Some remarks about the space of measures with uniformly bounded partial sums and Banach-Mazur distances between some spaces of polynomials

by P. Wojtaszczyk
Department of Mathematics
Texas A&M University
College Station, TX 77843

A lot of work have been done on the space of uniformly convergent Fourier series . In this note we attempt to present what is known about the analogous space of measures with uniformly bounded partial sums. Unfortunately very little is known, and many natural problems remain open. Let us start with the definition of the norm. For a formal Fourier series $\sum_{-\infty}^{\infty} a_n e^{in\theta}$ we put

$$\|\sum_{-\infty}^{\infty} a_n e^{in\theta}\| = \sup_{N\in\mathbf{N}} \frac{1}{2\pi}\int_0^{2\pi} |\sum_{-N}^{N} a_n e^{in\theta}| d\theta$$

. We will be interested in various spaces equipped with this norm. Anythime we use the notation $\|.\|$ in this note we mean the above norm. The spaces we will be dealing with are

$$M^b = \Big\{\mu \in M(\mathbf{T}) \ : \ \|\sum_{-\infty}^{\infty} \hat{\mu}(n)e^{in\theta}\| < \infty\Big\},$$

$$L_1^b = \Big\{f \in L_1(\mathbf{T}) \ : \ \|\sum_{-\infty}^{\infty} \hat{f}(n)e^{in\theta}\| < \infty\Big\}$$

$$H_1^b = \Big\{f \in H_1(\mathbf{T}) \ : \ \|\sum_{n=0}^{\infty} \hat{f}(n)e^{in\theta}\| < \infty\Big\}.$$

Clearly $H_1^b \subset L_1^b \subset M^b$ where the inclusions are isometric. S_N will always denote the symmetric partial sum projection i.e. $S_N(\sum_{-\infty}^{\infty} a_j e^{ij\theta} = \sum_{-N}^{N} a_j e^{ij\theta}$. This projection has norm one in each of the above spaces.

A few words about exposition. I will be rather informal (as is proper, I hope, in the seminar notes). In particular this will mean that I will not pay much attention to the constants and will assume that all numbers (like $\log N$ or $\sqrt{N}$) are integers, if needed.

We assume here the basic knowledge of harmonic analysis on the circle $\mathbf{T}$, in particular the F. M. Riesz Theorem and basic properties of the Dirichlet and Fejer kernels. Everything can be found in Zygmund or in any other book about trigonometric series or harmonic analysis. We also assume the standard knowledge of Banach space theory.

Theorem 1. (Helson) *If $\mu \in M_1^b$, then $\hat{\mu}(n) \to 0$ as $|n| \to \infty$.*

Proof. Suppose that for some $\mu \in M_1^b$ and for some sequence of integers (n_j) tending to ∞ with j we have $|\hat{\mu}(n_j)| > \varepsilon > 0$. We can assume that μ is singular with respect to the Lebesque measure (because if $\mu = \mu_s + f dm$ then clearly $\hat{f}(n) \to 0$, so we can replace μ by μ_s). Take ε_j, $j = 1, 2, \dots$ such that $|\varepsilon_j| = 1$ and $\varepsilon_j \hat{\mu}(n_j) > 0$. Let us call $\mu_j = \varepsilon_j e^{-in_j\theta}\mu$. Clearly $\hat{\mu}_j(0) > \varepsilon$. Let us consider also $\varphi_j = \varepsilon_j e^{-in_j\theta}\ S_{n_j}\mu$. We have $\|\varphi_j\|_1 \leq \|\mu\|$ and $\|\mu_j\| \leq \|\mu\|$ for all j, so we can choose the subsequence such that (we will not change the numbering) φ_j converges weak* to ν and μ_j converges weak* to ξ. Note that $\hat{\xi}(0) > \varepsilon$ so ξ is not zero and also ξ is singular with respect to the Lebesque measure, since μ was. But looking at Fourier coefficients we realise that $\hat{\xi}(s) = \hat{\nu}(s)$ for $s < 0$ and $\hat{\nu}(s) = 0$ for $s > 0$. Now the F. M. Riesz Theorem yields that $\nu \ll m$ and also $\xi - \nu \ll m$ so $\xi \ll m$. This contradiction finishes the proof. ♡

Remark 1. More quantitative estimates have been obtained by Pigno-Smith and J.J.F. Fournier. They showed that there exists a function $N(\varepsilon)$ such that for $\mu \in M_1^b$ with $\|\mu\| \leq 1$ we have

$$card\{n \in \mathbf{N}\ :\ |\hat{\mu}(n)| > \varepsilon\} \leq N(\varepsilon).$$

The exact asymptotics of $N(\varepsilon)$ as $\varepsilon \to 0$ is unknown.

Remark 2. It was shown by Mary Weiss and Y. Katznelson that there exists a singular measure μ on $\mathbf{T}$ with $\|\mu\| < \infty$.

Remark 3. The Helson theorem is equivalent to saying that the map $T : L_1^b \to c_0$ given as $T(f) = (\hat{f}(n))_{n=-\infty}^{\infty}$ is weakly compact.

To see it note that if T is a weakly compact operator then $T^{**}((L_1^b)^{**}) \subset c_0$ and clearly $M^b \hookrightarrow (L_1^b)^{**}$. If T is not weakly compact, then there exists a sequence $(f_n)_{n=1}^{\infty} \subset L_1^b$ with $\|f_n\| \leq 1$ such that $T(f)$ converges pointwise to some $\eta \in \ell_\infty \setminus c_0$. But clearly any $\sigma(M(\mathbf{T}), C(\mathbf{T}))$ cluster point μ of $(f_n)_{n=1}^{\infty}$ is in M^b and $(\hat{\mu}(n)) = \eta$, which contradicts the Helson theorem.

Remark 4. The space M^b does not have the Dunford-Pettis property, because $(e^{in\theta}dm)_{n=1}^{\infty} \subset M^b$ is weakly null but $(T(e^{in\theta}))_{n=1}^{\infty}$ is not norm convergent in c_0.

Theorem 2. (Salem-Zygmund) *If $\mu \in M^b$, then*

$$\sup_n \frac{\log n}{n} \sum_{k=1}^{n} |\hat{\mu}(k)| < \infty.$$

Proof. Applying the Hardy's inequality we get

$$\frac{\hat{\mu}(n)}{1} + \frac{\hat{\mu}(n-1)}{2} + ... + \frac{\hat{\mu}(1)}{n} \leq C\|\mu\|$$
$$\frac{\hat{\mu}(n-1)}{1} + ...\frac{\hat{\mu}(1)}{n-1} \leq C\|\mu\|$$
$$....$$

Adding those together we get

$$\sum_{k=1}^{n} \hat{\mu}(n-k+1) \left(\sum_{j=1}^{k} \frac{1}{j} \right) \leq Cn\|\mu\|$$

so

$$\sum_{k=1}^{\frac{n}{2}} \hat{\mu}(k) \log n \leq Cn\|\mu\|.$$

Since n was arbitrary this gives the claim. ♡

We can use the above Theorem to exhibit certain coefficient multipliers from M^b into ℓ_1.

Proposition 3.. *If $0 < \alpha \leq 1$, then the sequence $\left((n+1)^{-1} (\log(n+2))^{-\alpha}\right)_{n=0}^{\infty}$ is a coefficient multiplier from M^b into ℓ_1 i.e. for every $\mu \in M^b$ we have*

$$\sum_{n=0}^{\infty} |\hat{\mu}(n)| \, (n+1)^{-1} (\log(n+2))^{-\alpha} \leq C\|\mu\|.$$

Proof. From the Salem-Zygmund Theorem we infer that for each N we have

$$\sum_{2^N}^{2^{N+1}} |\hat{\mu}(n)| \leq CN^{-1}2^N\|\mu\|.$$

So

$$\sum_{-\infty}^{\infty} |\hat{\mu}(n)|(n+1)^{-1}\left(\log(n+2)\right)^{-\alpha} \leq \sum_{N} \sum_{2^N}^{2^{N+1}} |\hat{\mu}(n)|(n+1)^{-1}\left(\log(n+2)\right)^{-\alpha}$$

$$\leq \sum_{N} \frac{1}{2^{N+1}N^{\alpha}} \sum_{2^N}^{2^{N+1}} |\hat{\mu}(n)| \leq \sum_{N} N^{-(1+\alpha)}\|\mu\| \leq C\|\mu\|.$$

♡

Remark 5. It is known (cf Zygmund vol. I pp.184-5) that $\sum_{n=2}^{\infty} \frac{\cos nx}{\log n}$ represents a function in L_1^b, so $\frac{1}{n}$ is not a multiplier from M^b into ℓ_1.

All spaces M^b, L_1^b and H_1^b are convolution ideals in the sense that if $\mu \in M(\mathbf{T})$ and $\nu \in M^b$ (or $\nu \in L_1^b$ or $\nu \in H_1^b$), then $\mu * \nu \in M^b$ (or L_1^b or H_1^b). Since the series mentioned in Remark 1 is in L_1^b we infer that for every $\mu \in M(\mathbf{T})$ the series

$$\sum_{n=-\infty, n\neq 0,\pm 1}^{\infty} \frac{\hat{\mu}(n)}{\log |n|} e^{in\theta}$$

is a Fourier series of a function in L_1^b.

Let us consider the space LB of sequences $(\alpha_n)_{n=1}^{\infty}$ such that there exists a measure $\mu \in M^b$ with $\hat{\mu}(2^n) = \alpha_n$. We put $\|(\alpha_n)\| = \inf \|\mu\|$. It seems to be an interesting problem to describe this space. The following observations can be made :

1) The sequence space ℓ_2 embeds continuously into LB.

2) The unit vectors are unconditional in LB. This follows from the well known fact that for every bounded sequence $(\xi_n)_{n=1}^{\infty}$ there exists a measure $\mu \in M(\mathbf{T})$ such that $\hat{\mu}(2^n) = \xi_n$ and $\|\mu\|_1 \leq C\|\xi\|_\infty$ and from the observation that M^b is a convolution ideal. It is clear that LB is a dual space. I do not know if it is reflexive or if the sequence of unit vectors is a basis in LB.

Since the norm $\|.\|$ is built from the L_1 norm and the L_∞ norm the following fact is not surprising

Proposition 4. *The space H_1^b (so also M^b and L_1^b) contains the subspace isomorphic to ℓ_∞.*

Proof. Let us fix the sequence of polynomials $p_j = \sum_{n_j+1}^{n_{j+1}} a_k e^{ik\theta}$ for $j = 1, 2, ...$ such that $\|p_j\|_1 = 1$ but $\|p_j\| = \alpha_j$ with $\sum_{j=1}^{\infty} \frac{1}{\alpha_j} < \frac{1}{2}$. Then for any sequence of scalars $(\beta_j)_{j=1}^{\infty}$

with $|\beta_j| \leq 1$ and every integer N we have

$$\left\| S_N \left(\sum_j \frac{\beta_j}{\alpha_j} p_j \right) \right\|_1 \leq \sum_{j=1}^{s} \frac{|\beta_j|}{\alpha_j} \|p_j\|_1 + \left\| \sum_{n_{s_r}+1}^{N} \alpha_k e^{ik\theta} \right\|_1 \frac{|\beta_j|}{\alpha_j}$$

$$\leq \|\beta\|_\infty \sum_{j=1}^{s} \frac{1}{\alpha_j} + |\beta_j| \leq 2\|\beta\|_\infty$$

where s was chosen in such a way that $n_s \leq N < n_{s+1}$. For arbitrary s we fix l in such a way that $n_s \leq l < n_{s+1}$ and $\| \sum_{n_s+1}^{l} a_k e^{ik\theta} \|_1 = \alpha_s$. We have

$$\begin{aligned} \| \sum \frac{\beta_j}{\alpha_j} p_j \| &\geq \left\| \sum_{j=1}^{s-1} \frac{\beta_j}{\alpha_j} p_j + \sum_{n_s+1}^{l} a_k \frac{\beta_s}{\alpha_s} e^{ik\theta} \right\|_1 \\ &\geq \frac{|\beta_s|}{\alpha_s} \left\| \sum_{n_s+1}^{l} a_k e^{ik\theta} \right\|_1 - \sum_{j=1}^{s-1} \frac{|\beta_j|}{\alpha_j} \\ &\geq |\beta_s| - \frac{1}{2} \|\beta\|_\infty . \end{aligned}$$

Since s was arbitrary we get $\| \sum_j \beta_j \frac{1}{\alpha_j} p_j \| \geq \frac{1}{2} \|\beta\|_\infty$. ♡

Now we will turn to the local problems. We consider the space $(W_N, \|.\|)$ consisting of all trigonometric polynomials $\sum_{-N}^{N} a_n e^{in\theta}$ equipped with the norm $\|.\|$. We are interested in distances from $(W_N, \|.\|)$ to ℓ_p^{2N+1} for various values of p. We will also investigate how big ℓ_1 and ℓ_∞ subspaces W_N contains.

Lemma 5. $d(W_N, \ell_2^{2N+1}) \leq C\sqrt{\frac{N}{\log N}}$.

Proof. The operator $I \;:\; \ell_2^{2N+1} \longrightarrow W_N$ defined as $I((\xi_j)_{j=-N}^{N}) = \sum_{j=-N}^{N} \xi_j e^{ij\theta}$ has clearly norm 1. On the other hand for $p = \sum_{j=-N}^{N} \xi_j e^{ij\theta} \;\in W_N$ with $\|p\| \leq 1$ we have

$$\left(\sum_{j=-N}^{N} |\xi_j|^2 \right)^{1/2} \leq \left(\sum_{j=-n}^{n} |\xi_j| \right)^{1/2} (\sup |\xi_j|)^{1/2} .$$

Since $|\xi_j| \leq \|p\|$, the Salem-Zygmund theorem gives

$$\left(\sum_{j=-N}^{N} |\xi_j|^2 \right)^{1/2} \leq C \left(\frac{N}{\log N} \right)^{1/2} \|p\|.$$

so $\|I^{-1}\| \leq C\sqrt{\frac{N}{\log N}}$. ♡

Since $d(\ell_1^N, \ell_2^N) = \sqrt{N}$ we get from this Lemma that if $X \subset W_N$ and $d(X, \ell_1^k) \leq C$, then $k \leq C\frac{N}{\log N}$. I do not know if W_N actually contains such big ℓ_1 subspaces. We have however

Proposition 6. *For every $\alpha < 1$ there exist constants C_1 and C_2 such that for every N the space W_N has a subspace X_N such that*

$$dim\ X_N \geq C_1(2N+1)^\alpha$$

and

$$d(X_N, \ell_1^{dim X_N}) \leq C_2.$$

Proof. Let us take an integer K such that $(2N+1)^\alpha < \frac{2N+1}{K} \leq (2N+1)^\alpha + 1$ and let $D_N(t) = \sum_{j=-N}^{N} e^{ijt}$ be the usual Dirichlet kernel. Let us put $f_j = \frac{1}{\log N} \cdot D_N\big(t - \frac{2\pi K j}{2N+1}\big)$ for $j = 0, 1, ..., \frac{2N+1}{K} = L$. Clearly $\|\sum_{j=0}^{L} \alpha_j f_j\| \leq C \sum |\alpha_j|$. On the other hand

$$A = \int_0^1 \|\sum_{j=1}^{L} r_j(t) f_j\| dt \geq \int_0^1 \|\sum_{j=0}^{L} r_j(t) f_j\|_1 dt \geq C \int_0^1 \left(\sum_{j=0}^{L} |f_j|^2\right)^{1/2}$$

where r_j's are Rademacher functions.

If we consider disjoint intervals $I_j = (\frac{2\pi}{L} j - \frac{\pi}{L}, \frac{2\pi}{L} j + \frac{\pi}{L})$ and replace each f_j by $f_j \cdot \chi_{I_j}$ we get

$$A \geq \sum_{j=0}^{L} \int_{I_j} |f_j| = L \cdot \int_{I_o} |f_o| \quad .$$

Elementary and classical estimates of the Dirichlet kernel give

$$\int_{I_o} |f_o| \geq \frac{C}{\log N} \int_{\frac{-\pi}{L}}^{\frac{\pi}{L}} |D_N| \geq \frac{C}{\log N} \log(\frac{N}{L}) \geq C \quad .$$

This shows that $A \geq L \cdot C$ so by the theorem of Elton we can choose a subset of $(f_j)_{j=0}^{L}$ of cardinality proportional to L which spans ℓ_1 of the right dimension. This is our space X_N. ♡

Let us now discuss subspaces of W_N which are isomorphic to ℓ_∞^n. We have

Proposition 7. *Each space W_N contains a subspace X such that $dim X = (\log N)^2$ and $d(X, \ell_\infty^{(\log N)^2}) < C$ where C is an absolute constant.*

Proof. Let F be the Fejer kernel of order $\sqrt{N}$ and let $f_j = \frac{1}{\log N} e^{ijkt} F$ for $j = 1, 2, ..., (\log N)^2$, where k is such that intervals $supp \hat{f}_j$ are disjoint subsets of the set $\{0, 1, 2, ..., N\}$. Clearly $(f_j)_{j=1}^{(\log N)^2}$ is a basic sequence in W_N and $0 < c \leq \|f_j\| \leq C$ for some absolute constants c and C, so we have

$$\| \sum_{j=1}^{(\log N)^2} a_j f_j \| \geq C \sup |a_j| \quad .$$

We want to show the converse inequality. We have

$$\Big\| \sum_{j=1}^{(\log N)^2} a_j f_j \Big\| \leq \sup_r \int \Big| \sum_{j=1}^{r-1} a_j f_j + a_r \frac{1}{\log N} e^{irkt} PF \Big| \leq \sup_r \int | \sum_{j=1}^{r-1} a_j f_j | + C|a_r|$$

where PF is some partial sum of the Fejer kernel F.

In order to estimate this integral let us observe that it follows from the classical estimates of the Fejer kernel that if I_l's are disjoint intervals covering the circle, each of the length $\frac{1}{\sqrt{N}}$ then $\sum_l \frac{1}{\sqrt{N}} \sup_{I_l} |F(t)| \leq C$. Thus we have

$$\begin{aligned} \int | \sum_{j=1}^{r} a_j f_j | &= \int | \sum_{j=1}^{r} a_j \frac{1}{\log N} e^{ijkt} | F(t) dt \\ &\leq \frac{1}{\log N} \sum_l \int_{I_l} | \sum_{j=1}^{r} a_j e^{ijkt} | dt \cdot \sup_{I_l} F(t). \end{aligned}$$

Observe that the function $\sum_{j=1}^{r} a_j e^{ijkt}$ is periodic with the period $\frac{1}{k}$. Since $\frac{1}{k} < \frac{1}{\sqrt{N}}$ we get thet for each l

$$\int_{I_l} | \sum_{j=1}^{r} a_j e^{ijkt} | dt \leq |I_l| \int | \sum_{j=1}^{r} a_j e^{ijkt} | dt$$

so

$$\int | \sum_{j=1}^{r} a_j f_j | \leq \frac{C}{\log N} \Big\| \sum_{j=1}^{r} a_j e^{ijkt} \Big\|_1 \quad .$$

Since $r \leq (\log N)^2$ and we have

$$\| \sum_{j=1}^{r} a_j e^{ijkt} \|_1 \leq \| \sum_{j=1}^{r} a_j e^{ijkt} \|_2 \leq \sqrt{r} \max_{j=1,2,\ldots,r} |a_j|$$

we get the claim. ♡

There is one rather curious and unexpected consequence of the last argument. If we consider $\mathrm{span}(f_j)_{j=1}^{(\log N)^2}$ in $(W_N, \|.\|)$ it is isomorphic to $\ell_\infty^{(\log N)^2}$ but if we consider the same span in the space $(W_N, \|.\|_1)$ then we have

$$\Big\| \sum_{j=1}^{(\log N)^2} a_j f_j \Big\|_1 \sim \Big\| \sum_{j=1}^{(\log N)^2} a_j e^{ijt} \Big\|_1 \cdot \frac{1}{\log N} \quad .$$

This can be seen exactly like before using the fact that $\sum_{j=1}^{(\log N)^2} a_j e^{ijkt}$ oscillates fast compared to F. But this means that $\mathrm{span}(f_j)_{j=1}^{(\log N)^2}$ in $(W_N, \|.\|_1)$ is isomorphic (with constants independent of N) to $(W_{(\log N)^2}, \|.\|_1)$. But for $f \in W_N$ we have $\|f\|_1 \leq \|f\| \leq \log N \cdot \|f\|_1$, so $d\big((W_{(\log N)^2}, \|.\|_1), \ell_\infty^{(\log N)^2}\big) \leq \log N$. If we forget about logarithms, we get the upper estimate in the following

Corollary 8. $d\big((W_N, \|.\|_1), \ell_\infty^{2N+1}\big) \sim \sqrt{N}$.

The lower estimate is clear since $(W_N, \|.\|_1)$ is a subspace of L_1, so it has cotype 2.

Open problems. It follows from Proposition 7 that $d(W_N, X) \geq \log N$ for X any subspace of L_1, so in particular we have $d\big((W_N, \|.\|), (W_N, \|.\|_1)\big) \sim \log N$. It is not clear what is the correct order of the quantity $d\big((W_N, \|.\|), \ell_1^{2N+1}\big)$. From what we have said follows that $\log N \leq d\big((W_N, \|.\|), \ell_1^{2N+1}\big) \leq (\log N)^2$ because by the classical result of Marcinkiewicz (cf. Zygmund vol. II p. 30) we have $d\big((W_N, \|.\|_1), \ell_1^{2N+1}\big) \leq \log N$. Also the distance $d\big((W_N, \|.\|), \ell_2^{2N+1}\big)$ is unclear. One can easily check that it is greater than $\frac{\sqrt{N}}{\log N}$ but this and Lemma 5 leave an obvious gap.

Also I do not know if $(W_N, \|.\|)$ has local unconditional stucture. This is related to the question if M^b admits the lattice order. However if it has some, then it is not the natural one. Suppose that M^b is a Banach lattice with the natural order of measures. This would imply that multiplication by e^{int} would have uniformly bounded norms, so the maps $\sum_{-\infty}^{\infty} \hat{\mu}(n) e^{int} \mapsto \sum_{n=0}^{N} \hat{\mu}(n) e^{int}$ would be uniformly bounded. But this would imply that H_1^b is complemented in M^b, so by the F. M. Riesz Theorem we would get that $M^b \subset L_1^b$ as a set. This contradicts the existence of a singular measure in M^b.

I do not know if the following is true or false : if $\mu \in M^b$ and $\mu = \mu_s + f dm$ where μ_s is singular with respect to the Lebesque measure m, does $f \in L_1^b$ and $\mu_s \in M^b$?

Also the invariant projections in M^b and in H_1^b are unknown.

I do not know if ℓ_2 is isomorphic to a complemented subspace of M^b or L_1^b. This is not a local problem like in L_1 because W_N contains $\ell_2^{\log N}$ uniformly complemented. One has to take $\mathrm{span}(z^{N-2^k})_{k=1}^{\log N} \subset W_N$ and use the Paley's inequality to check that it is complemented. On the other hand the standard infinite dimensional argument also fail because M^b does not have the Dunford-Pettis property (cf. Remark 4)

REFERENCES

J. Elton *Sign-embedding of ℓ_1^n*, Trans. Amer. Math. Soc. vol. 279 (1983) pp.113-124

J.J.F. Fournier *On the coefficients in Fourier-Stieltjes series with norm-bounded partial sums*, Colloquium Math. vol. 49 (1985) pp.257-265

H. Helson *Proof of a conjecture of Steinhaus*, Proceedings of the National Academy of Sciences of USA vol. 40 (1954) pp.205-206

Y. Katznelson *Trigonometric series with positive partial sums*, Bull. Amer. Math. Soc. vol. 71 (1965) pp.718-719

L. Pigno and B. Smith *Quantitative behavior of transforms with norm-bounded partial sums*, American J. Math. vol. 102 (1980) pp.321-326

M. Weiss *On a problem of Littlewood*, Journal London Math. Soc. vol. 34 (1959) pp.217-221

A. Zygmund *Trigonometric series* Cambridge University Press