Wavelets as unconditional bases in $L_p(\mathbb{R})$ *†

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Abstract

We present weak sufficient conditions for decay of a wavelet so that the wavelet basis is an unconditional basis in $L_p(\mathbb{R})$, $1 < p < \infty$. We also prove that some unimodular wavelets yield unconditional bases in $L_p(\mathbb{R})$.

It is a part of the general wisdom that (good) wavelet bases are unconditional bases in $L_p(\mathbb{R})$ for $1 < p < \infty$. Results of this type are proven in almost every book on wavelets, see e.g. [3], [8], [9], [14]. In all those places, however, some assumptions on the smoothness of wavelets are used. Since the definition of L_p–spaces does not involve any smoothness, this may seem strange, and clearly is a result of the method of the proof, which in one way or the other involves the methods of the Calderon–Zygmund operators. The only result on the unconditionality of wavelet bases in $L_p(\mathbb{R})$ which uses *no* smoothness of wavelet is a theorem of Gripenberg [6]. One of the aims of this paper is to improve this result. In our second main result we deal with unimodular wavelets, also called in the literature minimally frequency supported wavelets (cf. [8]) or s-elementary wavelets (cf. [2]). It is an interesting class of wavelets which are very smooth but have quite bad decay.

Let us start with some definitions and notations. The symbol $\mathbb{R}$ will always denote the real line and the symbol $\mathbb{Z}$ will denote integers. The Lebesgue measure of a set $A \subset \mathbb{R}$ will be denoted as $|A|$. A *wavelet* is a function $\Psi(x) \in L_2(\mathbb{R})$ such that the system

$$\left\{2^{j/2}\Psi(2^j x - k)\right\}_{j,k\in\mathbb{Z}}$$

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is an orthonormal basis in $L_2(\mathbb{R})$. To simplify the notation, for any function f on $\mathbb{R}$ by $f_{jk}(x)$ we will denote the function $2^{j/2}f(2^j x - k)$. For a wavelet Ψ the system $\{\Psi_{jk}\}_{jk\in\mathbb{Z}}$ is called a wavelet basis.

Let us recal that a biorthogonal system $(x_n, x_n^*)_{n\in A}$ in a Banach space X is an unconditional basis if for every $x \in X$ the series $\sum_{n\in A} x_n^*(x)x_n$ converges to x unconditionally in norm. The detailed, elementary discussion of unconditional bases, especially in the context of wavelets, is presented in [14].

1 Wavelets with decay only

The assumptions on the wavelet we are making in this section are summarised in the following:

- There exists a function $\varphi(x)$ on $\mathbb{R}$ such that

  1. $\varphi(x) = \varphi(-x)$
  2. $\varphi(x)$ is decreasing on $(0,\infty)$
  3. $\varphi(x)$ is bounded on $\mathbb{R}$
  4. $\int_0^\infty \varphi(x)\ln(1+x)\,dx < \infty$
  5. $|\Psi(x)| \leq \varphi(x)$ for all $x \in \mathbb{R}$

Our assumptions about the function φ yield the following facts to be used later

Lemma 1 *For every $u \in \mathbb{R}$ we have*

$$\sum_{k\in\mathbb{Z}} \varphi(k)\varphi(u-k) \leq C\varphi(u/2) \tag{1}$$

and

$$\sum_{k\in\mathbb{Z}} \varphi(u-k) \leq C. \tag{2}$$

Proof Note that *4.* implies that $\int_{\mathbb{R}} \varphi(x)\,dx$ is finite. Since φ is monotone on $[0,\infty)$ and on $(-\infty,0]$ the second inequality easily follows. To prove the first note that without loss of generality we can assume that $u \geq 0$. Then we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \varphi(k)\varphi(u-k) &= \sum_{k\leq u/2} + \sum_{k>u/2} \varphi(k)\varphi(u-k)\\ &\leq 2\varphi(u/2)\sum_{k\leq u/2}\varphi(k) \leq C\varphi(u/2).\end{aligned}$$

♠

Our aim in this Section is to prove the following result

Theorem 1 *If a wavelet Ψ satisfies the above assumption $\bullet$, then the system $\{\Psi_{jk}\}_{j,k\in\mathbb{Z}}$ is an unconditional basis in $L_p(\mathbb{R})$ for $1<p<\infty$.*

REMARK This theorem improves upon the main result of G. Gripenberg [6] in two ways. First it replaces his condition $\int_0^\infty \varphi(x)(1+x)\,dx < \infty$ by a clearly weaker condition *4.* in $\bullet$. Second, we do not assume that the wavelet is associated with a multiresolution analysis. However the general structure and idea of our proof follows rather closely the paper [6] of G. Gripenberg.

REMARK A theorem stronger than our Theorem 1 is claimed in [1], but the proof presented there contains a serious error in the last few lines.

Now we can start the proof of Theorem 1. For a function $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ we can write

$$f = \sum_{j,k} \langle f, \Psi_{jk}\rangle \, \Psi_{jk}$$

where the series converges in L_2. For any sequence of signs $\boldsymbol{\epsilon} = \{\varepsilon_{jk}\}_{j,k\in\mathbb{Z}}$ with $\varepsilon_{jk} = \pm 1$ we can write

$$U_{\boldsymbol{\epsilon}} f = \sum_{jk} \varepsilon_{jk} \langle f, \Psi_{jk}\rangle \, \Psi_{jk}.$$

This clearly is an L_2–function and each operator $U_{\boldsymbol{\epsilon}}$ defined in this way is an isometry on $L_2(\mathbb{R})$.

The main technical ingredient of our proof is the following

Proposition 2 *Operators $U_{\boldsymbol{\epsilon}}$ are uniformly in $\boldsymbol{\epsilon}$ of weak type (1,1) i.e. there exists a constant C such that for each $f \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})$, for each $\lambda > 0$ and each $\boldsymbol{\epsilon}$ we have*

$$|\{x \in \mathbb{R} : |U_{\boldsymbol{\epsilon}} f(x)| > \lambda\}| \leq \frac{C}{\lambda}\|f\|_1. \tag{3}$$

By I_{rl} we will denote the dyadic interval $[2^{-r}l, 2^{-r}(l+1))$. Very often we will identify such an interval with an ordered pair of integers (r,l). Clearly two such dyadic intervals are either disjoint or one is contained in the other. To prove 3 we fix a function $f \in L_2 \cap L_1$, a sequence of signs $\boldsymbol{\epsilon}$ and a number $\lambda > 0$. From the Calderon–Zygmund decomposition (see e.g. [14] Th. 6.9.) we obtain a collection $\mathcal{S}$ of ordered pairs of integers such that the corresponding dyadic intervals are *disjoint* and such that for $(r,l) \in \mathcal{S}$

$$\lambda < \frac{1}{|I_{rl}|}\int_{I_{rl}} |f| \leq 2\lambda \tag{4}$$

and for $x \in \mathbb{R} \setminus \bigcup_{(r,l)\in\mathcal{S}} I_{rl}$ we have

$$|f(x)| \leq \lambda \text{ almost everywhere.} \tag{5}$$

It easily follows from 4 that

$$\left| \bigcup_{(r,l)\in\mathcal{S}} I_{rl} \right| \leq \frac{1}{\lambda} \|f\|_1. \tag{6}$$

We denote $F =: \mathbb{R} \setminus \bigcup_{(r,l)\in\mathcal{S}} I_{rl}$ and $f_{rl} =: f \cdot \chi_{I_{rl}}$.

By P_j we denote the orthonormal projection in $L_2(\mathbb{R})$ onto $\operatorname{span}\{\Psi_{ik}\}_{k\in\mathbb{Z}, i<j}$, so we can write

$$P_j h = \sum_{i<j} \sum_{k\in\mathbb{Z}} \langle h, \Psi_{ik} \rangle \, \Psi_{ik}. \tag{7}$$

By $Q_j h$ we will denote the complementary orthonormal projection i.e.

$$Q_j h = h - P_j h = \sum_{i\geq j} \sum_{k\in\mathbb{Z}} \langle h, \Psi_{ik} \rangle \, \Psi_{ik}.$$

Operators P_j and Q_j clearly commute with operators $U_{\boldsymbol{\epsilon}}$.

Using those notions we decompose f into two functions. We write

$$g = f \cdot \chi_F + \sum_{(r,l)\in\mathcal{S}} P_r(f_{rl}) \tag{8}$$

and

$$h = f - g = \sum_{(r,l)\in\mathcal{S}} f_{rl} - P_r(f_{rl}). \tag{9}$$

Later on we will prove the following two facts:

Fact 1 *There exists a constant C, independent of f, λ and $\boldsymbol{\epsilon}$ such that*

$$\left\| \sum_{(r,l)\in\mathcal{S}} P_r(f_{rl}) \right\|_2 \leq C\sqrt{\lambda \|f\|_1}. \tag{10}$$

Fact 2 *There exists an even, integrable function $\eta(x)$ monotone decreasing for $x > 10$ and a constant C, independent of f, r, l and $\boldsymbol{\epsilon}$ such that for each function $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ with* $\operatorname{supp} f \subset I_{rl}$ *we have*

$$|U_{\boldsymbol{\epsilon}} Q_r f(x)| \leq C\|f\|_1 2^r \eta(2^r x - l) \tag{11}$$

for all $x \in \mathbb{R}$ such that $|2^r x - l| > 10$.

Using the above two Facts we can give a proof of Proposition 2

Clearly

$$|\{x : |U_{\boldsymbol\epsilon} f(x)| > \lambda\}| \le |\{x : |U_{\boldsymbol\epsilon} g(x)| > \lambda/2\}| + |\{x : |U_{\boldsymbol\epsilon} h(x)| > \lambda/2\}|. \tag{12}$$

From 5 we obtain

$$\int_F |f(x)|^2 \, dx \le \lambda \int_F |f(x)| \, dx \le \lambda \|f\|_1.$$

From this and Fact 1 we obtain $\|g\|_2 \le C\sqrt{\lambda\|f\|_1}$. Thus using the Markov's inequality we obtain

$$\begin{aligned} |\{x : |U_{\boldsymbol\epsilon} g(x)| > \lambda/2\}| &= |\{x : |U_{\boldsymbol\epsilon} g(x)|^2 > \lambda^2/4\}| \le \tfrac{4}{\lambda^2}\|U_{\boldsymbol\epsilon} g\|_2^2 \\ &= \tfrac{4}{\lambda^2}\|g\|_2^2 \le \tfrac{C}{\lambda^2}\lambda\|f\|_1 = \tfrac{C}{\lambda}\|f\|_1. \end{aligned} \tag{13}$$

Let us denote $A =: \bigcup_{(r,l)\in\mathcal{S}} 10 \cdot I_{rl}$. It follows from 6 that

$$|A| \le \tfrac{10}{\lambda}\|f\|_1. \tag{14}$$

On the other hand

$$\begin{aligned} \int_{\mathbb{R}\setminus A} |U_{\boldsymbol\epsilon} h| &\le \sum_{(r,l)\in\mathcal{S}} \int_{\mathbb{R}\setminus A} |U_{\boldsymbol\epsilon} Q_r(f_{rl})| \\ &\le \sum_{(r,l)\in\mathcal{S}} \int_{\mathbb{R}\setminus 10 I_{rl}} |U_{\boldsymbol\epsilon} Q_r(f_{rl})|. \end{aligned}$$

Using now Fact 2 we get

$$\int_{\mathbb{R}\setminus A} |U_{\boldsymbol\epsilon} h| \le C \sum_{(r,l)\in\mathcal{S}} \|f_{rl}\|_1 \le C\|f\|_1. \tag{15}$$

From the Markov's inequality, 14 and 15 we get

$$\begin{aligned} |\{x : |U_{\boldsymbol\epsilon} h(x)| > \lambda/2\}| &\le |A| + |\{x \in \mathbb{R}\setminus A : |U_{\boldsymbol\epsilon} h(x)| > \lambda/2\}| \\ &\le \tfrac{10}{\lambda}\|f\|_1 + \tfrac{2C}{\lambda}\|f\|_1 = \tfrac{C}{\lambda}\|f\|_1. \end{aligned} \tag{16}$$

Putting together 13 and 16 we get Proposition 2

Now we will aim at Fact 1 and Fact 2.

Proposition 3 *If $f \in L_1(\mathbb{R})$ and* supp $f \subset [0,1]$, *then for $|x| > 10$ we have*

$$|U_{\boldsymbol\epsilon} Q_0 f(x)| \le C\|f\|_1 \sum_{j\ge 0} 2^j \varphi(\tfrac{1}{3} 2^j x). \tag{17}$$

Proof We have

$$U_{\boldsymbol{\epsilon}} Q_0 f(x) = \sum_{j\geq 0}\sum_{k\in\mathbb{Z}} \varepsilon_{jk} \langle f, \Psi_{jk}\rangle \Psi_{jk}.$$

We split this sum into two as follows

$$\begin{aligned} U_{\boldsymbol{\epsilon}} Q_0 f(x) &= \sum_{j\geq 0}\sum_{|k|<5\cdot 2^j} + \sum_{j\geq 0}\sum_{|k|\geq 5\cdot 2^j} \varepsilon_{jk} \langle f, \Psi_{jk}\rangle \Psi_{jk} \\ &=: \Sigma^1(x) + \Sigma^2(x) \end{aligned} \tag{18}$$

Note that for each $j\in\mathbb{Z}$ and $t\in\mathbb{R}$ we have

$$\sum_{k\in\mathbb{Z}} |\Psi(2^j t - k)| \leq \sum_{k\in\mathbb{Z}} \varphi(2^j t - k) \leq 4\int_0^\infty \varphi(x)\, dx \leq C \tag{19}$$

and also for $|k| < 5\cdot 2^j$ and $|x| > 10$

$$|\Psi(2^j x - k)| \leq \varphi(2^j x - k) \leq \varphi(|2^j x/2|) = \varphi(2^{j-1}|x|). \tag{20}$$

Those facts and 1 allow us to estimate $\Sigma^1(x)$ as follows:

$$\begin{aligned} |\Sigma^1(x)| &\leq \sum_{j\geq 0}\sum_{|k|<5\cdot 2^j} \int_{\mathbb{R}} |f(t) 2^{j/2}\Psi(2^j t - k)|\, dt \cdot 2^{j/2}\varphi(2^j x - k) \\ &\leq \sum_{j\geq 0} 2^j \varphi(2^{j-1}x) \int_{\mathbb{R}} |f(t)| \sum_{|k|<5\cdot 2^j} |\Psi(2^j t - k)|\, dt \\ &\leq \sum_{j\geq 0} 2^j\varphi(2^{j-1}x)\|f\|_1 \cdot \|\sum_{k\in\mathbb{Z}}\varphi(u-k)\|_\infty \\ &\leq C\|f\|_1 \sum_{j\geq 0} 2^j\varphi(2^{j-1}x). \end{aligned} \tag{21}$$

The summand $\Sigma^2(x)$ we estimate as follows:

$$\begin{aligned} |\Sigma^2(x)| &\leq \sum_{j\geq 0}\sum_{|k|\geq 5\cdot 2^j} 2^j \int_0^1 |f(t)||\Psi(2^j t - k)|\, dt\, |\Psi(2^j x - k)| \\ &\leq \sum_{j\geq 0}\sum_{|k|\geq 5\cdot 2^j} 2^j \|f\|_1 \sup_{t\in[0,1]} \varphi(2^j t - k)\, \varphi(2^j x - k) \\ &\leq \|f\|_1 \sum_{j\geq 0} 2^j \sum_{|k|\geq 5\cdot 2^j} \varphi(k - 2^j)\varphi(2^j x - k) \\ &\leq \|f\|_1 \sum_{j\geq 0} 2^j \sum_{k\in\mathbb{Z}} \varphi(k)\varphi(2^j(x-1) - k) \end{aligned}$$

The appeal to Lemma 1 completes the proof of Proposition 3. ♠

Proposition 4 *The series $\sum_{j\geq 0} 2^j\varphi(2^j x)$ is absolutely convergent for $x \neq 0$ and the function $F(x) =: \sum_{j\geq 0} 2^j\varphi(2^j x)$ is even, monotone decreasing on $(0,\infty)$ and integrable on each half-line (δ,∞) with $\delta > 0$.*

Proof Since $\varphi(x)$ is even, positive and monotone decreasing on $(0,\infty)$, once we show that $\int_\delta^\infty F(x)\,dx < \infty$ for every $\delta > 0$ all the rest follows.

$$\begin{aligned}\int_\delta^\infty F(x)\,dx &= \sum_{j\geq 0}\int_\delta^\infty \varphi(2^j x)2^j\,dx = \sum_{j\geq 0}\int_{2^j\delta}^\infty \varphi(u)\,du \\ &= \sum_{j\geq 0}\sum_{k\geq j}\int_{2^k\delta}^{2^{k+1}\delta}\varphi(u)\,du = \sum_{k\geq 0}(k+1)\int_{2^k\delta}^{2^{k+1}\delta}\varphi(u)\,du \\ &\leq C(\delta)\int_\delta^\infty \ln(u+1)\varphi(u)\,du \leq C(\delta).\end{aligned}$$

♠

Proof of Fact 2. Fact 2 follows immediately from Proposition 3, Proposition 4 and an appropriate change of variables. ♠

Proposition 5 *There exists a constant C such that for each $\boldsymbol{\epsilon}$ and every function $f \in L_1(\mathbb{R})$*

$$\|U_{\boldsymbol{\epsilon}} P_j f\|_\infty \leq C \cdot 2^j \cdot \|f\|_1 \tag{22}$$

Proof A change of variables argument shows that it suffices to prove 22 for $j = 0$. The following calculation which uses properties of φ and Lemma 1 shows 22 in this case:

$$\begin{aligned}|U_{\boldsymbol{\epsilon}} P_0 f(x)| &\leq \sum_{j<0} 2^j \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} |f(t)||\Psi(2^j t - k)|\,dt\,|\Psi(2^j x - k)| \\ &\leq C\sum_{j<0} 2^j \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} |f(t)|\varphi(2^j t - k)\,dt \\ &\leq C\sum_{j<0} 2^j \int_{\mathbb{R}} |f(t)|\left(\sum_{k\in\mathbb{Z}}\varphi(2^j t - k)\right)\,dt \\ &\leq C\sum_{j<0} 2^j \|f\|_1 \cdot \|\sum_{k\in\mathbb{Z}}\varphi(2^j t - k)\|_\infty \\ &\leq C\|f\|_1.\end{aligned}$$

♠

Proposition 6 *There exists a bounded, even, integrable decreasing on $(0,\infty)$ function $\alpha(t)$ constant on each interval $(2k+1, 2k+3)$ with $k \in \mathbb{Z}$, such that for $f \in L_2(\mathbb{R})$ with* supp $f \subset I_{jk}$ *we have*

$$|P_j f(t)| \leq 2^j \|f\|_1 \alpha(2^j t - k). \tag{23}$$

PROOF Using an appropriate change of variables we see that it suffices to check 23 for $j=0$ and $k=0$. From Proposition 5 we see that we can take $\alpha(t) = const$ for $|t|<10$. We make sure that this constant is $\geq F(10)$ where the function F is defined in Proposition 4. Since $P_0 f = f - Q_0 f$ we see that for $|t|>10$ we have $P_0 f(t) = -Q_0 f(t)$ and we apply Propositions 3 and 4. Next we modify the resulting function to make it constant where needed. We simply define $\alpha(x) = \alpha(|x|) = F(k)$ for $|x|>10$ and $k \leq |x| < k+1$. ♠

Now we are ready to establish Fact 1. It follows immediately from the following

Proposition 7 *Suppose I_{jk} are disjoint dyadic intervals and f_{jk} are functions such that* supp $f_{jk} \subset I_{jk}$ *and*

$$\lambda < \frac{1}{|I_{jk}|}\int_{I_{jk}} |f_{jk}| \leq 2\lambda.$$

Then

$$\int_{\mathbb{R}} \Big|\sum_{jk} P_j f_{jk}\Big|^2 \leq C\lambda \sum_{jk} \|f_{jk}\|_1. \tag{24}$$

PROOF Let us write

$$\int_{\mathbb{R}} \Big|\sum_{jk} P_j f_{jk}\Big|^2 = \sum_{jk}\sum_{j'k'} \int_{\mathbb{R}} P_j f_{jk} \overline{P_{j'} f_{j'k'}} = I.$$

Since P_j's are orthonormal projections and $P_j P_{j'} = P_{\min(j,j')}$, using 23 we get

$$\begin{aligned}
I &= \sum_{jk}\sum_{j'k'} \int_{\mathbb{R}} P_{\min(j,j')} f_{jk} \cdot \overline{f_{j',k'}} \\
&\leq 2\sum_{jk}\sum_{j'\geq j,k'} \Big|\int_{\mathbb{R}} P_j f_{jk} \cdot \overline{f_{j'k'}}\Big| \\
&\leq 2\sum_{jk}\sum_{j'\geq j,k'} \int_{I_{j'k'}} \|f_{jk}\|_1 2^j \alpha(2^j x - k)|f_{j'k'}(x)|\,dx \\
&\leq 2\sum_{jk} 2^j \|f_{jk}\|_1 \sum_{j'\geq j,k'} \int_{I_{j'k'}} \alpha(2^j x - k)|f_{j'k'}(x)|\,dx.
\end{aligned}$$

Since $j' \geq j$ we see that $\alpha(2^j x - k)$ is constant on interval $I_{j'k'}$, so we can write each integral as

$$\int_{I_{j'k'}} \alpha(2^j x - k)|f_{j'k'}(x)|\,dx = \frac{1}{|I_{j'k'}|}\int_{I_{j'k'}} |f_{j'k'}(x)\,dx \int_{I_{j'k'}} \alpha(2^j x - k)\,dx$$

so from our assumptions we get

$$I \leq 2\sum_{jk} \|f_{jk}\|_1 2^j \sum_{j'\geq j,k'} \int_{Ij'k'} \alpha(2^j x - k) \cdot 2\lambda$$

$$
\begin{aligned}
&\leq \quad 4\lambda \sum_{jk} \|f_{jk}\|_1 2^j \int_{\bigcup I_{j'k'}} \alpha(2^j x - k)\, dx \\
&\leq \quad 4\lambda \sum_{jk} \|f_{jk}\|_1 2^j \int_{\mathbb{R}} \alpha(2^j x - k)\, dx \\
&\leq \quad 4\lambda \int_{\mathbb{R}} \alpha(u)\, du \sum_{jk} \|f_{jk}\|_1 \\
&\leq \quad C\lambda \sum_{jk} \|f_{jk}\|_1.
\end{aligned}
$$

♠

Now we can formally conclude the proof of Theorem 1.

PROOF OF THEOREM 1 First observe that from Proposition 2, Marcinkiewicz interpolation theorem (see e.g. [11], [13] or [14] Theorem 6.23) and basic duality we infer that all operators $U_{\boldsymbol{\epsilon}}$ are continous on $L_p(\mathbb{R})$ for $1 < p < \infty$ and their norms on each $L_p(\mathbb{R})$ are uniformly bounded. Since those L_p spaces are reflexive we infer that for each f in $L_p(\mathbb{R})$ the series

$$\sum_{jk\in\mathbb{Z}} \langle f, \Psi_{jk} \rangle \Psi_{jk}$$

converges unconditionally in $L_p(\mathbb{R})$ to some function. Since for $f \in L_2 \cap L_p$ this series converges to f, one easily checks that it has to converge to f for all $f \in L_p(\mathbb{R})$. ♠

REMARK It follows from Theorem 2.1 from [5] that for $1 < p \leq 2$, and thus by duality for all p, $1 < p < \infty$, the basis $(\Psi_{jk})_{jk\in\mathbb{Z}}$ considered in Theorem 1 is equivalent in $L_p(\mathbb{R})$ with the classical Haar basis.

2 Unimodular wavelets

In this section we want to prove that some unimodular wavelets, among them the most important Shannon wavelet, give unconditional bases in $L_p(\mathbb{R})$, $1 < p < \infty$. Those wavelets have very bad decay; they are not in $L_1(\mathbb{R})$ and decay like $(1 + |x|)^{-1}$.

A *unimodular wavelet*, also called in the literature *minimally supported frequency wavelet [8]* or *s-elementary wavelet [2]*, is a wavelet $\Psi \in L_2(\mathbb{R})$ such that $\sqrt{2\pi}\hat{\Psi}(\xi)$ is a characteristic function of a set. Clearly the symbol $\hat{f}$ denotes the Fourier transform of f and the symbol $\check{f}$ denotes the inverse Fourier transform of f. For more information on unimodular wavelets see books [8], [14] and original papers [7], [4] or [2]. The most important of them is a *Shannon wavelet* S given by

$$\hat{S}(\xi) = \tfrac{1}{\sqrt{2\pi}} \chi_{[-2\pi,-\pi]\cup[\pi,2\pi]}. \tag{25}$$

From 25 we easily obtain the explicite formula for the Shannon wavelet, namely

$$S(x) = \frac{1}{\pi x}\left(\sin 2\pi x - \sin \pi x\right). \tag{26}$$

This formula easily gives that $S(0) = 1$ and $S(k) = 0$ for $k \in \mathbb{Z}$ and $k \neq 0$. In order to prove the unconditionality of wavelet bases associated with unimodular wavelets we will use the classical Littlewood–Paley Theorem (cf. [11] or [13]) which we now describe. Let $0 \leq a_1 < a_2 < \ldots$ and $0 \geq b_1 > b_2 > b_3 > \ldots$ be two sequences of real numbers and let

$$\begin{aligned} I_j &= [a_j, a_{j+1}) \quad\ \ \text{if } j > 0 \\ I_0 &= [b_1, a_1) \\ I_j &= [b_{-j-1}, b_{-j}) \quad \text{if } j < 0. \end{aligned} \tag{27}$$

Assume that there exists a number $\lambda > 1$ such that $\frac{a_{j+1}}{a_j} \geq \lambda$ and $\frac{b_{j+1}}{b_j} \geq \lambda$ for all $j \in \mathbb{Z}$.

Theorem 2 (Littlewood-Paley) *There exists a constant C_p such that for all sequences of $\boldsymbol{\epsilon} = (\varepsilon_j)_{j\in\mathbb{Z}}$ with $\varepsilon_j = 0, \pm 1$ and all $f \in L_p(\mathbb{R})$, $1 < p < \infty$*

$$g(\xi) = \sum_{j\in\mathbb{Z}} \varepsilon_j \chi_{I_j} \hat{f}(\xi)$$

is the Fourier transform of a function $f_{\boldsymbol{\epsilon}} \in L_p(\mathbb{R})$ and

$$\|f_{\boldsymbol{\epsilon}}\|_p \leq C_p \|f\|_p. \tag{28}$$

We also have

$$\Big(\int_{\mathbb{R}} (\sum_{j\in\mathbb{Z}} |(\chi_{I_j}\hat{f})^\vee(x)|^2)^{p/2} dx\Big)^{1/p} \leq C_p \|f\|_p. \tag{29}$$

First we will consider the special case of the Shannon wavelet.

Proposition 8 *The wavelet basis $(S_{jk})_{jk\in\mathbb{Z}}$ associated with the Shannon wavelet S is an unconditional basis in $L_p(\mathbb{R})$ for $1 < p < \infty$.*

PROOF We will compare the Shannon wavelet with the Haar wavelet h (cf. [14]) defined as

$$h(t) = \begin{cases} 1 & \text{for } t \in [0, \frac{1}{2}) \\ -1 & \text{for } t \in [\frac{1}{2}, 1) \\ 0 & \text{otherwise.} \end{cases}$$

It is classical and also follows from Theorem 1 that the Haar basis $(h_{jk})_{jk\in\mathbb{Z}}$ is an unconditional basis in $L_p(\mathbb{R})$ for $1 < p < \infty$. We will show that for any (finite) sequence of scalars $(a_{jk})_{jk\in\mathbb{Z}}$ we have

$$c_p\Big\| \sum_{jk\in\mathbb{Z}} a_{jk}h_{jk}\Big\|_p \leq \Big\| \sum_{jk\in\mathbb{Z}} a_{jk}S_{jk}\Big\|_p \leq C_p\Big\| \sum_{jk\in\mathbb{Z}} a_{jk}h_{jk}\Big\|_p \tag{30}$$

where constants $C_p \geq c_p > 0$ depend only on p. By duality it suffices to show only the left hand side inequality in 30. From the Littlewood-Paley Theorem and the definition of the Shannon wavelet we see that

$$\Big\| \sum_{jk\in\mathbb{Z}} a_{jk}S_{jk}\Big\|_p^p \geq c\int_{\mathbb{R}} \Big(\sum_{j\in\mathbb{Z}} |\sum_{k\in\mathbb{Z}} a_{jk}S_{jk}(x)|^2\Big)^{p/2} dx. \tag{31}$$

Now let M denotes the classical Hardy-Littlewood maximal function (for definition and basic properties see e.g. [11], [13], [14]). From the Fefferman-Stein result (see e.g. [13] XII.1 Theorem 1.1) we get that 31 can be minorized by (a constant times)

$$\int_{\mathbb{R}} \Big(\sum_{j\in\mathbb{Z}} \Big[M(\sum_{k\in\mathbb{Z}} a_{jk}S_{jk})(x)\Big]^2\Big)^{p/2} dx. \tag{32}$$

To minorize this we will need the following Lemma whose proof is momentarily postponed.

Lemma 9 *There exists a constant c such that for any (finite) sequence of scalars* $(a_k)_{k\in\mathbb{Z}}$

$$M(\sum_{k\in\mathbb{Z}} a_k S(\bullet - k))(x) \geq c\sum_{k\in\mathbb{Z}} |a_k||h(x-k)|. \tag{33}$$

Using this Lemma and a change of variables we immediately get that for each $j \in \mathbb{Z}$

$$M(\sum_{k\in\mathbb{Z}} a_{jk}S_{jk})(x) \geq c\sum_{k\in\mathbb{Z}} |a_{jk}||h_{jk}(x)| \tag{34}$$

so we can minorize 32 further by

$$c\int_{\mathbb{R}} \Big(\sum_{j\in\mathbb{Z}} (\sum_{k\in\mathbb{Z}} |a_{jk}||h_{jk}(x)|)^2\Big)^{p/2} dx = c\int_{\mathbb{R}} \Big(\sum_{jk\in\mathbb{Z}} |a_{jk}|^2|h_{jk}(x)|^2\Big)^{p/2} dx. \tag{35}$$

Since the Haar basis is unconditional we can finally minorize 35 to obtain

$$\| \sum_{jk\in\mathbb{Z}} a_{jk}S_{jk}\|_p \geq c\| \sum_{jk\in\mathbb{Z}} a_{jk}h_{jk}\|_p$$

which shows 30. To obtain the Proposition we need to observe that 30 implies that for each f in $L_p(\mathbb{R})$ the series $\sum_{jk} \langle f, S_{jk}\rangle S_{jk}$ converges unconditionally to some function and that for $f \in L_2(\mathbb{R}) \cap L_p(\mathbb{R})$ it converges to f, so it converges to f for all f's in $L_p(\mathbb{R})$.

♠

PROOF OF LEMMA 9 Let us denote $F(x) = \sum_{k\in\mathbb{Z}} a_k S(x-k)$. From 26 we see that

$$F(k) = a_k \text{ for } k \in \mathbb{Z}. \tag{36}$$

Also supp $\hat{F}(\xi) \subset [-2\pi, 2\pi]$. This implies that F is a restriction to $\mathbb{R}$ of an entire function of exponential type and thus the Bernstein type inequality holds i.e. we have

$$|F'(x)| \leq C\|F\|_p \text{ for all } x \in \mathbb{R}. \tag{37}$$

The proof of this inequality can be found in [10] sections 3.2.2 and 3.3. Now 37 and 36 and the definition of M easily imply 33. ♠

REMARK This proof is a modification of the proof of Proposition 3.6 of [15] which really is a periodic version of our Proposition 8.

Now let us consider unimodular wavelets Ψ such that $\sqrt{2\pi}\hat{\Psi}(\xi)$ is a characteristic function of a set K which is a union of a *finite* number of intervals. Note that almost all unimodular wavelets considered in [4], [7], [8], [2] are of this type.

Theorem 3 *Let $\Psi(x)$ be a unimodular wavelet as described above. Then $(\Psi_{jk})_{jk\in\mathbb{Z}}$ is an unconditional basis in $L_p(\mathbb{R})$, $1 < p < \infty$.*

PROOF We know that $\sqrt{2\pi}\hat{\Psi}(\xi) = \chi_K$ where $K = \bigcup_{r=1}^N I_r$ where I_r's are intervals. Since Ψ is a wavelet we infer that sets $2^j K$ with $j \in \mathbb{Z}$ are disjoint (up to a measure 0). Thus, may be increasing N, we can write $K = \bigcup_{r=1}^N 2^{s(r)} J_r$ where $s(r) \in \mathbb{Z}$, each J_r is a union of at most two intervals, one contained in $[-2\pi, -\pi]$ and the other contained in $[\pi, 2\pi]$ and sets $2^{s(r)} J_r$ are disjoint. This implies that for each r sets $2^j J_r$ are disjoint (up to a measure 0) for $j \in \mathbb{Z}$. Let us denote $A_r =: \bigcup_{j\in\mathbb{Z}} 2^j J_r$.

Let us use the convention that for a set $B \subset \mathbb{R}$ we consider an orthonormal projection $P(B)$ defined on $L_2(\mathbb{R})$ by

$$(P(B)f)^\wedge = \hat{f} \cdot \chi_B.$$

It follows from the Littlewood-Paley Theorem that all projections of this type used in this Section are actually continuous on $L_p(\mathbb{R})$ for $1 < p < \infty$.

Let us take $f = \sum_{jk\in\mathbb{Z}} a_{jk} \Psi_{jk}$. Since all the constants appearing later are independent of f in order to avoid formal problems we can assume that this sum is actually finite.

Then we have

$$
\begin{aligned}
\|f\|_p &\leq C(p,N)\sum_{r=1}^{N}\|P(A_r)f\|_p \\
&= C(p,N)\sum_{r=1}^{N}\|\sum_{jk\in\mathbb{Z}} a_{jk}P(A_r)\Psi_{jk}\|_p \\
&= C(p,N)\sum_{r=1}^{N}\|\sum_{jk\in\mathbb{Z}} a_{jk}P(2^{j+s(r)}J_r)\Psi_{jk}\|_p.
\end{aligned}
\tag{38}
$$

Now instead of the function

$$
P(A_r)(f)(x) = \sum_{jk\in\mathbb{Z}} a_{jk}P(2^{j+s(r)}J_r)(\Psi_{jk})(x)
$$

let us consider the function

$$
g_r(x) = P(A_r)(f)(2^{-s(r)}x).
$$

Then

$$
\|g_r\|_p = 2^{s(r)/p}\|P(A_r)f\|_p. \tag{39}
$$

On the other hand a standard calculations using elementary properties of the Fourier transform give

$$
\begin{aligned}
P(A_r)(f)(2^{-s(r)}x) &= \sum_{jk\in\mathbb{Z}} a_{jk}(\chi_{2^{j+s(r)}J_r})^{\vee}(2^{-s(r)}x) \\
&= 2^{s(r)}\sum_{jk\in\mathbb{Z}} a_{jk}(\chi_{2^{j}J_r})^{\vee}(x) \\
&= 2^{s(r)}P(A_r)\sum_{jk\in\mathbb{Z}} a_{jk}S_{jk}(x).
\end{aligned}
\tag{40}
$$

Thus from 38, 39 and 40 we get

$$
\begin{aligned}
\|f\|_p &\leq C(p,N)\sum_{r=1}^{N}2^{-s(r)/p}\|g_r\|_p \\
&= C(p,N)\sum_{r=1}^{N}2^{-s(r)/p}\|2^{s(r)}P(A_r)\sum_{jk\in\mathbb{Z}} a_{jk}S_{jk}\|_p \\
&\leq C(p,N)\sum_{r=1}^{N}2^{(1-\frac{1}{p})s(r)}\|P(A_r)\sum_{jk\in\mathbb{Z}} a_{jk}S_{jk}\|_p.
\end{aligned}
$$

Since N and $s(r)$ depend only on the wavelet and $P(A_r)$ are bounded projections in $L_p(\mathbb{R})$ by the Littlewood-Paley Theorem, this clearly implies that

$$
\|\sum_{jk\in\mathbb{Z}} a_{jk}\Psi_{jk}\|_p \leq C\|\sum_{jk\in\mathbb{Z}} a_{jk}S_{jk}\|_p. \tag{41}
$$

Since both systems $(\Psi_{jk})_{ij\in\mathbb{Z}}$ and $(S_{jk})_{jk\in\mathbb{Z}}$ are orthonormal bases, by duality we get the converse inequality

$$\|\sum_{jk\in\mathbb{Z}} a_{jk}S_{jk}\|_p \le C\|\sum_{jk\in\mathbb{Z}} a_{jk}\Psi_{jk}\|_p \tag{42}$$

so from Proposition 8 we infer (exactly like in the end of the proof of Proposition 8) that $(\Psi_{jk})_{jk\in\mathbb{Z}}$ is an unconditional basis in $L_p(\mathbb{R})$ with $1<p<\infty$. ♠

REMARK Let us note that our proofs show that bases discussed in Theorem 3 are equivalent to Haar basis.

Let us observe that we cannot adapt methods of Section 1 to unimodular wavelets discussed above. The reason is that the operators $U_{\boldsymbol{\epsilon}}$ when applied to the Shannon wavelet are not uniformly of weak type (1-1). To see it observe that if all $U_{\boldsymbol{\epsilon}}$ would be of weak type (1-1), then also operators

$$R_N(f) = \sum_{j=0}^{N}(-1)^j \langle f, S_{j0}\rangle S_{j0}$$

would be uniformly of weak type (1-1). If we take $f_\varepsilon(t) = \frac{1}{2\varepsilon}\chi_{(-\varepsilon,\varepsilon)}(t)$, i.e. f_ε is a sufficiently good approximation of the Dirac's delta, then we have

$$\lim_{\varepsilon\to 0} R_N(f_\varepsilon) = R_N(\delta_0) = \sum_{j=0}^{N}(-1)^j 2^{j/2} S_{j0}(x).$$

Thus we would have that

$$\begin{aligned} R_N(\delta_0) &= \sum_{j=0}^{N}(-1)^j 2^j S(2^j x) \\ &= \sum_{j=0}^{N}(-1)^j \frac{1}{\pi x}(\sin 2^{j+1}\pi x - \sin 2^j \pi x) \\ &= \frac{1}{\pi x}\Big[2\sum_{j=1}^{N}(-1)^j \sin 2^j\pi x - \sin \pi x + (-1)^N \sin 2^{N+1}\pi x\Big] \end{aligned}$$

is uniformly of weak type (1-1). Clearly

$$\frac{1}{\pi x}\Big[-\sin\pi x + (-1)^N \sin 2^{N+1}\pi x\Big]$$

is uniformly in N of weak type (1-1). Thus to justify our initial claim we need to show that

$$F_N =: \frac{1}{\pi x}\Big[2\sum_{j=1}^{N}(-1)^j \sin 2^j \pi x\Big]$$

is not uniformly in N of weak type (1-1). But by the classical result of Salem and Zygmund (see [12] or [16, Chap. XVI Th. 5.5]) we know that the distribution of $\frac{1}{\sqrt{N}}\sum_{j=1}^{N}(-1)^{j+1}\sin 2^{j}\pi x$ on the interval $[-1,1]$ tends to the standard gaussian distribution. This implies that for N big enough we have

$$\Big|\sum_{j=1}^{N}(-1)^{j+1}\sin 2^{j}\pi x\Big| \geq \frac{1}{2}\sqrt{N}$$

on a set A_N with $|A_N| \geq c > 0$. But then for $x \in A_N$ we have $|F_N(x)| \geq \frac{1}{\pi}\sqrt{N}$ which shows that functions F_N are not uniformly of weak type (1-1).

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