \rangle$. But:

$$
\begin{aligned}
\langle v_{mn}, v_{mn}^* \rangle &= \langle v_{mn}, V^* \psi_m r_{k(m,n)} \rangle \\
&\ge \langle U(\phi_m r_{k(m,n)}), V^*(\psi_m r_{k(m,n)}) \rangle - \eta V\psi_{mX^*} \\
&= \langle \phi_m, \psi_m \rangle - \eta V\psi_{mX^*} \\
&\ge 1 - \eta VU \ge 1/2.
\end{aligned}
$$

Thus TS is invertible and so it follows that $Y(\ell_2)$ is isomorphic to a complemented subspace of Y. It then follows from the Pelczynski decomposition technique that $Y \sim Y(\ell_2)$; more precisely $Y \sim Y(\ell_2) \oplus W$ for some W and so $Y \sim Y(\ell_2) \oplus (Y(\ell_2) \oplus W) \sim Y(\ell_2) \oplus Y \sim Y(\ell_2)$. $\square$

Remark. The order continuity of the Banach lattice X is essential. In [14] a non-atomic Banach lattice X (actually an M-space) was constructed which is isomorphic to c_0. In particular X has an unconditional basis but is not isomorphic to $X(\ell_2)$.

Theorem 2.4. *Let Y be a Banach space with an unconditional basis. Then Y is isomorphic to an order-continuous nonatomic Banach lattice if and only if $Y \sim Y(\ell_2)$.*

Remark. Here again we regard Y as an order-continuous Banach lattice.

Proof. One direction follows immediately from Theorem 2.3 and Corollary 2.2. For the other direction, it is only necessary to show that if $Y \sim Y(\ell_2)$ then Y is isomorphic to order-continuous nonatomic Banach lattice. To this we introduce the space $Y(L_2)$; this is the space of sequences of functions (f_n) in $L_2 0,1$ such that $\sum f_{n2} e_n$ converges in Y. We set $(f_n)_{Y(L_2)} = \sum f_{n2} e_{nY}$. It is clear that $Y(L_2)$ is an order-continuous Banach lattice. Now if (g_n) is an orthonormal basis of L_2 we define $W : Y(\ell_2) \to Y(L_2)$ by $W(\sum_{m,n} a_{mn}e_{mn}) = (\sum_n a_{mn} g_n)_{m=1}^{\infty}$ and it is easy to see that W is an isometric isomorphism. $\square$

Proposition 2.5. *If X is a non-atomic order continuous Banach lattice with unconditional basis, then $X \sim X \oplus X$ and $X \sim X \oplus R$.*

Proof. Both facts follow from Theorem 2.3. □

Note that for spaces with unconditional basis both properties do not hold in general (see [5] and [6])

Proposition 2.6. *Let X be an order continuous non-atomic Banach lattice with an unconditional basis and let Y be a complemented subspace of X. Assume that Y contains a complemented subspace isomorphic to X. Then $X \sim Y$.*

Proof. The proof is a repetition of the proof of Proposition 2.d.5. of [16]. □

3. Hardy spaces

We recall that $H_1(\mathbf{T}^n)$ is defined to be the space of boundary values of functions f holomorphic in the unit disk $\mathbf{D}$ and such that

$$\sup_{0<r<1} \int_{\mathbf{T}^n} !f(re^{it_1}, re^{it_2}, \ldots, re^{it_n})! \, dt_1 dt_2 \ldots dt_n < \infty.$$

The basic theory of such spaces is explained in [18].

Let us consider first the case $n = 1$. Then $\Re H_1$ is defined be the space of real functions $f \in L_1(\mathbf{T})$ such that for some $F \in H_1(\mathbf{T})$ we have $\Re F = f$. $\Re H_1$ is normed by $f_1 + \min\{F_{H_1} : \Re F = f\}$. Then H_1 is isomorphic to the complexification of $\Re H_1$, and further when considered as a real space is isomorphic to $\Re H_1$. Further it was shown in [17] that $\Re H_1$ has an unconditional basis and is isomorphic a space of martingales $H_1(\delta)$. To define the space $H_1(\delta)$ let $(h_n)_{n \geq 1}$ be the usual enumeration of the Haar functions on $I = 0, 1$ normalized so that $h_{n\infty} = 1$. Then suppose $f \in L_1$ is of the form $f = \sum a_n h_n$. We define $f_{H_1(\delta)} = \int (\sum_n !a_n!^2 h_n^2)^{1/2} dt$ and $H_1(\delta) = \{f : f_{H_1(\delta)} < \infty\}$.

These considerations can be extended to the case $n > 1$. In a similar way, $H_1(\mathbf{T}^n)$ is isomorphic to the complexification of, and is also real-isomorphic to, a martingale space $H_1(\delta^n)$. Here we define for $\alpha \in \mathcal{M} = \mathbf{N}^n$ the function $h_\alpha \in L_1(I^n)$ by $h_\alpha(t_1, \ldots, t_n) = \prod h_{\alpha_k}(t_k)$. Then $H_1(\delta^n)$ consists of all $f = \sum_{\alpha \in \mathcal{M}} a_\alpha h_\alpha$ such that $f_{H_1(\delta^n)} = \int (\sum !a_\alpha!^2 h_\alpha^2)^{1/2} dt < \infty$.

It is clear from the definition that the system $(h_\alpha)_{\alpha \in \mathcal{M}}$ is an unconditional basis of $H_1(\delta^n)$. We can thus define a space $H_1(\delta^n, \ell_2) = H(\delta^n)(\ell_2)$ as in Section 1; since $H_1(\delta^n)$ has cotype two, this space is isomorphic to Rad $H_1(\delta^n)$. The following theorem is due to Bourgain [2]:

Theorem 3.1. *$H_1(\delta, \ell_2)$ is not isomorphic to a complemented subspace of $H_1(\delta)$.*

In a subsequent paper [3] Bourgain implicitly extended this result to higher dimensions.

Theorem 3.2. *For every $n = 1, 2, \ldots$ the space $H_1(\delta^n, \ell_2)$ is not isomorphic to any complemented subspace of $H_1(\delta^n)$.*

Sketch of proof. For $n = 1$ this Theorem is proved in detail in [2]. The subsequent paper [3] states only the weaker fact that $H_1(\delta^n)$ is not isomorphic to $H_1(\delta^{n+1})$. His proof however gives the above Theorem as well. All that is needed is to change in Section 3 of [3] condition (m+1) and Lemma 4. Before we formulate the appropriate condition we need some further notation. By $BMO(\delta^n)$ we will denote the dual of $H_1(\delta^n)$ and by $BMO(\delta^n,\ell_2)$ we will denote the dual of $H_1(\delta^n,\ell_2)$. The space $H_1(\delta^n,\ell_2)$ has an unconditional basis given by $(h_\alpha\otimes e_k)_{\alpha\in\mathcal{M},\,k\in\mathbf{N}}$. In our notation from Section 2 $h_\alpha\otimes e_k$ is a sequence of $H_1(\delta^n)$-functions which consists of zero functions except at the k-th place where there is h_α. The same element can be treated as an element of the dual space. Note that the natural duality gives

$$\langle h_\alpha\otimes e_k, h_{\alpha'}\otimes e_{k'}\rangle = \begin{cases} \int_{I^n} !h_\alpha!, & \text{when } \alpha=\alpha' \text{ and } k=k' \\ 0, & \text{otherwise.}\end{cases}$$

Now we are ready to state the new condition (m+1):

Let $\Phi : H_1(\delta^n,\ell_2) \longrightarrow H_1(\delta^n)$ and $\Phi^\times : BMO(\delta^n,\ell_2) \longrightarrow BMO(\delta^n)$ be bounded linear operators (note that $\Phi^\times$ is *not* the adjoint of Φ). Then for every $\varepsilon>0$ there exists a set $A\subset\mathcal{M}$ such that $\sum_{\alpha\in A} !h_\alpha! = 1$ and integers k_α for $\alpha\in A$ such that

$$\sum_{\alpha\in A}\int_{I^n} !\Phi(h_\alpha\otimes e_{k_\alpha})!\cdot!\Phi^\times(h_\alpha\otimes e_{k_\alpha})! < \varepsilon.$$

With this condition one can repeat the proof from [3] and obtain the Theorem. $\square$

Corollary 3.3. *We have the following*

$$\ell_2 \overset{c}{\subset} H_1(\delta) \overset{c}{\subset} H_1(\delta,\ell_2) \overset{c}{\subset} H_1(\delta^2) \overset{c}{\subset} H_1(\delta^2,\ell_2) \overset{c}{\subset} \dots$$

where $X \overset{c}{\subset} Y$ means that X is isomorphic to a complemented subspace of Y but Y is **not** *isomorphic to a complemented subspace of X.*

Proof. It is well known and easy to check that the map $h_\alpha\otimes e_k \mapsto h_\alpha(t_1,\dots,t_n)\cdot r_k(t_{n+1})$ where r_k is the k-th Rademacher function gives the desired complemented embedding. That no smaller space is isomorphic to a complemented subspace of a bigger one is the above theorem of Bourgain. $\square$

Corollary 3.4. *The spaces $H_1(\delta^n)$ is not isomorphic to a nonatomic Banach lattice for $n=1,2,\dots$. The spaces $H_1(\delta^n,\ell_2)$ are each isomorphic to a nonatomic Banach lattice.*

Proof. The first claim follows directly from Theorem 3.1, 3.2 and Theorem 2.3. We only have to observe that (since $H_1(\delta^n)$ does not contain any subspace isomorphic to c_0 and indeed has cotype two) any Banach lattice isomorphic as a Banach space to $H_1(\delta^n)$ is order continuous (see Theorem 1.c.4 of [16]). The second claim follows from Corollary 2.4. $\square$

Remark. For $H_p(\mathbf{T}^n)$ with $0<p<\infty$ we have the following situation. When $1<p<\infty$ the orthogonal projection from $L_p(\mathbf{T}^n)$ onto $H_p(\mathbf{T}^n)$ is bounded so then $H_p(\mathbf{T}^n)$ is isomorphic to $L_p(\mathbf{T}^n)$. This implies in particular that these spaces are isomorphic to nonatomic lattices. When $0<p<1$ then $H_p(\mathbf{T}^n)$ admit only purely atomic orders as a p-Banach lattices. To see this observe that if X is not a purely atomic p-Banach lattice then its Banach envelope (for definition and properties see [11]) is a Banach lattice which is not purely atomic. On the other hand it is known that the Banach envelope of $H_p(\mathbf{T}^n)$ is isomorphic to ℓ_1. For $n = 1$ this can be found in [11] Theorem 3.9, for $n > 1$ the proof uses Theorem $2'$ of [19] but otherwise is the same; alternatively see [11] Theorem 3.5, for a proof using bases. When we compare it with the observation from [1] mentioned in the Introduction, that ℓ_1 is not isomorphic to any nonatomic Banach lattice, we conclude that the spaces $H_p(\mathbf{T}^n)$ cannot be isomorphic to any nonatomic p-Banach lattice.

Remark. For the dual spaces $H_1(\mathbf{T}^n)^* = BMO(\mathbf{T}^n)$ the situation is rather different. We first observe the following proposition:

Proposition 3.5. *For any Banach space X the spaces $\ell_1(X)^*(= \ell_\infty(X^*))$ and $L_1(0,1,X)^*$ are isomorphic.*

Proof. Clearly $\ell_1(X)^*$ is isomorphic to a 1-complemented subspace of $L_1(X)^*$. Now let $\chi_{n,k} = \chi_{((k-1)2^{-n},k2^{-n})}$ for $1 \le k \le 2^n$ and $n = 0,1,\ldots$. Let $T : \ell_1(X) \to L_1(X)$ be defined by $T((x_n)) = \sum x_n\chi_{m,k}$ where $n = 2^m + k - 1$. Let $L_1(\mathcal{D}_N, X)$ be the subspace of all functions measurable with respect to the finite algebra generated by the sets $((k-1)2^{-N}, k2^{-N})$ for $1 \le k \le 2^N$ and define $S_N : L_1(\mathcal{D}_N; X) \to \ell_1(X)$ by setting $S(x \otimes \chi_{N,k})$ to be the element with x in position $2^N + k - 1$ and zero elsewhere. Then applying Exercise 7 of II.E of [22] (cf. [8] Proposition 1), we obtain that $L_1(X)^*$ is isomorphic to a complemented subspace of $\ell_1(X)^*$. Then by the Pelczynski decomposition technique we obtain the proposition. $\square$

Now from the Proposition, observe that, since $H_1(\mathbf{T}^n) \sim \ell_1(H_1(\mathbf{T}^n))$, we have $L_1(H_1(\mathbf{T}^n))^* \sim BMO(\mathbf{T}^n)$ and clearly this isomorphism induces a nonatomic (but not order-continuous) lattice structure on $BMO(\mathbf{T}^n)$. (It is easy to see that a space which contains a copy of ℓ_∞ cannot have an order-continuous lattice structure, because it fails the separable complementation property.)

4. Rad H_1 and tent spaces

The space $H_1(\delta, \ell_2)$ is, as observed in Section 2, isomorphic to Rad H_1 and has a structure as a nonatomic Banach lattice. The complex space Rad H_1 is easily seen to be isomorphic to the vector-valued space $H_1(\mathbf{T}, \ell_2)$ consisting of the boundary values of the space of all functions F analytic in the unit disk $\mathbf{D}$ with values in a Hilbert space ℓ_2 and such that:

$$\sup_{0<r<1} \int_0^{2\pi} F(re^{i\theta}\frac{d\theta}{2\pi} = F < \infty.$$

To see this isomorphism just note that $H_1(\mathbf{T}, \ell_2)$ can be identified with the space of sequences (f_n) in H_1 such that

$$(f_n) = \int_0^{2\pi} (\sum_{n=1}^{\infty} !f_n(e^{i\theta})!^2)^{1/2}\frac{d\theta}{2\pi} < \infty.$$

This is in turn easily seen to be equivalent to the norm of $\sum r_n f_n$ in $L_2(0,1;H_1)$ (see [16] Theorem 1.d.6).

We now show that a nonatomic Banach lattice isomorphic to Rad H_1 arises naturally in in harmonic analysis. More precisely we will show that tent space T^1 which was introduced and studied by R. Coifman, Y. Meyer and E. Stein in [4] is isomorphic to Rad H_1. Tent spaces are useful in some questions of harmonic analysis (cf. [7] or [21]). They can be defined over $\mathbf{R}^n$ but for the sake of simplicity we will consider them only over $\mathbf{R}$.

Let us fix $\alpha > 0$. For $x \in \mathbf{R}$ we define

$$\Gamma_\alpha(x) = \{(y,t) \in \mathbf{R} \times \mathbf{R}^+ \ : |x-y| < \alpha t\}.$$

Given a function $f(y,t)$ defined on $\mathbf{R} \times \mathbf{R}^+$ we put

$$f_\alpha = \int_{\mathbf{R}} \Big(\int_{\Gamma_\alpha(x)} |f(y,t)|^2 t^{-2}\, dy\, dt \Big)^{1/2} dx.$$

It was shown in [4] Proposition 4 that for different α's the norms $_\alpha$ are equivalent i.e. for $0 < \alpha < \beta < \infty$ there is a $C = C(\alpha, \beta)$ such that for every f we have

$$f_\alpha \le f_\beta \le C f_\alpha. \tag{4.1}$$

This implies that the space $T^1 = \{f(y,t) \ : \ f_\alpha < \infty\}$ does not depend on α. Observe that T^1 is clearly a non-atomic Banach lattice.

Our main result of this Section is

Theorem 4.1. *The space T^1 is lattice-isomorphic to $H_1(\delta, L_2)$ and hence isomorphic to Rad H_1.*

Actually for the proof of this Theorem it is natural to work with the dyadic H_1 space on $\mathbf{R}$. This space, which we denote $H_1(\delta_\infty)$ can be defined as follows:

Let $I_{nk} = k \cdot 2^n, (k+1)\cdot 2^n$ for $n,k = 0, \pm1, \pm2 \dots$ and let h_{nk} be the function which is equal to 1 on the left hand half of I_{nk}, -1 on the right hand half of I_{nk} and zero outside I_{nk}. In other words h_{nk} is the Haar system on $\mathbf{R}$. The system $\{h_{nk}\}_{n,k=0,\pm1,\pm2\dots}$ is a complete orthogonal system. For a function $f = \sum_{n,k} a_{nk} h_{nk}$ we define its $H_1(\delta_\infty)$-norm by

$$f = \int_{\mathbf{R}} \Big(\sum_{n,k} |a_{nk}|^2 |h_{nk}|^2\Big)^{1/2} dt \tag{4.2}$$

That this space is isomorphic to the space $H_1(\delta)$ follows from the work of Sjölin and Stromberg [20]. However, slighlty more is true:

Lemma 4.2. *The atomic Banach lattices $H_1(\delta)$ and $H_1(\delta_\infty)$ are lattice-isomorphic (or, equivalently the natural normalized unconditional bases of these spaces are permutatively equivalent).*

Proof. For any subset $\mathcal{A}$ of $\mathbf{Z}^2$ write $H_{\mathcal{A}}$ for the closed linear span of $\{h_{nk} : (n,k) \in \mathcal{A}\}$ in $H_1(\delta_\infty)$. For $m \in \mathbf{Z}$ let $\mathcal{A}_m = \{(n,k) : I_{nk} \subset 2^{-m-1}, 2^{-m}\}$ and $\mathcal{B}_m = \{(n,k) : I_{nk} \subset -2^{-m}, -2^{-m-1}$. Let $\mathcal{D} = \cup_{m\in\mathbf{Z}}(\mathcal{A}_m \cup \mathcal{B}_m)$ and $\mathcal{D}_+ = \cup_{m\ge0}\mathcal{A}_m$. Then it is clear that $H_{\mathcal{D}}$ and $H_{\mathcal{D}_+}$ are each lattice isomorphic to $\ell_1(H_1(\delta))$. Now $H_1(\delta_\infty)$ is lattice isomorphic to $H_{\mathcal{D}} \oplus H_{\mathcal{E}}$ where $\mathcal{E} = \{(m,0),\ (m,-1) : m \in \mathbf{Z}\}$. It is easy to show that $H_{\mathcal{E}}$ is lattice isomorphic to ℓ_1. Similarly $H_1(\delta)$ is lattice-isomorphic to $H_1(\mathcal{D}_+) \oplus \ell_1$ and this completes the proof of the lemma. □

Remark. Note also that $H_1(\delta)$ is lattice-isomorphic to $\ell_1(H_1(\delta))$.

Proof of the Theorem. We will prove that T^1 is lattice-isomorphic to $H_1(\delta_\infty, L_2)$. Let us introduce squares $A_{nk} \subset \mathbf{R} \times \mathbf{R}^+$ defined as $A_{nk} = I_{nk} \times 2^n, 2^{n+1}$ for

$n, k = 0, \pm 1, \pm 2, \ldots$. It is geometrically clear that squares $\{A_{nk}\}_{n,k=0,\pm1,\pm2,\ldots}$ are essentially disjoint and that they cover $\mathbf{R} \times \mathbf{R}^+$. For $j = 0, 1, 2$ we define

$$A^j_{nk} = (k + \tfrac{j}{3})2^n, (k + \tfrac{j+1}{3})2^n \times 2^n, 2^{n+1}.$$

Note that in this way we divide each A_{nk} into three essentially disjoint rectangles. Let $D^j = \bigcup_{n,k} A^j_{nk}$. Let T^1_j be the subspace of T^1 consisting of all functions whose support is contained in D^j. Clearly $T^1 = T^1_0 \oplus T^1_1 \oplus T^1_2$, so it is enough to show that T^1_j is lattice-isomorphic to $H_1(\delta_\infty, L_2)$.

We write $f^j \in T^1_j$ as $f^j = \sum_{n,k} f^j_{nk}$ where $f^j_{nk} = f^j \cdot \chi_{A^j_{nk}}$. We start with $j = 1$. For any $\alpha > 0$ we have

$$
\begin{aligned}
(4.3) \qquad f^1{}_\alpha &= \int_{\mathbf{R}} (\int_{\Gamma_\alpha(x)} !f^1(y,t)!^2 t^{-2} dy\, dt)^{1/2} dx \\
&= \int_{\mathbf{R}} (\int_{\Gamma_\alpha(x)} \sum_{n,k} !f^1_{nk}(y,t)!^2 t^{-2} dy\, dt)^{1/2} dx \\
&= \int_{\mathbf{R}} (\sum_{nk} \int_{\Gamma_\alpha(x)} !f^1_{nk}(y,t)!^2 t^{-2} dy\, dt)^{1/2} dx.
\end{aligned}
$$

If we now take $\alpha = \frac{2}{3}$ we have $\Gamma_\alpha(x) \supset A^1_{nk}$ for all $x \in I_{nk}$, so from (4.3) we get

$$(4.4) \qquad f^1{}_\alpha \geq \int_{\mathbf{R}} (\sum_{nk} \chi_{I_{nk}}(x) \int_{A^1_{nk}} !f^1_{nk}(y,t)!^2 t^{-2} dy\, dt)^{1/2} dx.$$

On the other hand when we take $\alpha = \frac{1}{6}$ we have $\Gamma_\alpha(x) \cap A^1_{nk} = \emptyset$ for all $x \notin I_{nk}$, so from (4.3) we get

$$(4.5) \qquad f^1{}_\alpha \leq \int_{\mathbf{R}} (\sum_{n,k} \chi_{I_{nk}}(x) \int_{A^1_{nk}} !f^1_{nk}(y,t)!^2 t^{-2} dy\, dt)^{1/2} dx.$$

For each (n, k) the subspace of T^1 consisting of functions supported on A^1_{nk} is easily seen to be isometric to the Hilbert space. If we fix an isometry between this space and ℓ_2 we obtain from (4.2), (4.3) and (4.4) that T^1_1 is lattice-isomorphic to $H_1(\delta_\infty, L_2)$. In order to complete the proof of the Theorem it is enough to show that T^1_0 and T^1_2 are lattice-isomorphic to T^1_1. This isomorphism can be given by $\sum_{nk} f^j_{nk} \mapsto \sum_{nk} f^1_{nk}$. The fact that this map is really an isomorphism follows from:

Lemma 4.3. *Let $\phi(t)$ be a uniformly bounded measurable function on $\mathbf{R}^+$. For a function f defined on $\mathbf{R} \times \mathbf{R}^+$ we define*

$$A_\phi(f)(y, t) = f(y + t\phi(t), t).$$

Then $A_\phi : T^1 \longrightarrow T^1$ is a continuous linear operator.

Proof of the Lemma. Since

$$\begin{aligned}
&\int_{\Gamma_\alpha(x)} !A_\phi(f)(y,t)!^2 t^{-2} dy\, dt \\
&= \int_{\mathbf{R}^+} (t^{-2} \int_{x-\alpha t}^{x+\alpha t} !A_\phi(f)(y,t)!^2 dy) dt \\
&= \int_{\mathbf{R}^+} (t^{-2} \int_{x-\alpha t - t\phi(t)}^{x+\alpha t - t\phi(t)} !f(y,t)!^2 dy) dt \\
&\leq \int_{\mathbf{R}^+} (t^{-2} \int_{x-(\phi_\infty+\alpha)t}^{x+(\phi_\infty+\alpha)t} !f(y,t)!^2 dy) dt \\
&= \int_{\Gamma_{\alpha+\phi_\infty}(x)} !f(y,t)!^2 t^{-2} dy\, dt
\end{aligned}$$

the Lemma follows. □

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