Ergodic Characterizations of Reflexivity of Banach Spaces

Vladimir P. Fonf
Ben-Gurion University of the Negev, Beer-Sheva, Israel
e-mail: fonf@math.bgu.ac.il

Michael Lin
Ben-Gurion University of the Negev, Beer-Sheva, Israel
e-mail: lin@math.bgu.ac.il

Przemyslaw Wojtaszczyk
University of Warsaw, Warsaw, Poland
e-mail: wojtaszczyk@mimuw.edu.pl

Abstract

Let X be a Banach space with a basis. We prove the following characterizations:
(i) X *is finite-dimensional if and only if every power-bounded operator is uniformly ergodic.*
(ii) X *is reflexive if and only if every power-bounded operator is mean ergodic.*
(iii) X *is quasi-reflexive of order one if and only if for every power-bounded operator* T*,* T *or* T^* *is mean ergodic.*

1. Introduction

Using the spectral theorem, von-Neumann (1931) proved that for every unitary operator T in a complex Hilbert space,

$$Px := \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} T^k x \qquad \text{exists} \ \ \forall x. \tag{$*$}$$

A linear operator T on a (real or complex) Banach space X is called *mean ergodic* if $(*)$ is satisfied, and *uniformly ergodic* if the convergence in $(*)$ is uniform on the unit ball, i.e., $\lim_{n\to\infty} ||\frac{1}{n}\sum_{k=1}^{n} T^k - P|| = 0$. A Banach space X will be called *mean ergodic* if every power-bounded operator $T \in B(X)$ satisfies $(*)$.

A simple proof of von-Neumann's mean ergodic theorem, due to F. Riesz, appeared in 1937 in Hopf's *Ergodentheorie*, and was followed by more general results: Riesz (1938) showed that the L_p spaces $(1 < p < \infty)$ are mean ergodic, and Lorch (1939) proved that *all reflexive Banach spaces are mean ergodic.* In general Banach spaces, Kakutani (1938) and Yosida (1938) obtained characterizations of the convergence of the sequence $\{\frac{1}{n}\sum_{k=1}^{n} T^k x\}$ for a given $x \in X$. A power-bounded

T in a Banach space is mean ergodic if and only if X has the following *ergodic decomposition*

$$X = \{y \in X : \; Ty = y\} \oplus \overline{(I-T)X} \tag{$**$}$$

In general, the right-hand side of $(**)$ is precisely the set of $x \in X$ for which the sequence $\{\frac{1}{n}\sum_{k=1}^{n} T^k x\}$ converges. We denote by $F(T)$ the set of fixed points of the linear operator T. Sine (1970) proved that *a power bounded T is mean ergodic if and only if $F(T)$ separates $F(T^*)$.* We refer the reader to [K] for the proofs and for the references of the above results. Note that since $F(T) \cap \overline{(I-T)X} = \{0\}$, the Hahn-Banach Theorem yields that $F(T^*)$ always separates $F(T)$.

Brunel and Sucheston [BSu_1], [BSu_2] proved that *a Banach space X is super-ergodic if and only if it is super-reflexive.* For the definitions see those papers. It is known [Da, p. 169] that super-reflexivity characterizes the existence of an equivalent norm, in which the space is uniformly convex, and thus also super-ergodicity characterizes that property. It follows from the work of Brunel and Sucheston that super-ergodicity with respect to contractions is the same as super-ergodicity with respect to power-bounded operators (operators which are contractions in some *equivalent* norm).

Sucheston [Su] posed the following question, concerning the converse of Lorch's result: *If every contraction in a Banach space X is mean ergodic, is X reflexive?* The weaker assumption, that only all isometries are mean ergodic, is not sufficient for reflexivity, since Davis [D] had constructed an equivalent norm on the real ℓ_1, for which the only isometries are I and $-I$. Even under the stronger assumption, that all power-bounded operators are mean ergodic, i.e., X is mean ergodic, the problem is still unsolved.

In this paper, we obtain a positive solution to this last problem for Banach spaces with bases (throughout this paper, a basis means a *Schauder basis*). From this result we conclude that *a Banach space X is reflexive if and only if every closed subspace is mean ergodic.* Our construction also yields that a Banach space with basis is finite-dimensional if and only if every power-bounded operator is uniformly ergodic. We show that a non-reflexive Banach space with basis is 1-quasi-reflexive if and only if for every power-bounded T, T or T^* is mean ergodic, and such a space is *not* mean ergodic.

Recently, Emel'yanov and Wolff [EW] have proved that on any (not necessarily separable) Banach space X which contains c_0 there is a power-bounded operator which is not mean ergodic. Our methods yield a different proof of this result.

We mention that Eeml'yanov [E] proved that if every power-bounded operator on a Banach lattice E is mean ergodic, then E is reflexive. For a dual Banach lattice, Zaharopol [Z] proved that if all power-bounded *positive* operators are mean ergodic, then the Banach lattice is reflexive.

2. Ergodic characterizations of reflexivity and 1-quasi-reflexivity

Definition 1. A *Schauder decomposition* of a Banach space X is an infinite sequence $\{E_k\}_{k=1}^{\infty}$ of closed subspaces $\{0\} \neq E_k \subset X$ such that each $x \in X$ has a unique representation $x = \sum_{k=1}^{\infty} x_k$, with $x_k \in E_k$, $(k = 1, 2, ...)$. We denote it by $X = \sum_k E_k$.

Note that the definition (see [S, vol. II, pp. 485-489]) does not require the spaces E_k to be finite-dimensional.

When $X = \sum_k E_k$, the corresponding "coordinate" projectors $Q_k : X \to E_k$ are defined for $x = \sum x_j$ $(x_j \in E_j)$ by $Q_k x = x_k$ $(k = 1, 2, ...,)$. The "partial sum" operators $P_n = \sum_{k=1}^{n} Q_k$ $(n = 1, 2, ...)$ satisfy $\lim_n P_n x = x$ for every x. An adaptation of the proof given in [S, vol. I, pp. 18-20] for bases shows that the partial sums operators are continuous and uniformly bounded (see [S, vol. II, p. 499]). Hence also the coordinate projectors are continuous and uniformly bounded. By introducing the following norm (which is equivalent to the original one)

$$|||x||| = \sup\{||Q_k x||,\ ||P_k x|| :\ k = 1, 2, ...\},$$

we get

$$|||Q_k||| = |||P_k||| = 1 \qquad \forall k \geq 1. \tag{1}$$

Since power-boundedness of a linear operator is the same in all equivalent norms, whenever necessary we may assume that the original norm $||\cdot||$ satisfies (1).

Definition 2. A Schauder decomposition $X = \sum_{k=1}^{\infty} E_k$ is called *shrinking* if for each $f \in X^*$ we have $\lim_{n\to\infty} ||f_{|\sum_{i=n}^{\infty} E_i}|| = 0$.

Note that if each subspace E_k of a Schauder decomposition is spanned by one vector e_k, then $\{e_k\}$ is a basis of X, and when this decomposition is shrinking we call $\{e_k\}$ a *shrinking basis.*

Lemma. *Let $X = \sum_k X_k$ be a non-shrinking Schauder decomposition of a Banach space X. Then there is a Schauder decomposition $X = \sum_k E_k$ with the following property: there exist a linear functional $h \in X^*$ and a sequence $\{e_k\}$, such that for every $k \geq 1$ we have $e_k \in E_k$, $||e_k|| \leq 1$, and $h(e_k) = 1$.*

Proof. Since the decomposition $X = \sum_k X_k$ is not shrinking, there is a functional $f \in X^*$ with $||f|| = 1$ and $\limsup_n ||f_{|\sum_{k\geq n} X_k}|| = \alpha > 0$ (obviously $\alpha \leq 1$). Take a vector y_1 such that

$$y_1 = \sum_{k=n_1+1}^{\infty} a_k^{(1)} x_k^{(1)}, \quad x_k^{(1)} \in X_k, \quad ||y_1|| = 1, \quad |f(y_1)| > \frac{\alpha}{2}.$$

Find $n_2 > n_1$ with $||\sum_{k=n_2+1}^{\infty} a_k^{(1)} x_k^{(1)}|| < \alpha/4$, and take a vector y_2 such that

$$y_2 = \sum_{k=n_2+1}^{\infty} a_k^{(2)} x_k^{(2)}, \quad x_k^{(2)} \in X_k, \quad ||y_2|| = 1, \quad |f(y_2)| > \frac{\alpha}{2}.$$

We continue inductively and obtain a strictly increasing sequence of integers $\{n_j\}$ and a sequence of vectors $\{y_j\}$, such that for each j,

$$y_j = \sum_{k=n_j+1}^{\infty} a_k^{(j)} x_k^{(j)}, \quad x_k^{(j)} \in X_k, \quad ||y_j|| = 1, \quad |f(y_j)| > \frac{\alpha}{2},$$

and $|| \sum_{k=n_j+1}^{\infty} a_k^{(j-1)} x_k^{(j-1)} || < \frac{\alpha}{4}$.

Define $E_1 = \sum_{i=1}^{n_2} X_i$, and $E_j = \sum_{i=n_j+1}^{n_{j+1}} X_i$ for $j \geq 2$. Clearly $\{E_j\}$ is a Schauder decomposition. Put $z_j = \sum_{k=n_j+1}^{n_{j+1}} a_k^{(j)} x_k^{(j)}$. Then $z_j \in E_j$, and, by the construction, $1 - \alpha/4 \leq ||z_j|| \leq 1 + \alpha/4$ and $\alpha/4 \leq |f(z_j)| \leq 1 + \alpha/4$. Finally, let $h = \frac{4+\alpha}{\alpha} f$, and define $e_j = \frac{\alpha}{(4+\alpha) f(z_j)} z_j$. Then $||e_j|| \leq 1$, and $h(e_j) = 1$ for every j.

Theorem 1. *If a Banach space X admits a non-shrinking Schauder decomposition, then there exists a power bounded linear operator $T \in B(X)$ which is not mean ergodic.*

Proof. Let $X = \sum_{k=1}^{\infty} E_k$ be the decomposition given by the Lemma, so we have $h \in X^*$ and a sequence $\{e_k\}$ such that $e_k \in E_k,\ h(e_k) = 1,\ ||e_k|| \leq 1,\ k = 1, 2, ...$ The change to a norm satisfying (1) yields that $||e_k|| \leq M$, so we replace e_k by $M^{-1} e_k$ and h by Mh. Thus, we can assume that the norm satisfies (1) (for the projectors defined by $\{E_k\}$).

Take an arbitrary sequence $a = \{a_j\}_{j=1}^{\infty}$ of positive numbers with

$$\sum_{j=1}^{\infty} a_j = 1, \quad a_j > 0,\ j = 1, 2, ..., \tag{2}$$

and denote $A_n = \sum_{j=1}^{n} a_j$. For $x \in X$ and $m > n \geq 2$ we then have

$$\sum_{k=n}^{m} A_k Q_k x = \sum_{k=n}^{m} Q_k (\sum_{j=1}^{n-1} a_j + \sum_{j=n}^{k} a_j) x = (\sum_{j=1}^{n-1} a_j)(\sum_{k=n}^{m} Q_k x) + \sum_{j=n}^{m} a_j (\sum_{k=j}^{m} Q_k x).$$

Since $\sum_k Q_k x$ converges, we see that $\{\sum_{k=1}^{m} A_k Q_k x\}_m$ is a Cauchy sequence in the norm, hence converges. Denoting $P_0 = 0$, we obtain by (1) that

$$|| \sum_{k=1}^{m} A_k Q_k x || = || \sum_{j=1}^{m} a_j (\sum_{k=j}^{m} Q_k x) || = || \sum_{j=1}^{m} a_j (P_m - P_{j-1}) x || \leq 2 ||x||. \tag{3}$$

We now define an operator $T_a : X \to X$ by

$$T_a x = \sum_{k=1}^{\infty} A_k Q_k x + \sum_{j=2}^{\infty} h(P_{j-1} x) a_j e_j. \tag{4}$$

Since $||e_j|| \leq 1$ for every j, (1), (2) and (3) yield $||T_a x|| \leq (2 + ||h||) ||x||$, so

$$||T_a|| \leq 2 + ||h|| \tag{5}$$

The bound $(2 + ||h||)$ for $||T_a||$ does not depend on the choice of the sequence $\{a_j\}$ satisfying (2), so in order to prove that the operator T_a is power-bounded, it is enough to show that for sequences a and b satisfying (2), the composition T_aT_b is of the same type (say T_c). We formulate it precisely:

Claim. *Let the sequences $a = \{a_j\}$ and $b = \{b_j\}$ satisfy (2), and define the operators T_a and T_b by (4) (with $B_0 = 0$ and $B_n = \sum_{j=1}^n b_j$). Then the sequence $c = \{c_j\}$, defined by $c_j = A_jb_j + B_{j-1}a_j$, $j = 1, 2, ...$, satisfies (2), and the composition satisfies $T_aT_b = T_c$.*

Proof. Clearly $C_1 = A_1B_1$. We obtain that c satisfies (2), since for $n \geq 2$ we have

$$C_n = \sum_{j=1}^n c_j = \sum_{j=1}^n (B_{j-1}a_j + b_jA_j) = \sum_{k=1}^{n-1} b_k \sum_{j=k+1}^n a_j + \sum_{k=1}^n b_k \sum_{j=1}^k a_j$$

$$= \sum_{k=1}^{n-1} b_kA_n + b_nA_n = A_nB_n.$$

Now we show that $T_aT_b = T_c$. In view of the decomposition $X = \sum_k E_k$, it is enough to show that $T_aT_be = T_ce$ for each vector $e \in E_k$, $k = 1, 2, ...$ Fix k, and take $x_k \in E_k$. The definiton (4) yields

$$T_ax_k = A_kx_k + \sum_{j=k+1}^\infty h(x_k)a_je_j = A_kx_k + h(x_k) \sum_{j=k+1}^\infty a_je_j \tag{6}$$

We apply (6) to T_b and to T_a, and obvious computations yield

$$T_a(T_bx_k) = T_a(B_kx_k + h(x_k) \sum_{j=k+1}^\infty b_je_j) = A_kB_kx_k + h(x_k) \sum_{j=k+1}^\infty (B_{j-1}a_j + b_jA_j)e_j$$

Since $A_kB_k = C_k$, an application of (6) to c yields $T_a(T_bx_k) = T_cx_k$, and the claim is proved.

To prove that the power-bounded operator T_a is not mean ergodic, it is enough to show (by the above mentioned Sine's criterion [K]) that the *non-zero* functional h is a fixed point for T_a^*, while zero is the only fixed point for T_a.

Suppose that $T_ax = x$. Using the definition (4), we have

$$\sum_{k=1}^\infty Q_kx = x = T_ax = \sum_{k=1}^\infty A_kQ_kx + \sum_{k=2}^\infty h(P_{k-1}x)a_ke_k.$$

We look at the components in each E_k. For $k = 1$ we have $Q_1x = A_1Q_1x$, so $Q_1x = 0$ since $A_1 = a_1 < 1$. For $k > 1$ we obtain $(1 - A_k)Q_kx = h(P_{k-1}x)a_ke_k$. Assume now that $Q_jx = 0$ for every $j < k$; then $P_{k-1}x = 0$, and thus $(1 - A_k)Q_kx = 0$, yielding $Q_kx = 0$. Hence by induction we have $Q_kx = 0$ for every $k \geq 1$, so $T_ax = x$ implies $x = 0$.

Fix an arbitrary $k \geq 1$ and take an arbitrary $e \in E_k$. Applying h to (6) and using $h(e_j) = 1$ for every j, we obtain

$$(T_a^*h)(e) = h(T_a e) = h(A_k e + h(e) \sum_{j=k+1}^{\infty} a_j e_j) = A_k h(e) + h(e) \sum_{j=k+1}^{\infty} a_j = h(e).$$

In view of the decomposition $X = \sum_k E_k$, we have $T_a^* h = h$. The Theorem is now proved.

Remarks. 1. Clearly, $\sum_{j=1}^{\infty} a_j P_{j-1} x$ converges in norm for $\{a_j\}$ satisfying (2), and the equality of the vector sums appearing in the first and third terms of (3) yields

$$\sum_{k=1}^{\infty} A_k Q_k x = \sum_{j=1}^{\infty} a_j (I - P_{j-1}) x = x - \sum_{j=1}^{\infty} a_j P_{j-1} x \quad \forall x \in X. \tag{7}$$

2. In fact, the functional h of the previous proof is the *only* fixed point for T_a^* (up to a scalar multiplier). We now prove this fact, though not needed for Theorem 1, since it will be important for Theorem 4.

So, we assume that $T_a^* f = f$, and prove that $f = th$ for some scalar t. With $P_0 = 0$, we can write (4) as

$$T_a x = \sum_{m=1}^{\infty} (A_m Q_m x + h(P_{m-1} x) a_m e_m).$$

Duality yields (with w^*-convergence of the series)

$$T_a^* f = \sum_{m=1}^{\infty} (A_m Q_m^* f + a_m f(e_m) P_{m-1}^* h). \tag{8}$$

Since $f = \sum_{m=1}^{\infty} Q_m^* f$ (again, w^*-convergence of the series), the assumption $T_a^* f = f$ and (8) yield

$$\sum_{m=1}^{\infty} [(1 - A_m) Q_m^* f - a_m f(e_m) P_{m-1}^* h] = 0. \tag{9}$$

Now fix an integer n, and apply the functional of the left side of (9) to a vector $z_n \in E_n$:

$$\sum_{m=1}^{\infty} [(1 - A_m)(Q_m^* f)(z_n) - a_m f(e_m)(P_{m-1}^* h)(z_n)] =$$

$$\sum_{m=1}^{\infty} [(1 - A_m) f(Q_m z_n) - a_m f(e_m) h(P_{m-1} z_n)] =$$

$$(1 - A_n) f(z_n) - \sum_{m=n+1}^{\infty} a_m f(e_m) h(z_n) = 0. \tag{10}$$

Since $h(e_n) = 1$, (10) with $z_n = e_n$ yields the following system of linear equations in the unknowns $t_m = f(e_m)$ $m = 1, 2, ...$:

$$(1 - A_n)t_n = \sum_{m=n+1}^{\infty} a_m t_m \qquad n = 1, 2, ... \tag{11}$$

Substraction of equation number n from equation number $(n + 1)$ shows that the only solution of the system (11) is $t_1 = t_2 = ... = t$, so

$$f(e_n) = t = th(e_n), \; n = 1, 2, ... \tag{12}$$

In order to prove that $f = th$, we show the equality on each E_n. Fix $x \in E_n$. Then $h(x - h(x)e_n) = h(x) - h(x)h(e_n) = 0$ since $h(e_n) = 1$. Denote $z = x - h(x)e_n$. By (10) with $z_n = z$ we have

$$(1 - A_n)f(z) - \sum_{m=n+1}^{\infty} a_m f(e_m)h(z) = 0.$$

Since $h(z) = 0$, this yields $f(z) = 0$, so $f(x) = h(x)f(e_n) = th(x)$.

Corollary 1. *Let X be a (separable) Banach space with a basis. Then X is reflexive if and only if every power-bounded operator in X is mean ergodic.*

Proof. Zippin [Zi] proved that if a non-reflexive Banach space has a basis, then it has a non-shrinking basis. Thus, if X is not reflexive, Theorem 1 yields a power-bounded operator which is not mean ergodic. If X is reflexive, apply Lorch's Theorem.

Corollary 2. *For every Banach space X the following assertions are equivalent:*

(i) X is reflexive.

(ii) Every closed subspace of X is mean ergodic (i.e., each power bounded operator defined on a closed subspace is mean ergodic).

Proof. $(ii) \Rightarrow (i)$: Suppose that X is non-reflexive. By a result of Pelczynski [Di, p. 54], X has a non-reflexive (separable) closed subspace with a basis, and Corollary 1 yields a contradiction. $(i) \Rightarrow (ii)$ follows from Lorch's Theorem, since a closed subspace of a reflexive Banach space is reflexive.

Theorem 2. *If an infinite-dimensional Banach space X admits a Schauder decomposition, then there is a mean ergodic power-bounded operator $T \in B(X)$ which is not uniformly ergodic.*

Proof. We may assume that the norm satisfies (1). For a sequence $\{a_j\}$ satisfying (2), let $T_a x = \sum_{k=1}^{\infty} A_k Q_k x$ (T_a is defined as in (4) with $h = 0$). By the proof of Theorem 1, T_a is power-bounded, and has no fixed points except 0 (this part of the proof did not require the special properties of h, which were used only to show that T_a^* had h as fixed point).

Let $f \in X^*$ satisfy $T_a^* f = f$. Then for $z_n \in E_n$ we have

$$f(z_n) = f(T_a z_n) = \sum_{k=1}^{\infty} A_k f(Q_k z_n) = A_n f(z_n).$$

Since $A_n < 1$ for each n, we have $f(z_n) = 0$ for any $z_n \in E_n$. Hence $T_a^* f = f$ implies $f = 0$, which yields (Hahn-Banach) that $\overline{(I - T_a)X} = X$, so T_a is mean ergodic.

Since T_a has no non-zero fixed points, it is uniformly ergodic if and only if $I - T_a$ is invertible on X [L2]. By definition,

$$(I - T_a)x = x - \sum_{k=1}^{\infty} A_k Q_k x = \sum_{k=1}^{\infty} (1 - A_k) Q_k x = \sum_{k=1}^{\infty} \Big(\sum_{j=k+1}^{\infty} a_j \Big) Q_k x.$$

We now take $a_j = 2^{-j}$ for $j \geq 1$, and put $T = T_a$. Then $(I - T)x = \sum_{k=1}^{\infty} 2^{-k} Q_k x$. Take a sequence $e_k \in E_k$ with $\|e_k\| = 1$ for every k. Then $\sum_{k=1}^{\infty} \frac{1}{k^2} e_k$ converges, say to y. When we try to solve $(I - T)x = y$, we obtain the equations $Q_k x = \frac{2^k}{k^2} e_k$, which imply $\|Q_k x\| \to \infty$. Since for $x \in X$ we have $Q_k x \to 0$, there is no $x \in X$ with $(I - T)x = y$, so $I - T$ is not invertible, and therfore T is not uniformly ergodic.

Remark. The existence of T which is not uniformly ergodic in X with an unconditional basis was proved in [FLR].

Corollary 3. *Let X be a Banach space with basis. Then the following conditions are equivalent:*

(i) X is finite-dimensional.

(ii) Every power-bounded operator is uniformly ergodic.

(iii) Every mean ergodic power-bounded operator is uniformly ergodic.

Proof. Clearly, if X is finite-dimensional, every power-bounded $T \in B(X)$ is uniformly ergodic. Obviously, $(ii) \Rightarrow (iii)$, and $(iii) \Rightarrow (i)$ follows from Theorem 2.

The ideas in the proof of Theorem 1 can be used to obtain the following generalization, which applies also to non-separable spaces:

Theorem 3. *Let a Banach space X admit a sequence of projectors $\{P_n\}$ such that*

(i) $\sup ||P_n|| < \infty$

(ii) $P_n P_m = P_{\min(m,n)}$

(iii) There exists a functional $h \in X^*$ such that for each $n \geq 1$ there is a vector $e_n \in (P_n - P_{n-1})X$ with $||e_n|| \leq 1$ and $h(e_n) = 1$.

Then, for a sequence $\{a_n\}$ which satisfies (2), the operator

$$Sx = x - \sum_{n=2}^{\infty} a_n P_{n-1}x + \sum_{n=2}^{\infty} a_n h(P_{n-1}x)e_n, \tag{13}$$

is power-bounded and not mean ergodic.

Proof. It is immediate from the assumptions that S is well defined. Denote $Y_n = P_nX$. By (ii), $\{Y_n\}$ is an increasing sequence of subspaces, and $Y = \overline{\bigcup_{n\geq 1} Y_n}$ is a S-invariant subspace. By (ii), $\lim_n P_n y = y$ for $y \in Y_k$, so by (i) $\lim_n P_n y = y$ for every $y \in Y$. Let $Q_1 = P_1$, and $Q_k = P_k - P_{k-1}$ for $k \geq 2$. It is easily checked, using (ii), that each Q_k is a projection, and $Q_kQ_j = 0$ for $j \neq k$. Since $\sum_{k=1}^{n} Q_k y = P_n y \to y$ for every $y \in Y$, the sequence $\{E_k\}$ with $E_k = Q_kX = Q_kY$ is a Schauder decomposition of Y. Assumption (iii) allows us to apply the proof of Theorem 1 to Y – the restriction of S to Y is the operator T_a constructed in that proof, when we substitute (7) into (4). Hence there is a vector $y \in Y$ such that the sequence $\{\frac{1}{n}\sum_{k=1}^{n} T_a^k y\}$ does not converge, which shows that S is not mean ergodic.

To complete the proof, we have to show that S is power-bounded on all of X (this does not follow from the proof of Theorem 1, since Y is not necessarily complemented in X).

Denote S by S_a to indicate the dependence on $\{a_j\}$ (which satifies (2)). Clearly

$$||S_a x|| \leq ||x|| + ||x||(1 + ||h||) \sup_n ||P_n||$$

so we have an estimate of the norm of S_a, which is independent of a. As in the proof of Theorem 1, the power-boundedness follows from the following claim.

Claim. *Let the sequences $a = \{a_j\}$ and $b = \{b_j\}$ satisfy (2), and define the sequence $c = \{c_j\}$ by $c_j = A_j b_j + B_{j-1} a_j$, $j = 1, 2, ...,$. Then $\{c_j\}$ satisfies (2), and the operators S_a, S_b and S_c defined by (13) satisfy $S_aS_b = S_c$.*

Proof. $\{c_j\}$ satisfies (2) by the claim in the proof of Theorem 1. Apply property (ii) to (13), we obtain

$$P_n(S_b x) = P_n x - \sum_{i=2}^{\infty} b_i P_n P_{i-1} x + \sum_{i=2}^{\infty} b_i h(P_{i-1}x) P_n e_i$$

$$= P_n x - \sum_{i=1}^{n} b_i P_{i-1} x - (1 - B_n) P_n x + \sum_{i=1}^{n} b_i h(P_{i-1}x) e_i.$$

We substitute this into

$$S_a(S_b x) = S_b x - \sum_{n=2}^{\infty} a_n P_{n-1}(S_b x) + \sum_{n=2}^{\infty} a_n h(P_{n-1}(S_b x)) e_n,$$

and some straight forward (tedious) calculations prove the claim.

Corollary 4. *Let X be a Banach space which contains a closed subspace isomorphic to c_0. Then there exists a power-bounded $T \in B(X)$ which is not mean ergodic.*

Proof. Let Y be a closed subspace of X isomorphic to c_0, and let $y_n \in Y$ be the image of the of the unit vector $e_n \in c_0$. Then $\{y_n\}$ is a basis of Y, and there is $K > 0$ such that $\| \sum_{j=1}^{\infty} a_j y_j \| \leq K \sup_j |a_j|$. Let $\{y_n^*\} \subset Y^*$ be the coefficient functionals, which are uniformly bounded, and take $f_n \in X^*$ a Hahn-Banach extension of y_n^*. We now define $x_n = \sum_{j=1}^{n} y_j$ and $g_n = f_n - f_{n+1}$. Then $g_k(x_n) = \delta_{kn}$, and the operators $P_n x = \sum_{k=1}^{n} g_k(x) x_k$ are commuting projections satisfying assumption (ii) of Theorem 3. The functional $h = f_1$ satisfies assumption (iii) since $h(x_n) = 1$. Finally, the isomorphism of Y and c_0 yields that $\sup_n \|P_n\| \leq 2K \sup_n \|f_n\|$, since

$$P_n x = f_1(x) y_1 - f_{n+1}(x) x_n + \sum_{k=2}^{n} f_k(x)(x_k - x_{k-1}) = \sum_{k=1}^{n} [f_k(x) - f_{n+1}(x)] y_k.$$

Remark. The Corollary was first proved in [EW] using a different method. Note that if X is separable (as any space with a basis is) and contains c_0, then (even without a basis), we easily obtain a power-bounded operator $T \in B(X)$ which is not mean ergodic, since c_0 is complemented in X [Di, p. 71], and $T_0 \in B(c_0)$ defined by $T_0(a_1, a_2, a_3, ...) = (a_1, a_1, a_2, ...)$ (i.e., $T_0(\sum_{j=1}^{\infty} a_j e_j) = a_1 e_1 + \sum_{j=2}^{\infty} a_{j-1} e_j$ in terms of the standard basis $\{e_j\}$) is power-bounded and not mean ergodic (T_0 has no non-zero fixed points in c_0, but $T_0^* e_1 = e_1$ in ℓ_1). Thus, the novelty of the result is for non-separable spaces, in which c_0 need not be complemented.

For a basis $\{x_i\}$ of a Banach space X, we denote by $\{x_i^*\}$ the associated coefficient functionals. Recall [S, vol. I p. 268] that a basis $\{x_i\}$ is called *k-shrinking* if codim $[x_i^*]_{i=1}^{\infty} = k$ (where $[y_j]_{j=1}^{\infty}$ denotes the closed linear manifold generated by the sequence $\{y_j\}_{j=1}^{\infty}$). It is well known [Da],[S, vol. I p. 272] that a basis is 0-shrinking if and only if it is shrinking in the sense of Definition 2.

Definition 3. A Banach space X is called *quasi-reflexive of order k* if $\dim X^{**}/X = k < \infty$ (we identify X with its natural embedding in X^{**}). The original construction of the James space [Ja], valid over the real or complex field, yields an example of a Banach space with basis which is quasi-reflexive of order 1.

Theorem 4. *Let X be a Banach space with a basis, such that* $\dim X^{**}/X \geq 2$. *Then there exists a power-bounded operator $T \in B(X)$ such that neither T nor T^* are mean ergodic.*

Proof. According to Zippin's result [Zi] mentioned above, the (non-reflexive) space X has a non-shrinking basis, say $\{u_i\}$; that is, $\{u_i\}$ is a basis which is not 0-shrinking. If $\{u_i\}$ is k-shrinking with $k \geq 2$, we keep it. If $\{u_i\}$ is 1-shrinking, we use Theorem 1 of [DeLS]: *Let X be a Banach space which is not quasi-reflexive of order k* (in our case $\dim X^{**}/X \geq 2$, so X is not quasi-reflexive of order 1). *If X has a k-shrinking basis, then X has a $(k+1)$-shrinking basis.* Thus, we have established that there exists in X a basis $\{x_i\}$ such that

$$\operatorname{codim}[x_i^*]_{i=1}^{\infty} \geq 2. \tag{14}$$

Since this basis $\{x_i\}$ is not shrinking, the Lemma (with $X_k = \{tx_k : t \in \mathbf{R}\}$) yields a Schauder decomposition $X = \sum_k E_k$ with the following property: there exist a functional $h \in X^*$ and a sequence $\{e_k\}$, $e_k \in E_k$, $k = 1, 2, ...$ such that $h(e_k) = 1$, $||e_k|| \leq 1, k = 1, 2, ...$ By the construction in the proof of the Lemma, each E_k is finite-dimensional, and the decomposition $X = \sum_k E_k$ has the following additional property: the "partial sum" operators P_m are of the form

$$P_m x = \sum_{i=1}^{n_m} x_i^*(x) x_i, \quad x \in X, \quad m = 1, 2, ...$$

This yields $P_m^* f = \sum_{i=1}^{n_m} f(x_i) x_i^*$ for $f \in X^*$, and so, $\bigcap_m \ker P_m^{**} = [x_i^*]_{i=1}^{\infty\perp}$. By (14),

$$\dim \bigcap_{m=1}^{\infty} \ker P_m^{**} = \dim(X^*/[x_i^*]_{i=1}^{\infty})^* \geq \operatorname{codim}[x_i^*]_{i=1}^{\infty} \geq 2. \tag{15}$$

We now proceed as in the proof of Theorem 1. For a sequence $a = \{a_j\}$ satisfying (2), define the operator T_a according to (4). It was shown that T_a is power-bounded and not mean ergodic, $F(T_a) = \{0\}$, and $F(T_a^*) = \{th\}$. We will choose $\{a_j\}$ satisfying (2) such that $\sum_{n=1}^{\infty}(1 - A_n) < \infty$ (e.g., $a_j = 2^{-j}$). Since $\dim F(T_a^*) = 1$, to prove that the operator T_a^* is not mean ergodic we have to show (by Sine's criterion) that $\dim F(T_a^{**}) \geq 2$. By (15), it is enough to show

$$F(T_a^{**}) = \bigcap_{m=1}^{\infty} \ker P_m^{**}. \tag{16}$$

¿From (4), (1), and the condition $\sum_n (1 - A_n) < \infty$, it follows that

$$T_a^{**}\psi = \psi + \sum_{m=1}^{\infty}(A_m - 1)Q_m^{**}\psi + \sum_{m=2}^{\infty} a_m P_{m-1}^{**}\psi(h) e_m.$$

Hence $\psi \in F(T_a^{**})$ is equivalent to

$$\sum_{m=1}^{\infty}(1 - A_m)Q_m^{**}\psi = \sum_{m=2}^{\infty} a_m P_{m-1}^{**}\psi(h) e_m. \tag{17}$$

If $P_n^{**}\psi = 0$ for every $n \geq 1$, then clearly (17) holds, so $\psi \in F(T_a^{**})$. Suppose now that $\psi \in F(T_a^{**})$; we apply the operators Q_n^{**} to both sides of (17), and obtain the equations $(1 - A_n)Q_n^{**}\psi = a_n P_{n-1}^{**}\psi(h) e_n$, $n = 1, 2, ...,$. Solving successively, we obtain $Q_n^{**}\psi = 0$ for $n \geq 1$, which proves (16) and completes the proof of the theorem.

Remark. Every operator T on X is the restriction of T^{**} to its invariant subspace X, so if T^{**} is mean ergodic, so is T. Hence, if both operators T and T^* are not mean ergodic, then automatically all the next conjugates (T^{**}, $T^{***}, \ldots$) are not mean ergodic.

Theorem 5. *Let X be a non-reflexive Banach space with a basis. Then the following assertions are equivalent:*

(i) X is quasi-reflexive of order one.

(ii) For each power-bounded operator $T \in B(X)$, T or T^* is mean ergodic.

Proof. $(ii) \Rightarrow (i)$: If dim $X^{**}/X \geq 2$, then Theorem 4 yields a contradiction to (ii).

$(i) \Rightarrow (ii)$: Let T be a power-bounded operator on X which is not mean ergodic. By Sine's criterion, $F(T)$ does not separate $F(T^*)$, so there is $f_0 \in F(T^*)$ such that $f_0(y) = 0$ for every $y \in F(T)$. To show that T^* is mean ergodic, we will prove that $F(T^*)$ separates $F(T^{**})$. As mentioned in the introduction, $F(T^*)$ always separates $F(T)$. Hence $F(T^{**})$ separates $F(T^*)$, so there is $\psi_0 \in F(T^{**})$ such that $\psi_0(f_0) \neq 0$. By the definition of f_0, ψ_0 is not in $F(T)$, so $\psi_0 \notin X$. Since dim $X^{**}/X = 1$, every $\phi \in X^{**}$ is of the form $\phi = \alpha\psi_0 + x$ with $x \in X$, so each $\psi \in F(T^{**})$ is of the form $\psi = \alpha\psi_0 + y$ with $y \in F(T)$. We then have $\psi(f_0) = \alpha\psi_0(f_0) \neq 0$ for $\psi \in F(T^{**})$ with $\alpha \neq 0$. If $\alpha = 0$, then ψ is in $F(T)$, and the separation of $F(T)$ by $F(T^*)$ provides an $f \in F(T^*)$ with $\psi(f) = f(\psi) \neq 0$. Hence $F(T^*)$ separates $F(T^{**})$, so T^* is mean ergodic by Sine's criterion.

Remark. The implication $(i) \Rightarrow (ii)$ does not require a basis for X.

If $T \in B(X)$ is power-bounded, then it is easily shown that

$$(I-T)X \subset \{y: \sup_n \| \sum_{k=1}^n T^k y\| < \infty\} \subset \overline{(I-T)X}.$$

When T is uniformly ergodic, then [L$_2$] $(I-T)X$ is closed, which yields

$$(I-T)X = \{y: \sup_n \| \sum_{k=1}^n T^k y\| < \infty\} \tag{18}$$

If X is a *dual* space and T is a power-bounded *dual* operator, then (18) holds [L$_1$]. It now follows from Theorem 2 that in every infinite-dimensional reflexive Banach space X with a basis there is a power-bounded T which is not uniformly ergodic, but satisfies (18). It was shown in [FLR] that if X is a separable Banach space which does not contain infinite-dimensional dual spaces, then (18) implies uniform ergodicity. This result is true also in complex Banach spaces, since the needed result of [F_3], stated for real spaces, is valid also in complex spaces, with the same proof.

Proposition. *Let Z be an infinite-dimensional Banach space which is the dual of a separable Banach space (e.g., Z is a separable dual space). Then there exists an infinite-dimensional Banach space E with a basis such that E^* is isomorphic to a closed subspace of Z.*

Proof. This proposition is an immediate consequence of the results of [JR]: Let F be separable, with $F^* = Z$. Since the unit ball of Z is compact in the weak-* topology and not in the norm, there is a sequence $\{y_n\}$ in Z which is weak-* convergent to 0, such that $\limsup_n \|y_n\| > 0$. Combining Theorem III.1 and Proposition II.1(a) of [JR], we obtain a separable Banach space E with a basis, such that E^* is isometrically isomorphic to the weak-* closed subspace generated in Z by a subsequence $\{y_{n_k}\}$.

Theorem 6. *Let X be a Banach space. Then the following conditions are equivalent:*

(i) X does not contain an infinite-dimensional closed subspace isomorphic to the dual of a separable Banach space.

(ii) Every power-bounded operator T defined on a closed subspace Y, which satisfies $F(T) = \{0\}$ and $(I - T)Y = \{y \in Y :\ \sup_n \|\sum_{k=1}^n T^k y\| < \infty\}$, is uniformly ergodic.

If X is separable, each of the previous conditions is equivalent to

(iii) Every power-bounded operator T defined on a closed subspace Y which satisfies $(I - T)Y = \{y \in Y :\ \sup_n \|\sum_{k=1}^n T^k y\| < \infty\}$ is uniformly ergodic.

Proof. The proof of (i) $\Rightarrow$ (ii) is the same as that of Corollary 3.4(ii) of [FLR], noting that the results of [F_1], [F_2] used there yield a dual of a *separable* space. For the complex case, we observe that the proof of Proposition 6.7 in [FLiP] is valid also for complex Banach spaces, and it implies the required result of [F_1].

We now assume that (ii) holds. If (i) does not hold, then X has an infinite-dimensional closed subspace Z which is isomorphic to the dual F^* of a separable Banach space F. By the Proposition, there is an infinite-dimensional Banach space E with a basis, such that E^* is isomorphic to a subspace of F^*. Hence E^* is isomorphic to a closed subspace of Z, say Y. By Theorem 2, there is a power-bounded $S \in B(E)$ which is not uniformly ergodic, with $F(S^*) = \{0\}$. Let $T \in B(Y)$ correspond to S^*. Then $(I - T)Y = \{y \in Y :\ \sup_n \|\sum_{k=1}^n T^k y\| < \infty\}$ by [L_1], but T is not uniformly ergodic since S is not – contradicting (ii). Hence (i) must hold.

When X is separable, the proof of (i)$\Rightarrow$(iii) runs along the lines of the proof of Theorem 3.3 of [FLR], applied to any closed subspace Y (which also satisfies (i)). For the complex case, in the proof of [FLR] we should replace [F_3] by Theorem 3.2 of [FLi], the proof of which is valid also for complex Banach spaces. Clearly (iii)$\Rightarrow$(ii).

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