ON APPROXIMATE ℓ_1 SYSTEMS IN BANACH SPACES

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Abstract. Let X be a real Banach space and let $(f(n))$ be a positive nondecreasing sequence. We consider systems of unit vectors $(x_i)_{i=1}^{\infty}$ in X which satisfy $\|\sum_{i\in A} \pm x_i\| \geq |A| - f(|A|)$, for all finite $A \subset \mathbb{N}$ and for all choices of signs. We identify the spaces which contain such systems for bounded $(f(n))$ and for all unbounded $(f(n))$. For arbitrary unbounded $(f(n))$, we give examples of systems for which $[x_i]$ is H.I., and we exhibit systems in all isomorphs of ℓ_1 which are not equivalent to the unit vector basis of ℓ_1. Several other results are proved.

Contents

1. Introduction

The following elementary isometric characterization of the unit vector basis of ℓ_1^n provides the motivation for this paper.

Theorem 0. *Let $(x_i)_{i=1}^n$ be unit vectors in a real Banach space X such that $\|\sum_{i=1}^n \pm x_i\| = n$ for all choices of signs. Then $\|\sum_{i=1}^n a_i x_i\| = \sum_{i=1}^n |a_i|$ for all scalars $a_1, \dots, a_n$.*

Date: June 5, 2001.

2000 *Mathematics Subject Classification.* Primary: 46B25; Secondary: 40A30, 41A65, 46B04, 46B15, 46B20.

The research of the third author was done while he was visiting the Industrial Mathematics Institute. He would like to thank Prof. P. Petrushev and Prof. R. DeVore for the invitation and warm hospitality. His research was supported by grant no. 2 PO3A 00113 from Komitet Bada'n Naukowych.

We examine the 'stability' of the above result with respect to small changes in the hypothesis. Accordingly, we study two classes of 'approximate ℓ_1 systems'. The following definition corresponds to the mildest possible weakening of the hypothesis.

Definition 1.1. Let $(x_i)_{i\in I}$ be a sequence of unit vectors in a Banach space X (where $I = \{1, 2, \dots, n\}$ or $I = \mathbb{N}$), and let $\mu \geq 0$. We say that (x_i) is a μ-approximate ℓ_1 system if

$$\left\| \sum_{i\in A} \pm x_i \right\| \geq |A| - \mu \tag{1}$$

for all finite sets $A \subset I$ and for all choices of signs.

A system which does not satisfy the above for any choice of μ will satisfy the following for some choice of $(f(n))$.

Definition 1.2. Suppose that $(f(n))_{n=1}^{\infty}$ is a strictly positive nondecreasing sequence satisfying $\lim_{n\to\infty} f(n) = \infty$. Let $(x_i)_{i\in I}$ be a sequence of unit vectors in a Banach space X. We say that (x_i) is an $f(n)$-approximate ℓ_1 system if

$$\left\| \sum_{i\in A} \pm x_i \right\| \geq |A| - f(|A|)$$

for all finite sets $A \subset I$ and for all choices of signs.

The first two sections after the Introduction concern μ-approximate ℓ_1 systems. In Section 2 we characterize the Banach spaces which contain an infinite μ-approximate ℓ_1 system: they are precisely the spaces whose duals contain an isometric copy of L_1.

In Section 3 we examine the problem of extracting a large subsystem that is $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_1^n. We show that there exists such a subsystem with finite complement. The size of the complement, however, does not depend on μ and ε alone. To show this we exhibit systems of size n for which the size of the complementary set is necessarily of order cn for any $c < 1/4$. These examples give a partial answer to a question raised by Elton.

The next two sections concern $f(n)$-approximate ℓ_1 systems. In Section 4 we characterize the Banach spaces which contain $f(n)$-approximate ℓ_1 systems for all $(f(n))$: they are precisely the spaces which have a spreading model isometric to the the unit vector basis of ℓ_1. We use this result to give examples of $f(n)$-approximate ℓ_1 systems whose closed linear spans do not contain any unconditional basic sequence.

In Section 5 we exhibit nontrivial examples of $f(n)$-approximate ℓ_1 systems in the space ℓ_1 (and all its isomorphs). In particular, we construct, for any given $(f(n))$, examples of both conditionally basic and unconditionally basic $f(n)$-approximate ℓ_1 systems in ℓ_1 which are not equivalent to the unit vector basis of ℓ_1.

The results which we present in Section 5 are based on some observations about 'lacunary Haar' systems in L_1 and H_1. In Section 6 we pursue these ideas, proving that there is a lacunary Haar system in L_1 which is a quasi-greedy basis for its linear span.

We use standard Banach space notation and terminology throughout: for example, $[x_i]_{i=1}^{\infty}$ denotes the closed linear span of the sequence $(x_i)_{i=1}^{\infty}$.

2. μ-approximate ℓ_1 Systems

First we characterize the Banach spaces which contain an infinite μ-approximate ℓ_1 system for some $\mu > 0$. In this regard, note that Theorem 0 tells us that X contains an infinite 0-approximate ℓ_1 system if and only if X contains an isometric copy of ℓ_1.

For $A, B \subset \mathbb{N}$ and $n \in \mathbb{N}$, we write $A < B$ (respectively, $A < n$) if $\max\{i : i \in A\} < \min\{i : i \in B\}$ (respectively, $\max\{i : i \in A\} < n$).

Theorem 2.1. *Suppose that $(x_i)_{i=1}^{\infty}$ is a μ-approximate ℓ_1 system. Then, given any decreasing sequence (ε_i) of positive numbers, there is a subsequence (y_i) of (x_i) such that*

$$\left\| \sum_{i=1}^{\infty} a_i y_i \right\| \geq \sum_{i=1}^{\infty} (1 - \varepsilon_i) |a_i| \tag{2}$$

for all $(a_i) \in \ell_1$.

Proof. Set $n_0 = 0$ and suppose that $n_0 < n_1 < \cdots < n_k$ have been chosen to satisfy the following recursive hypothesis. For each choice of signs (η_i), and for all finite $A > n_k$, there exists $x^* \in B(X^*)$ such that

$$x^*(\eta_i x_{n_i}) \geq 1 - \varepsilon_i \qquad (1 \leq i \leq k) \tag{3}$$

and

$$x^* \left(\sum_{i=1}^{k} \eta_i x_{n_i} + \sum_{i \in A} \eta_i x_i \right) \geq k + |A| - \mu. \tag{4}$$

For $k = 0$, note that (3) is vacuously true and that (4) just follows from the Hahn-Banach theorem and the definition of a μ-approximate ℓ_1 system. So the recursive definition starts. The following claim will establish the inductive step.

Claim. There exists $n_{k+1} > n_k$ such that if $A \geq n_{k+1}$ and (η_i) is any choice of signs then there exists $x^* \in B(X^*)$ satisfying (3), (4), and

$$x^*(\eta_i x_i) > 1 - \varepsilon_{k+1} \qquad (i \in A). \tag{5}$$

Proof of Claim. We argue for a contradiction. If not, then there exists a choice of signs (η_i) and an infinite sequence (A_j), with $n_k < A_1 < \cdots < A_j < \ldots$, such that, for each $j \geq 1$, whenever (3) and (4) are satisfied for $A = A_j$ then (5) is not satisfied, i.e.

$$\min_{i \in A_j} x^*(\eta_i x_i) \leq 1 - \varepsilon_{k+1}. \tag{6}$$

Fix $N \geq 1$. By the recursive hypothesis there exists $x^* \in B(X^*)$ satisfying (3) and

$$x^* \left(\sum_{i=1}^{k} \eta_i x_{n_i} + \sum_{j=1}^{N} \sum_{i \in A_j} \eta_i x_i \right) \geq k + \left(\sum_{j=1}^{N} |A_j| \right) - \mu. \tag{7}$$

Thus (4) is satisfied by x^* for $A = A_j$ and for each $1 \leq j \leq N$. Hence (6) is satisfied for each $1 \leq j \leq N$. This implies

$$x^* \left(\sum_{i=1}^{k} \eta_i x_{n_i} + \sum_{j=1}^{N} \sum_{i \in A_j} \eta_i x_i \right) \leq k + \sum_{j=1}^{N} |A_j| - N\varepsilon_{k+1}. \tag{8}$$

But (8) contradicts (7) when $N > \mu/\varepsilon_{k+1}$.

For n_{k+1} as given by the claim, the recursive hypothesis is satisfied for $k+1$. Let $y_i = x_{n_i}$. Fix $(a_i) \in \ell_1$. Let $\eta_i = \operatorname{sgn}(a_i)$. Then by (3) and a weak compactness argument there exists $x^* \in B(X^*)$ such that $x^*(\eta_i y_i) \geq 1 - \varepsilon_i$ for all $i \geq 1$. Thus

$$\left\| \sum_i a_i y_i \right\| \geq x^* \left(\sum_i a_i y_i \right) \geq \sum (1 - \varepsilon_i)|a_i|.$$

$\square$

Recall that a normalized sequence (y_i) which satisfies (2) for some sequence (ε_i) of positive numbers decreasing to zero is called *an asymptotically isometric copy of* ℓ_1. This class of sequences was introduced by Hagler

[12] and has been used recently by Dowling and Lennard [7] in fixed point theory.

Theorem 2.2. *Let X be a Banach space. The following are equivalent:*

(a) *X contains an infinite μ-approximate ℓ_1 system for some $\mu \geq 0$.*
(b) *X contains an asymptotically isometric copy of ℓ_1.*
(c) *X^* contains an isometric copy of $L_1(0,1)$.*

Proof. Theorem 2.1 yields (a) $\Rightarrow$ (b). Suppose that (x_i) is an asymptotically isometric copy of ℓ_1 in X which satisfies (2) for some sequence (ε_i) of positive numbers decreasing to zero. By passing to a subsequence we may assume that $\sum_{i=1}^{\infty} \varepsilon_i < \infty$. This implies that (x_i) is a μ-approximate ℓ_1 system for $\mu = \sum_{i=1}^{\infty} \varepsilon_i$, so (b) $\Rightarrow$ (a). The equivalence of (b) and (c) was proved in [12] (cf. also [5] for the complex version). □

Remark 2.3. For several further equivalences see [12] and [5].

We do not know, however, if a μ-approximate ℓ_1 system is *itself* an asymptotically isometric copy of ℓ_1.

Question 2.4. Suppose that (x_i) is a μ-approximate ℓ_1 system. Does there exist a sequence (ε_i) of positive numbers decreasing to zero such that

$$\left\| \sum_{i=1}^{\infty} a_i x_i \right\| \geq \sum_{i=1}^{\infty} (1 - \varepsilon_i)|a_i|.$$

for all $(a_i) \in \ell_1$?

The following 'global' result will be used in Section 6.

Proposition 2.5. *Let (x_i) be a basic μ-approximate ℓ_1 system with basis constant K. Then*

$$\min\left(\frac{1}{6}, \frac{1}{24K\mu}\right) \sum_{i=1}^{\infty} |a_i| \leq \left\| \sum_{i=1}^{\infty} a_i x_i \right\| \leq \sum_{i=1}^{\infty} |a_i| \tag{9}$$

for all $(a_i) \in \ell_1$.

Proof. Suppose that $\sum_{i=1}^{n} |a_i| = 1$. Let $\eta_i = \operatorname{sgn} a_i$ $(1 \leq i \leq n)$. There exists $x^* \in Ba(X^*)$ such that $x^*\left(\sum_{i=1}^{n} \eta_i x_i\right) \geq n - \mu$. Let $A = \{1 \leq i \leq n : \eta_i x^*(x_i) \leq 3/4\}$. Then $(n - |A|) + (3/4)|A| \geq n - \mu$, i.e. $|A| \leq 4\mu$. We now consider two cases. For the first case suppose that $\sum_{i \in A} |a_i| < 1/3$. Then

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| \geq \sum_{i \notin A} a_i x^*(x_i) - \sum_{i \in A} |a_i| > \frac{2}{3}\frac{3}{4} - \frac{1}{3} = \frac{1}{6}.$$

For the second case suppose that $\sum_{i\in A}|a_i| \geq 1/3$. Then, since (x_i) is basic with basis constant K, we have

$$\left\|\sum_{i=1}^n a_i x_i\right\| \geq \frac{1}{2K}\max_{i\in A}|a_i| \geq \frac{1}{2K}\frac{\sum_{i\in A}|a_i|}{|A|} \geq \frac{1}{2K}\frac{1/3}{4\mu} = \frac{1}{24K\mu}.$$

$\square$

Remark 2.6. The factor $1/\mu$ in (9) is best possible up to a multiplicative constant. To see this, let $(x_i)_{i=1}^{\mu+1}$ be the usual basis of $\ell_\infty^{\mu+1}$, where $\mu \in \mathbb{N}$. Then $(x_i)_{i=1}^{\mu+1}$ is a μ-approximate ℓ_1 system for which $\|1/(\mu+1)\sum_{i=1}^{\mu+1} x_i\| = 1/(\mu+1)$.

3. Almost Isometric Results

Our goal in this section is to understand how well a μ-approximate ℓ_1 system compares with the standard unit vector basis of ℓ_1. The following result shows that, given $\varepsilon > 0$ and a μ-approximate ℓ_1 system, one can obtain a $(1+\varepsilon)$-isometric copy of the unit vector basis of ℓ_1 by removing a *finite* set of vectors from the system.

Proposition 3.1. *Suppose that (x_i) is a μ-approximate ℓ_1 system. Then, given $\varepsilon > 0$, there exists a finite set A such that*

$$\left\|\sum_{i\in B} \pm x_i\right\| \geq |B| - \varepsilon \tag{10}$$

whenever $A \cap B = \emptyset$. In particular, if $0 < \varepsilon < 1$, then

$$(1-\varepsilon)|\sum_{i\in B}|a_i| \leq \left\|\sum_{i\in B} a_i x_i\right\| \leq \sum_{i\in B}|a_i| \tag{11}$$

for all scalars (a_i) whenever $A \cap B = \emptyset$.

Proof. We may assume that μ is the least constant satisfying (1). There exist $N \geq 1$ and a choice of signs $(\eta_i)_{i=1}^N$ such that

$$\left\|\sum_{i=1}^N \eta_i x_i\right\| < N + \varepsilon - \mu.$$

Let $(\eta_i)_{N+1}^\infty$ be any choice of signs. Then, for $n > N$, we have

$$\begin{aligned}\left\|\sum_{i=N+1}^n \eta_i x_i\right\| &\geq \left\|\sum_{i=1}^n \eta_i x_i\right\| - \left\|\sum_{i=1}^N \eta_i x_i\right\| \\ &\geq (n-\mu) - (N+\varepsilon-\mu) = (n-N) - \varepsilon.\end{aligned}$$

Taking $A = \{1, \ldots, N\}$ gives (10), and (11) follows from (10) by the triangle inequality. $\square$

The following example shows that the cardinality of a set A which satisfies (10) does not depend only on ε and μ, even when (x_i) is 1-unconditional. (Recall that (x_i) is α-unconditional if $\|\sum_{i=1}^\infty \pm a_i x_i\| \le \alpha \|\sum_{i=1}^\infty a_i x_i\|$ for all scalars (a_i) and for all choices of signs.)

Example 3.2. For each $n \ge 1$, let $(e_i)_{i=1}^n$ be the unit vector basis of ℓ_p^n, where p is chosen so that $n^{1/p} = n - 1$. Then $(e_i)_{i=1}^n$ is a 1-approximate ℓ_1 system. But it is clear that if A satisfies (10) for $\varepsilon = 1/2$, then $|A| \to \infty$ as $n \to \infty$.

For (11), on the other hand, the cardinality of A depends only on ε and μ, provided that (x_i) is 1-unconditional.

Proposition 3.3. *(a) Let (x_i) be a μ-approximate ℓ_1 system. There exists a nonnegative sequence (ε_i) such that $\sum_{i=1}^\infty \varepsilon_i \le \mu$ and*

$$\left\| \sum_{i=1}^\infty a_i x_i \right\| \ge \sum_{i=1}^\infty a_i (1 - \varepsilon_i) \tag{12}$$

for all $(a_i) \in \ell_1$.
(b) Let $\varepsilon > 0$. If $(x_i)_{i=1}^\infty$ is 1-unconditional then A can be chosen to satisfy (11) with $|A| \le \lfloor \mu/\varepsilon \rfloor$.

Proof. (a) For each $n \in I$, select $x_n^* \in B(X^*)$ such that $x_n^*\left(\sum_{i=1}^n x_i\right) \ge n - \mu$. Note that, if $m \le n$, then $x_n^*\left(\sum_{i=1}^m x_i\right) \ge m - \mu$. By passing to a subsequence, we may assume that $x_n^*(x_m) \to 1 - \varepsilon_m$ as $n \to \infty$. Thus, for each $m \in I$,

$$\sum_{i=1}^m (1 - \varepsilon_i) = \lim_{n \to \infty} x_n^* \left(\sum_{i=1}^m x_i \right) \ge m - \mu.$$

Hence $\sum_{i=1}^\infty \varepsilon_i \le \mu$. If $(a_i) \in \ell_1$, then

$$\left\| \sum_{i=1}^\infty a_i x_i \right\| \ge \lim_{n \to \infty} x_n^* \left(\sum_{i=1}^n a_i x_i \right) = \sum_{i=1}^\infty a_i (1 - \varepsilon_i).$$

(b) For $\varepsilon > 0$, let $A = \{i \in I : \varepsilon_i > \varepsilon\}$. Then $|A| \le \lfloor \mu/\varepsilon \rfloor$. If B is disjoint from A then (12) and 1-unconditionality give (11). $\square$

Recall that a sequence (y_i) is *suppression* 1*-unconditional* if, whenever $A \subseteq B \subseteq I$, then $\|\sum_A a_i y_i\| \le \|\sum_B a_i y_i\|$ for all scalars (a_i). We shall now show that Proposition 3.3 (b) does not hold if "1-unconditional" is replaced by "suppression 1-unconditional". The following theorem is a 'local' formulation of this fact.

Theorem 3.4. *Let $\alpha \in (0, 1/4)$ and $\mu \in (0,1)$. If n is a power of 2 then there exists a norm $\|\cdot\|$ on $\mathbb{R}^n$ with the following properties:*

(i) *$(e_i)_{i=1}^n$ is a suppression 1-unconditional normalized basis of $(\mathbb{R}^n, \|\cdot\|)$.*

(ii)
$$\left\| \sum_{i=1}^{n} \pm e_i \right\| \geq n - \mu \tag{13}$$
for all choices of signs.

(iii) *For every $A \subseteq \{1, \dots, n\}$, with $|A| = 1 + \lceil (3/4 + \alpha) n \rceil$, there exists a nonzero vector x, with $\operatorname{supp} x \subseteq A$, such that*
$$\|x\|_1 \geq \left(1 + \frac{4\alpha\mu}{3 + 8\alpha - 4\alpha\mu}\right) \|x\|. \tag{14}$$

Let $I = \{1, 2, \dots, n\}$. Since n is a power of 2, there exist n sets $S_i \subseteq I$ $(1 \leq i \leq n)$ such that (here $\triangle$ denotes the *symmetric difference*)
$$|S_i \Delta S_j| = |S_i \Delta (I \setminus S_j)| = n/2 \qquad (i \neq j).$$
Indeed, one can simply take the S_i's (under the obvious correspondence) to be the rows of the Hadamard matrix of order n (see e.g. [3]).

For $1 \leq i \leq n$, we say that a set $S \subseteq I$ is *i-large* if either $|S \triangle S_i| < n/4$ or $|(I \setminus S) \triangle S_i| < n/4$. Note that, for each $1 \leq i \leq n$, the collection of all i-large sets is closed under complementation.

First we prove that every $S \subseteq I$ is i-large for at most one value i. So suppose that S is i_0-large and that $j \neq i_0$. Then either $|S \Delta S_{i_0}| < n/4$ or $|S \Delta (I \setminus S_{i_0})| < n/4$. We shall assume that the former holds (the proof in the latter case is similar). Since $|S_{i_0} \Delta S_j| = n/2$, the triangle inequality gives
$$|S \Delta S_j| \geq |S_{i_0} \Delta S_j| - |S \Delta S_{i_0}| > \frac{n}{2} - \frac{n}{4} = \frac{n}{4}.$$
Similarly, $|(I \setminus S) \Delta S_j| > n/4$. Thus, S is *not* j-large.

Let $y = (y_i)_{i \in I}$ be a vector whose coordinates belong to the interval $[-1, 1]$. We set $P(y) = \{i \in I : y_i > 1 - \mu\}$ and $N(y) = \{i \in I : y_i < -1 + \mu\}$. For $S \subseteq I$, we say that y is *S-admissible* and that y *is obtained from* S if the following conditions hold:

(a) $|y_i| \leq 1 - \mu$ whenever S is i-large.

(b) $P(y) \subseteq S$ and $N(y) \subseteq I \setminus S$.

Note that if y is S-admissible then $-y$ is $(I\setminus S)$-admissible. This follows from the fact that the collection of i-large sets is closed under complementation.

A vector y is said to be *admissible* if y is S-admissible for some $S \subseteq I$. Let F denote the collection of all admissible vectors. Then F is symmetric, i.e. if $y \in F$ then $-y \in F$. Now we can define the norm $\|\cdot\|$:

$$\left\| \sum_{i \in I} x_i e_i \right\| = \max_{y \in F} \sum_{i \in I} x_i y_i. \tag{15}$$

The symmetry of F guarantees that (15) defines a norm. The fact that this norm is suppression 1-unconditional is an immediate consequence of the following easily checked property of F: if $y \in F$ and z is obtained from y by replacing some of the coordinates of y by zeros, then $z \in F$. It is also easy to check that $\|e_i\| = 1$ for all $1 \le i \le n$.

¿From now on the proof is similar to that of [8]. We include it here for the sake of completeness.

Proof of (ii). Let $\eta = (\eta_i)_{i=1}^n$ be a choice of signs Define $y = (y_i)$ thus:

$$y_i = \begin{cases} \eta_i & \text{if } P(\eta) \text{ is } \textit{not } i\text{-large,} \\ (1-\mu)\eta_i & \text{if } P(\eta) \text{ is } i\text{-large.} \end{cases}$$

Clearly, y is $P(\eta)$-admissible, so $y \in F$. Since $P(\eta)$ is i-large for at most one index i_0, we have

$$\left\| \sum_{i=1}^n \eta_i e_i \right\| \ge \sum_{i=1}^n \eta_i y_i \ge \left(\sum_{i=1}^n \eta_i^2 - 1 \right) + (1-\mu)\eta_{i_0}^2 = n - \mu,$$

which proves (13). □

Proof of (iii). Suppose $A \subset I$ with $|A| = 1 + \lceil (3/4 + \alpha)n \rceil$. Choose $i_0 \in A$ and let $\tilde{A} = A \setminus \{i_0\}$ (so that $|\tilde{A}| = \lceil (3/4+\alpha)n \rceil$). We define a vector x, with $\operatorname{supp} x = A$, thus:

$$x_i = \begin{cases} |\tilde{A}| - (3/4)n & \text{for } i = i_0, \\ 1 & \text{for } i \in \tilde{A} \cap S_{i_0}, \\ -1 & \text{for } i \in \tilde{A} \cap (I \setminus S_{i_0}), \\ 0 & \text{otherwise.} \end{cases}$$

Now let us show that $\|x\|$ satisfies (14). Let y be an admissible vector that is obtained from $S \subseteq I$. Suppose that

$$|\tilde{A} \cap S_{i_0} \cap P(y)| + |\tilde{A} \cap (I \setminus S_{i_0}) \cap N(y)| > 3n/4. \tag{16}$$

Since $P(y) \subseteq S$ and $N(y) \subseteq I \setminus S$, we have

$$|S_{i_0} \cap S| + |(I \setminus S_{i_0}) \cap (I \setminus S)| > 3n/4.$$

Thus,

$$|S_{i_0} \Delta S| < n/4.$$

So S is i_0-large. Hence $|y_{i_0}| \leq 1 - \mu$, and so

$$\begin{aligned} \sum_{i \in I} x_i y_i &= x_{i_0} y_{i_0} + \sum_{i \in \tilde{A}} x_i y_i \\ &\leq (|\tilde{A}| - (3/4)n)(1-\mu) + |\tilde{A}|. \end{aligned} \tag{17}$$

On the other hand, if (16) does not hold, then

$$\begin{aligned} \sum_{i \in I} x_i y_i &\leq |x_{i_0}| + |\tilde{A}| - (|\tilde{A}| - (3/4)n)\mu \\ &\leq (|\tilde{A}| - (3/4)n)(1-\mu) + |\tilde{A}|. \end{aligned} \tag{18}$$

It follows from (17) and (18) that

$$\|x\| \leq |\tilde{A}| + (1-\mu)(|\tilde{A}| - (3/4)n). \tag{19}$$

But

$$\begin{aligned} \|x\|_1 &= |x_{i_0}| + |\tilde{A}| \\ &= (|\tilde{A}| - (3/4)n) + |\tilde{A}| \\ &\geq \|x\| + \mu(|\tilde{A}| - (3/4)n) \end{aligned}$$

(by (19))

$$\geq \left(1 + \mu\left((1-\mu) + \frac{|\tilde{A}|}{|\tilde{A}| - (3n/4)}\right)^{-1}\right)\|x\|$$

(by (19) again)

$$\geq \left(1 + \mu\left((1-\mu) + \frac{3+4\alpha}{4\alpha}\right)^{-1}\right)\|x\|$$

(since $|\tilde{A}| \geq (3/4 + \alpha)n$)

$$= \left(1 + \frac{4\alpha\mu}{3 + 8\alpha - 4\alpha\mu}\right)\|x\|,$$

which proves (14). $\square$

Remark 3.5. The construction of the norm in Theorem 3.4 is explicit and determinstic. A 'random' argument can be given (see [6] for the details) to extend Theorem 3.4 to all A satisfying $|A| > (1/2+\alpha)n$, although this argument has the defect that it does not give the norm explicitly. By the 'almost isometric' part of a theorem of Elton [8, Theorem 1], it is *not* possible to extend the result to sets with $|A| < (1/2-\alpha)n$.

4. $f(n)$-approximate ℓ_1 Systems

Henceforth $(f(n))_{n=1}^{\infty}$ will denote a strictly positive nondecreasing sequence satisfying $\lim_{n\to\infty} f(n) = \infty$. Let us first observe that the norm of the linear span of an $f(n)$-approximate ℓ_1 system does indeed behave like the ℓ_1 norm for moderately sized coefficients. It is convenient here to extend the definition of f to $\mathbb{R}^+$ by taking $f(x) = f(\lceil x \rceil)$

Proposition 4.1. *Let (x_i) be an $f(n)$-approximate ℓ_1 system and suppose that $0<\delta<M<\infty$. Then, for all scalars (a_i) satisfying $\delta \le |a_i| \le M$, we have*

$$\left\|\sum_{i\in A} a_i x_i\right\| \ge \sum_{i\in A} |a_i| - Mf\left(\left(\sum_{i\in A}|a_i|\right)/\delta\right).$$

Proof. Let $\eta_i = \operatorname{sgn} a_i$. Then, by the triangle inequality,

$$\begin{aligned}\left\|\sum_{i\in A} a_i x_i\right\| &= \left\|\sum |a_i|\eta_i x_i\right\| \\ &\ge M\left\|\sum_{i\in A}\eta_i x_i\right\| - \left(\sum_{i\in A}(M-|a_i|)\right) \\ &\ge M(|A| - f(|A|)) - M|A| + \sum_{i\in A}|a_i| \\ &\ge \sum_{i\in A}|a_i| - Mf\left(\left(\sum_{i\in A}|a_i|\right)/\delta\right).\end{aligned}$$

The last inequality follows from the fact that $\min|a_i| \ge \delta$, which implies $|A| \le \left(\sum_{i\in A}|a_i|\right)/\delta$. □

We now aim to characterize the Banach spaces which contain an infinite $f(n)$-approximate ℓ_1 system for slowly increasing $f(n)$. To that end let us recall the notion of *spreading model* (see e.g. [4]). Let (x_i) be a sequence in a Banach space X and let (s_i) be a basis for a Banach space $(Y, |\cdot|)$. Then

$(Y, |\cdot|)$ is said to be a spreading model for (x_i) if, for all $k \geq 1$ and for all scalars $a_1, \dots, a_k$, we have

$$\left|\sum_{i=1}^{k} a_i s_i\right| = \lim_{\substack{n_1 \to \infty \\ n_1 < \cdots < n_k}} \left\|\sum_{i=1}^{k} a_i x_{n_i}\right\|.$$

Recall that every normalized sequence which is not norm-convergent has a subsequence (x_i) which has a spreading model.

Proposition 4.2. *Suppose that $(x_i)_{i=1}^{\infty}$ is a normalized sequence which has spreading model isometrically equivalent to the unit vector basis of ℓ_1. Then, given $(f(n))$, (x_i) has a subsequence (y_i) which is an $f(n)$-approximate ℓ_1 system.*

Proof. Set $n_0 = 0$. For $k \geq 1$, select $n_k > n_{k-1}$ with $f(n_k) > 2k$. Since (s_i) (the basis of the spreading model) is isometrically equivalent to the unit vector basis of ℓ_1, we may choose a strictly increasing sequence $(m_k)_{k=1}^{\infty}$ of positive integers such that if $1 \leq n \leq n_{k+1}$ and $m_k < j_1 < \cdots < j_n$, then

$$\left\|\sum_{i=1}^{n} \pm x_{j_i}\right\| \geq n - (f(n_k) - 2k).$$

Set $y_k = x_{m_k}$. Suppose that $k \geq 0$, that $n_k < n \leq n_{k+1}$, and that $1 \leq j_1 < \cdots < j_n$. Then

$$\begin{aligned}\left\|\sum_{i=1}^{n} \pm y_{j_i}\right\| &\geq \left\|\sum_{j=k+1}^{n} \pm y_j\right\| - k \\ &\geq (n-k) - (f(n_k) - 2k) - k \\ &= n - f(n_k) \geq n - f(n).\end{aligned}$$

□

Proposition 4.3. *Suppose that $(x_i)_{i=1}^{\infty}$ is an $f(n)$-approximate ℓ_1 system, where $\inf_{n \geq 1} f(n)/n = 0$. Then the spreading model associated to (x_i) is isometrically equivalent to the unit vector basis of ℓ_1.*

Proof. Fix $\varepsilon \in (0,1)$ and $N \geq 1$. It suffices to show that there exists $M \in \mathbb{N}$ such that if $M \leq n_1 < \cdots < n_N$ then $\left\|\sum_{i=1}^{N} \pm x_{n_i}\right\| \geq N - \varepsilon$ for all choices of signs (for then it follows that $\left\|\sum_{i=1}^{N} a_i x_{n_i}\right\| \geq (1-\varepsilon)\sum_{i=1}^{N} |a_i|$ for all real scalars (a_i)). Suppose, to derive a contradiction, that there is no such M. Then there exist finite sets A_i $(i = 1, 2, \dots)$, with $|A_i| = N$

and with $A_1 < A_2 < \dots$, and there exists a choice of signs (η_j) such that $\|\sum_{j\in A_i}\eta_j x_j\| < N-\varepsilon$ for each i. Thus, for each $k \geq 1$, we have

$$\begin{aligned} Nk - f(Nk) &\leq \left\| \sum_{i=1}^{k} \sum_{j\in A_i} \eta_j x_j \right\| \\ &\leq \sum_{i=1}^{k} \left\| \sum_{j\in A_i} \eta_j x_j \right\| \\ &\leq \sum_{i=1}^{k} (N-\varepsilon) = Nk - k\varepsilon. \end{aligned}$$

So $f(Nk) \geq \varepsilon k$ for all k, which contradicts the fact that $(f(n))$ is a positive nondecreasing sequence satisfying $\inf_{n\geq 1} f(n)/n = 0$. $\square$

Combining the previous two results, we obtain the following characterization of Banach spaces which contain $f(n)$-approximate ℓ_1 systems for all $(f(n))$.

Theorem 4.4. *Let X be a Banach space. The following are equivalent:*

(i) *X has a spreading model isometrically equivalent to the unit vector basis of ℓ_1.*
(ii) *X contains an $f(n)$-approximate ℓ_1 system for some $(f(n))$ satisfying* $\inf_{n\geq 1} f(n)/n = 0$.
(iii) *For all $(f(n))$, X contains a basic $f(n)$-approximate ℓ_1 system.*

Proof. (iii) $\Rightarrow$ (ii) is clear and (ii) $\Rightarrow$ (i) is Proposition 4.3. To show (i) $\Rightarrow$ (iii), let (x_i) have spreading model isometrically equivalent to the usual ℓ_1 basis. If (x_i) has no weakly convergent subsequence, then (x_i) has a basic subsequence [19]. If (x_i) has a weakly convergent subsequence, then we may assume, after passing to a subsequence, that $y_i = x_{2i} - x_{2i-1}$ is weakly null and hence has a basic subsequence [14]. Clearly, $(y_i/2)$ also has spreading model isometrically equivalent to the ℓ_1 basis. So we can take (x_i) to be basic. Arguing as in Proposition 4.2, we see that (x_i) has a subsequence that is an $f(n)$-approximate ℓ_1 system. $\square$

Remark 4.5. Beauzamy and Lapresté [4] characterized the existence of a spreading model *isomorphically* equivalent to the unit vector basis of ℓ_1.

The following renorming result is now an immediate consequence of a deep result of Odell and Schlumprecht [18, Corollary 3.3] and of the equivalence of (i) and (ii) in Theorem 4.4.

Corollary 4.6. *Let X be a separable Banach space. The following are equivalent:*

(i) *X does not contain a subspace isomorphic to ℓ_1.*
(ii) *There exists an equivalent norm $|||\cdot|||$ on X such that if (x_n) is an $f(n)$-approximate ℓ_1 system with respect to $|||\cdot|||$ then $f(n) > \delta n$ for some $\delta > 0$.*

Our next goal is to use Theorem 4.4 to give examples of $f(n)$-approximate ℓ_1 systems which have no unconditional basic subsequences. To that end, we must recall the definition of the *mixed Tsirelson* spaces. Given a sequence $(\mathcal{M}_j)_{j=0}^{\infty}$ of compact families of finite subsets of $\mathbb{N}$, and given a sequence $(\theta_j)_{j=0}^{\infty}$ of real numbers converging to 0, the mixed Tsirelson space $T[(\mathcal{M}_j, \theta_j)_{j=0}^{\infty}]$ is defined in [2] as the completion of the linear space c_{00} under the norm $\|\cdot\|$ given as follows. For $x \in c_{00}$,

$$\|x\| = \max\left\{\|x\|_\infty, \sup_j \theta_j \sup\left\{\sum_{i=1}^{n} \|E_i x\| : (E_i)_{i=1}^{n} \text{ is } \mathcal{M}_j\text{-admissible}\right\}\right\},$$

where, for $E \subset \mathbb{N}$, Ex is the restriction of x to E and, for a family $\mathcal{M}$, an $\mathcal{M}$-admissible sequence $(E_i)_{i=1}^{n}$ is a sequence of subsets of $\mathbb{N}$ such that $E_1 < E_2 < \cdots < E_n$ and such that the set $\{\min E_1, \min E_2, \ldots, \min E_n\}$ belongs to $\mathcal{M}$.

In the definition of the spaces X and X_u below, $(\mathcal{M}_j)_{j=0}^{\infty}$ is an appropriate subsequence of $(\mathcal{S}_n)_{n=1}^{\infty}$, where $\mathcal{S}_n$ denotes the nth *Schreier family* (introduced in [1]) defined inductively as follows:

$$\mathcal{S}_0 = \{\emptyset\} \cup \{\{n\} : n \in \mathbb{N}\},$$

and, for $k \geq 0$,

$$\mathcal{S}_{k+1} = \{\emptyset\} \cup \{\cup_{i=1}^{n} A_i : n \in \mathbb{N}, A_i \in \mathcal{S}_k, n \leq A_1 < A_2 < \ldots A_n\}.$$

Let us also recall that an infinite-dimensional Banach space is *hereditarliy indecomposable* (H.I.) if X does not have a subspace which can be expressed as a topological direct sum $Y \oplus Z$, with Y and Z infinite-dimensional.

Proposition 4.7. *The H.I. space X introduced in* [2] *has a spreading model isometrically equivalent to the unit vector basis of ℓ_1.*

Proof. We shall assume that the reader is familiar with [2], to which we refer for the detailed definition of the spaces X and X_u. Briefly, $X_u =$

$T[(\mathcal{M}_j, 1/m_j)_{j=0}^{\infty}]$, where $m_0 = 2$, and, for $j \geq 1$, $m_j > m_{j-1}^{m_{j-1}}$ In particular, $m_1 > 4$, which we use below.

These spaces are the completions of c_{00} equipped with norms defined by certain classes of linear functionals. In particular, there exist classes of linear functionals K_j^n ($n \geq 0$, $j \geq 0$), $A_j = \cup_{n=1}^{\infty} K_j^n$ ($j \geq 0$), $K^n = \cup_{j=0}^{\infty} K_j^n$ ($n \geq 0$), and $K = \cup_{n=0}^{\infty} K^n$, for which the norm $\|\cdot\|_u$ of X_u is defined thus:

$$\|x\|_u = \sup\{f(x) : f \in K\}.$$

To obtain X one defines $L_j^n \subset K_j^n$, $B_j = \cup_{n=1}^{\infty} L_j^n \subset A_j$, and $L = \cup_{j=0}^{\infty} B_j \subset K$. The norm $\|\cdot\|$ of X is defined thus:

$$\|x\| = \sup\{f(x) : f \in L\}.$$

The construction in X is the same as the standard spreading model isometric to ℓ_1 in the Tsirelson space T. Indeed, let $e_i = x_{2i-1} + x_{2i}$, where (x_i) is the usual basis of X. Since $m_1 > 4$, the norm of e_i is achieved by partitions from the level $\mathcal{M}_0 = \mathcal{S}_1$, therefore $\|e_i\| = 1$. For any $k < n_1 < n_2 < \cdots < n_k$, we have that $\{2n_1 - 1, 2n_1, 2n_2 - 1, 2n_2, \ldots, 2n_k - 1, 2n_k\} \in \mathcal{S}_1$, whence $\|\sum_{i=1}^k x_{n_i}\| = k$. Therefore, the spreading model generated by (x_i) is isometric to ℓ_1. $\square$

Remark 4.8. Odell and Schlumprecht [17] constructed spreading models isometric to ℓ_1 in T hereditarily, i.e. in every infinite dimensional subspace Y of T there exists (y_i) with spreading model 1-equivalent to the unit vector basis of ℓ_1. We can show a similar hereditary result for X_u and X. The proof requires more technical details from [2] and thus we chose not to include it here, since it is further away from the main topic of the present paper.

Remark 4.9. The H.I. space GM (introduced in [10]) has isometrically the same spreading models as the space S (introduced in [20]) [21]. S has a spreading model isomorphic to ℓ_1 [13]. Modifying the construction of [13] slightly shows that S has ℓ_1 isometrically as a spreading model [16].

Corollary 4.10. *Given $(f(n))$, there exists an H.I. Banach space X which has a basis (x_i) that is an $f(n)$-approximate ℓ_1 system. In particular, X does not contain any unconditional basic sequence.*

Remark 4.11. Konyagin and Temlyakov [15] defined a basis (x_i) to be *superdemocratic* if there exists a positive constant C such that, whenever

$|A| = |B|$, then

$$\frac{1}{C}\left\|\sum_{i\in A}\eta_i x_i\right\| \le \left\|\sum_{i\in B}\eta_i' x_i\right\| \le C\left\|\sum_{i\in A}\eta_i x_i\right\|$$

for all choices of signs $(\eta_i)_{i\in A}$ and $(\eta_i')_{i\in B}$. An example is given in [15] of a superdemocratic basis which is not unconditional. Note that the bases given by Corollary 4.10 are superdemocratic and their linear spans do not contain any unconditional basic sequence. In fact, provided $f(n) = o(n)$, we have

$$\lim_{n\to\infty}\sup_{|A|=|B|=n}\frac{\left\|\sum_{i\in A}\eta_i x_i\right\|}{\left\|\sum_{i\in B}\eta_i' x_i\right\|} = 1,$$

where the supremum is taken over all possible choices of signs.

5. $f(n)$-approximate ℓ_1 Systems in ℓ_1

In this section we construct some nontrivial examples of $f(n)$-approximate ℓ_1 systems in the space ℓ_1 itself. Actually, our construction can be carried out without much extra complication in any space that is isomorphic to ℓ_1. For any given $(f(n))$, we shall construct two examples of $f(n)$-approximate ℓ_1 systems: first, an unconditional basic sequence which is not equivalent to the ℓ_1-basis; secondly, a conditional basic sequence.

As motivation for these results let us recall that a sequence in an L_1 space is equivalent to the unit vector basis of ℓ_1 if it is, roughly speaking, 'sufficiently disjoint'. Here is one such criterion for sufficient disjointness which we state without proof.

Proposition 5.1. *Let $(f_n)_{n=1}^\infty$ be a normalized sequence in an L_1 space. Suppose that there exist $t > 0$ and a sequence $(A_n)_{n=1}^\infty$ of disjoint measurable sets such that*

$$\int_{A_n}|f_n| - \sum_{m\neq n}\int_{A_m}|f_n| \ge t \qquad (n \ge 1).$$

Then $\|\sum a_n f_n\| \ge t\sum |a_n|$ for all $(a_n) \in \ell_1$.

So our examples show that an $f(n)$-approximate ℓ_1 system, even for very slowly increasing $(f(n))$, does not necessarily possess enough disjointness to be equivalent to the ℓ_1 basis. It is ususally possible, on the other hand, to extract an ℓ_1 subsequence.

Proposition 5.2. *Suppose that $k \in (0, 1)$ and that $(x_i)_{i=1}^\infty$ is an $f(n)$-approximate ℓ_1 system in an L_1 space for $f(n) = kn$. Then $(x_i)_{i=1}^\infty$ has an ℓ_1 subsequence.*

Proof. Suppose, to derive a contradiction, that (x_i) has a weakly convergent subsequence. Then, passing to a subsequence, we may assume that $(y_i)_{i=1}^\infty$ is weakly null, where $y_i = x_{2i} - x_{2i-1}$. Since L_1 spaces have the *weak Banach-Saks property* [22], it follows that (after passing to a subsequence of (y_i) and relabelling) $(1/n)\sum_{i=1}^n y_i \to 0$. On the other hand,

$$\left\|\sum_{i=1}^n y_i\right\| = \left\|\sum_{i=1}^n (x_{2i} - x_{2i-1})\right\| \geq 2n - f(2n) = 2n(1-k),$$

which yields the contradiction. Finally, it is well-known that every sequence in an L_1 space which has no weakly convergent subsequence has an ℓ_1 subsequence. □

Let us recall the definition of the L_1-normalized Haar system on $[0,1]$. Let $h_0^0 \equiv 1$. For $n \geq 0$ and $1 \leq k \leq 2^n$, we define h_k^n thus:

$$h_k^n = \begin{cases} 2^n & \text{on } [(k-1)/2^n, (2k-1)/2^{n+1}) \\ -2^n & \text{on } [(2k-1)/2^{n+1}, k/2^n) \\ 0 & \text{elsewhere.} \end{cases}$$

The dyadic Hardy space H_1 has the following norm:

$$\left\|a_0^0 h_0^0 + \sum_{n=0}^\infty \sum_{k=1}^{2^n} a_k^n h_k^n\right\| = \int_0^1 \left((a_0^0 h_0^0)^2 + \sum_{n=0}^\infty \sum_{k=1}^{2^n} (a_k^n h_k^n)^2\right)^{1/2} dt.$$

Note that H_1 is isometrically isomorphic to a subspace of $L_1(0,1)$. To see this, observe that the mapping

$$f = a_0^0 h_0^0 + \sum_{n=0}^\infty \sum_{k=1}^{2^n} a_k^n h_k^n \mapsto (a_k^n h_k^n)_{n,k}$$

defines an isometric embedding from H_1 into $L_1(\ell_2(\mathbb{Z} \times \mathbb{Z}))$. Since $L_1(0,1)$ contains a subspace linearly isometric to ℓ_2, namely the closed linear span of a sequence of independent gaussian random variables, it follows that $L_1(\ell_2)$ is linearly isometric to a subspace of $L_1(L_1)$, which in turn is linearly isometric to $L_1(0,1)$.

Note also that each 'layer' $(h_i^n)_{i=1}^{2^n}$ of the Haar system is isometrically equivalent to the unit vector basis of $\ell_1^{2^n}$:

$$\left\|\sum_{k=1}^{2^n} a_k h_k^n\right\| = \sum_{k=1}^{2^n} |a_k|. \tag{20}$$

Proposition 5.3. *Given $(f(n))$, there exists an increasing sequence $(n_i)_{i=0}^\infty$ of nonnegative integers such that the 'lacunary Haar system' $((h_j^{n_i})_{j=1}^{2^{n_i}})_{i=0}^\infty$ in H_1 is an $f(n)$-approximate ℓ_1 system.*

Proof. Set $n_0 = 0$. Suppose that $n_0 < n_1 < \cdots < n_k$ have been chosen so that $\mathcal{H}_k = ((h_j^{n_i})_{j=1}^{2^{n_i}})_{i=0}^k$ satisfies

$$\left\| \sum_A \pm h_j^n \right\| > |A| - f(|A|) \tag{21}$$

for all $A \subseteq \mathcal{H}_k$ and for all choices of signs. Then there exists $\varepsilon_k > 0$ such that

$$\left\| \sum_A \pm h_j^n \right\| > |A| - f(|A|) + \varepsilon_k \tag{22}$$

for all $A \subseteq \mathcal{H}_k$ and for all choices of signs. By a uniform integrability argument there exists $\delta > 0$ such that if $g \in H_1$ satisfies $\lambda(\operatorname{supp} g) < \delta$, where λ denotes Lebesgue measure, then

$$\left\| g + \sum_A \pm h_j^n \right\| > \|g\| + \left\| \sum_A \pm h_j^n \right\| - \varepsilon_k$$

for all $A \subseteq \mathcal{H}_k$ and for all choices of signs. Select n_{k+1} so large that $f(\lfloor \delta 2^{n_{k+1}} \rfloor) > 2|\mathcal{H}_k|$, and let $\mathcal{H}_{k+1} = \mathcal{H}_k \cup (h_i^{n_{k+1}})_{i=1}^{2^{n_{k+1}}}$. Suppose that $A \subseteq \mathcal{H}_{k+1}$. We shall show that (21) is satisfied by A. Write $A = B \cup C$, where $B \subseteq \mathcal{H}_k$ and $C \subseteq (h_i^{n_{k+1}})_{i=1}^{2^{n_{k+1}}}$. There are two cases to consider. First, suppose that $|C| < \lfloor \delta 2^{n_{k+1}} \rfloor$. Then $\lambda(\operatorname{supp}(\sum_C \pm h_i^n)) = |C| 2^{-n_{k+1}} < \delta$. Hence, by the choice of δ, we have

$$\begin{aligned}\left\| \sum_A \pm h_i^n \right\| &> \left\| \sum_B \pm h_i^n \right\| + \left\| \sum_C \pm h_i^n \right\| - \varepsilon_k \\ &> (|B| - f(|B|) + \varepsilon_k) + |C| - \varepsilon_k\end{aligned}$$

(by (22) and (20))

$$\begin{aligned}&= |A| - f(|B|) \\ &\geq |A| - f(|A|).\end{aligned}$$

On the other hand, if $|C| \geq \lfloor \delta 2^{n_{k+1}} \rfloor$, then

$$\begin{aligned}\left\| \sum_A \pm h_i^n \right\| &> \left\| \sum_C \pm h_i^n \right\| - \left\| \sum_B \pm h_i^n \right\| \\ &\geq |C| - |B| \\ &= |A| - 2|B| \\ &\geq |A| - 2|\mathcal{H}_k| \\ &> |A| - f(\lfloor \delta 2^{n_{k+1}} \rfloor)\end{aligned}$$

(by the choice of n_{k+1})

$$\geq |A| - f(|A|).$$

$\square$

Remark 5.4. For $k = 1, 2, \ldots$, let $r_k = 2^{1-k} \sum_{i=1}^{2^{k-1}} h_i^{k-1}$ be the sequence of Rademacher functions. Then (r_k) is equivalent in H_1 to the unit vector basis of ℓ_2 and its span is complemented by the orthogonal projection. Since the linear span of every lacunary Haar system contains a subsequence of the Rademacher functions, it follows that the closed linear span of a lacunary Haar system contains a complemented block subspace isomorphic to ℓ_2.

We shall now transfer the above example from H_1 to ℓ_1 by a localization argument.

Theorem 5.5. *Suppose that X is isomorphic to ℓ_1. Then, given $(f(n))$ and $\alpha > 1$, X contains a normalized basic sequence (y_i) satisfying the following:*
(i) *(y_i) is an $(f(n))$-appproximate ℓ_1 system.*
(ii) *(y_i) is α-unconditional.*
(iii) *for $k = 1, 2, \ldots$, the unit vector basis of ℓ_2^k is uniformly equivalent to a complemented block basis of (y_i). (This implies, in particular, that $[y_i]$ is not isomorphic to ℓ_1.)*

The proof of Theorem 5.5 requires two technical lemmas.

Lemma 5.6. *Let $(f(j))_{j=1}^{\infty}$, $\alpha > 1$, and $k \in \mathbb{N}$ be given. There exist $m \in \mathbb{N}$, $N \in \mathbb{N}$ and $\varepsilon > 0$ such that, whenever $d(X, \ell_1^N) < 1 + \varepsilon$, then there exist m unit vectors $(x_i)_{i=1}^m$ in X satisfying the following:*
(i) *$(x_i)_{i=1}^m$ is an $f(j)$-approximate ℓ_1 system.*

(ii) *$(x_i)_{i=1}^m$ is an α-unconditional basic sequence.*

(iii) *ℓ_2^k is uniformly equivalent to a uniformly complemented block basis of $(x_i)_{i=1}^m$.*

Proof. By Proposition 5.3, there is a lacunary Haar system which is an $f(n)$-approximate ℓ_1 system. Let $(y_i)_{i=1}^m$ be an enumeration of the the first k layers of this system. Then $(y_i)_{i=1}^m$ is 1-unconditional and satisfies conditions (i) and (iii). Since H_1 is isometric to a subspace of L_1, given $\eta > 0$, we can find a positive integer N such that $[y_i]_{i=1}^m$ is $(1+\eta)$-isomorphic to a subspace of ℓ_1^N. The lemma now follows by a standard perturbation argument. $\square$

The following lemma is implicit in James's proof that ℓ_1 is not distortable [11].

Lemma 5.7. *Let $(\varepsilon_i)_{i=1}^\infty$ and $(\alpha_i)_{i=1}^\infty$ be sequences of positive numbers, and let $(N_i)_{i=1}^\infty$ be a sequence of positive integers. Let X be isomorphic to ℓ_1. There exist subspaces $F_i \subseteq X$ satisfying the following:*

(i) $d(F_i, \ell_1^{N_i}) < 1 + \varepsilon_i$;

(ii)
$$\left\| \sum_{i=n}^\infty x_i \right\| \geq (1 - \alpha_n) \sum_{i=n}^\infty \|x_i\|$$
for all $n \geq 1$ and all $x_i \in F_i$.

Proof of Theorem 5.5. It will be clear from the construction (and from Lemmas 5.6 and 5.7) that the sequence (y_i) can always be chosen to be α-unconditional provided $\alpha > 1$. Therefore, to avoid unnecessary complication, we shall omit the verifiication of (ii).

Select $\beta > 0$ and a positive sequence $((g(n))$ such that
$$\beta + 3g(n) = f(n) \qquad (n \geq 1).$$
Fix $k \geq 1$ and let N_k, m_k, and ε_k be as given by Lemma 5.6 when applied to the sequence $(g(n)/2^k)_{n=1}^\infty$. Now choose positive integers $p_m \uparrow \infty$ $(m \geq 1)$ such that
$$g(n_1 + n_2 + \cdots + n_{p_m}) > 2(n_1 + n_2 + \cdots + n_m). \tag{23}$$
Next choose positive numbers α_i $(i \geq 1)$ such that
$$(n_1 + n_2 + \cdots + n_{p_{i+1}})\alpha_i < \beta. \tag{24}$$
Apply Lemma 5.7 to (ε_i), (α_i), and (N_i), to find subspaces F_i satisfying the conclusion of the lemma.

By Lemma 5.6, for each $k \geq 1$ there exist vectors $(x_i^k)_{i=1}^{m_k}$ in F_k satisfying (i)-(iii) of Lemma 5.6 (applied to $(g(n)/2^k)_{n=1}^\infty$). Let $(y_i)_{i=1}^\infty$ be an enumeration of the sequence $x_1^1, \ldots, x_{m_1}^1, x_1^2, \ldots, x_{m_2}^2, \ldots$, and for $k \geq 1$, let $B_k = \{i : y_i \in F_k\}$.

By Lemma 5.6, it is clear that (y_i) satisfies condition (iii) of Theorem 5.5. So it remains only to verify that (y_i) is an $f(n)$-approximate ℓ_1 system. Suppose that $A \subset \mathbb{N}$ and that $N = |A|$ satisfies
$$n_1 + n_2 + \cdots + n_{p_m} < N \leq n_1 + n_2 + \cdots + n_{p_{m+1}}.$$

(The verification is similar but easier if $N \le n_1 + \cdots + n_{p_1}$.) Then

$$\left\| \sum_{i \in A} \pm y_i \right\| \ge \left\| \sum_{j=m+1}^{\infty} \left(\sum_{i \in A \cap B_j} \pm y_i \right) \right\| - (n_1 + \cdots + n_m)$$

(by the triangle inequality since $|B_j| = m_j$)

$$\ge (1 - \alpha_m) \sum_{j=m+1}^{\infty} \left\| \sum_{i \in A \cap B_j} \pm y_i \right\| - (n_1 + \cdots + n_m)$$

(by Lemma 5.7)

$$\ge (1 - \alpha_m) \sum_{j=m+1}^{\infty} \left(|A \cap B_j| - \frac{g(|A \cap B_j|)}{2^j} \right) - (n_1 + \cdots + n_m)$$

(since $(y_i)_{i \in B_j}$ is a $(g(n)/2^j)$-approximate ℓ_1 system)

$$\ge (1 - \alpha_m) \left(\sum_{j=1}^{\infty} |A \cap B_j| - (n_1 + \cdots + n_m) \right)$$

$$- g(|A|) \left(\sum_{j=1}^{\infty} 2^{-j} \right) - (n_1 + \cdots + n_m)$$

$$\ge (1 - \alpha_m)|A| - g(|A|) - 2(n_1 + \cdots + n_m)$$

$$\ge |A| - \beta - g(|A|) - 2(n_1 + \cdots + n_m)$$

(by (24) since $|A| = N \le n_1 + n_2 + \cdots + n_{p_{m+1}}$.)

$$\ge |A| - \beta - 3g(|A|)$$

(by (23) since $|A| = N > n_1 + \cdots + n_{p_m}$)

$$> |A| - f(|A|)$$

by the choice of β and $(g(n))$. This proves that (y_n) is an $f(n)$-approximate ℓ_1 system. □

We obtain a strengthening of Theorem 4.4.

Corollary 5.8. *Let X be a Banach space. The following are equivalent:*

(i) *X has a spreading model that is isometrically eqivalent to the unit vector basis of ℓ_1.*

(ii) *For all $(f(n))$, X contains an $f(n)$-approximate ℓ_1 basic sequence whose closed linear span is not isomorphic to ℓ_1.*

Proof. When X does not contain ℓ_1, the result is an immediate consequence of Theorem 4.4. When X does contain ℓ_1, the result follows from (iii) of Theorem 5.5. $\square$

We can also consider the Haar system in L_1 instead of H_1. Let us observe that every lacunary Haar system in L_1 is a *conditional* monotone basis for its linear span. The conditionality follows from the easily verified fact that the full Haar system is equivalent to a block basis of every lacunary Haar system. All the proofs of this section go through mutatis mutandis for the lacunary Haar system in L_1 to yield the following analogue of Theorem 5.5.

Theorem 5.9. *Suppose that X is isomorphic to ℓ_1. Then, given $(f(n))$ and $\alpha > 1$, X contains a normalized basic sequence (y_n) satisfying the following:*

(i) *(y_n) is an $(f(n))$-appproximate ℓ_1 system.*

(ii) *(y_n) is a conditional basis for its closed linear span with basis constant at most α.*

(iii) *The unit vector basis of ℓ_2^k ($k \geq 1$) is uniformly equivalent to a block basis of (y_n).*

If $X = \ell_1$, we may also take (y_i) to be a monotone basic sequence.

6. Lacunary Haar Systems are Quasi-greedy

Let $(x_n)_{n=1}^\infty$ be a normalized basis for X with biorthogonal functionals $(x_n^*)_{n=1}^\infty$. For each $x \in X$ and $m = 1, 2, \dots$ we define

$$\mathcal{G}_m(x) = \sum_{n \in A} x_n^*(x) x_n,$$

where A is a set of cardinality m such that $\min\{|x_n^*(x)| : n \in A\} \geq \max\{|x_n^*(x)| : n \in \mathbb{N} \setminus A\}$. Note that A is not necessarily uniquely defined. We say that the basis (x_n) is *quasi-greedy* if $\mathcal{G}_m(x) \to x$ for each $x \in X$.

Theorem 6.1. *Let $\varepsilon > 0$. There exists an increasing sequence of integers $(n_j)_{j=0}^\infty$ such that the lacunary Haar system $((h_j^{n_i})_{j=1}^{2^{n_i}})_{i=0}^\infty$ in $L_1(0,1)$ is a quasi-greedy basis satisfying $\|\mathcal{G}_m(x)\| \leq (1+\varepsilon)\|x\|$ for all x in its closed linear span and for all $m \geq 1$.*

The proof of Theorem 6.1 uses two auxiliary results. The first is an obvious symmetry property of the Haar system.

Lemma 6.2. *Every reaarangement of the Haar system which merely changes the order of terms within each layer of the Haar system (i.e., so that every h_i^k on layer k comes before every h_i^{k+1} on layer $k+1$ after the rearrangement) is a monotone basis.*

Proof. Clearly, every such rearrangement of the Haar system is a martingale difference sequence with respect to the standard dyadic filtration. □

For a proof of the following we refer the reader to [23].

Theorem A. *Let (x_n) be a basis for the Banach space X. The following are equivalent:*

(i) *(x_n) is a quasi-greedy basis.*
(ii) *There exists a constant C such that $\|\mathcal{G}_m(x)\| \le C\|x\|$ for all x in the linear span of (x_n) and for all $m \ge 1$.*

Proof of Theorem 6.1. Select $\varepsilon_i \downarrow 0$ such that $\prod_{i=1}^{\infty}(1+\varepsilon_i) < 1+\varepsilon$. Set $n_0 = 1$ and suppose that $n_0 < n_1 < \cdots < n_k$ have been chosen. Let $(x_i)_{i=1}^{N_k}$ be the lexicographical ordering of the elements of $\cup_{j=0}^{k} \cup_{i=1}^{2^{n_j}} h_i^{n_j}$ and let $F_k = [x_i]_{i=1}^{N_k}$. We shall assume as an inductive hypothesis that

$$\|\mathcal{G}_m(x)\| \le \left(\prod_{i=1}^{k}(1+\varepsilon_i)\right)\|x\| \qquad (x \in F_k). \tag{25}$$

Let $\delta_{k+1} = \varepsilon_{k+1}/N_k$. By uniform integrability there exists $\alpha_{k+1} > 0$ such that if $x \in F_k$ then

$$\|x+y\| \ge \frac{1}{1+\varepsilon_{k+1}}(\|x\|+\|y\|) \tag{26}$$

whenever $y \in L_1$ satisfies $\lambda(\operatorname{supp} y) < \alpha_{k+1}$.

Choose n_{k+1} such that $2^{1-n_{k+1}}/\delta_{k+1} < \alpha_{k+1}$. Fix $m \ge 1$. Suppose that $\|x+y\| = 1$, where $x \in F_k$ and $y \in [h_i^{n_{k+1}}]_{i=1}^{2^{n_{k+1}}}$. Note that $\|y\| \le 2$ (since the Haar system is monotone) and that

$$\mathcal{G}_m(x+y) = \mathcal{G}_{m_1}(x) + \mathcal{G}_{m_2}(y),$$

for some m_1, m_2 with $m = m_1 + m_2$. We now consider two cases. First suppose that the smallest nonzero coefficient in the basis expansion of $\mathcal{G}_m(x+y)$ has absolute value at least δ_{k+1}. Since $\lambda(\operatorname{supp}(h_i^{n_{k+1}})) = 2^{-n_{k+1}}$, we have

$$\lambda(\operatorname{supp} \mathcal{G}_{m_2}(y)) < \frac{\|y\|}{2^{n_{k+1}}\delta_{k+1}} \le \frac{2}{2^{n_{k+1}}\delta_{k+1}} < \alpha_{k+1}.$$

Thus

$$\begin{aligned}\|\mathcal{G}_m(x+y)\| &\le \|\mathcal{G}_{m_1}(x)\| + \|\mathcal{G}_{m_2}(y)\|\\ &\le \left(\prod_{i=1}^{k}(1+\varepsilon_i)\right)(\|x\| + \|\mathcal{G}_{m_2}(y)\|)\end{aligned}$$

(by (25))

$$\le \left(\prod_{i=1}^{k+1}(1+\varepsilon_i)\right)\|x + \mathcal{G}_{m_2}(y)\|$$

(by (26))

$$\le \left(\prod_{i=1}^{k+1}(1+\varepsilon_i)\right)\|x + y\|,$$

where the last line follows from Lemma 6.2. For the second case we assume that the smallest nonzero coefficient in the basis expansion of $\mathcal{G}_m(x+y)$ has absolute value at most δ_{k+1}. Then $|x_i^*(x - \mathcal{G}_{m_1}(x))| \le \delta_{k+1}$ for $1 \le i \le N_k$. Thus, by the choice of δ_{k+1}, we have

$$\|x - \mathcal{G}_{m_1}(x)\| \le N_k\delta_{k+1} < \varepsilon_{k+1}.$$

Thus

$$\begin{aligned}\|\mathcal{G}_m(x+y)\| &= \|\mathcal{G}_{m_1}(x) + \mathcal{G}_{m_2}(y)\|\\ &\le \|x + \mathcal{G}_{m_2}(y)\| + \|x - \mathcal{G}_{m_1}(x)\|\\ &\le \|x+y\| + \varepsilon_{k+1}\end{aligned}$$

(by Lemma 6.2)

$$\begin{aligned}&= (1+\varepsilon_{k+1})\|x+y\|\\ &\le \left(\prod_{i=1}^{k+1}(1+\varepsilon_i)\right)\|x + y\|,\end{aligned}$$

which establishes the inductive hypothesis for $k+1$. Thus (x_n) satisfies (ii) of Theorem A, with $C = \prod_{i=1}^{\infty}(1+\varepsilon_i) < 1+\varepsilon$. □

Remark 6.3. Recall that the full Haar system is *not* a quasi-greedy basis of L_1. To see this most easily, fix $n \ge 1$ and $\varepsilon > 0$. Let $x_n = h_0^0 + \sum_{k=0}^{n-1}((1+\varepsilon)h_1^{2k} + h_1^{2k+1}))$. Then, for sufficiently small ε, we have $\|x_n\| \le 2$. But $\mathcal{G}_n(x) = \sum_{k=0}^{n-1}((1+\varepsilon)h_1^{2k})$, so $\|\mathcal{G}_n(x)\| \ge n/4$. Since n is arbitrary, Theorem A implies that the Haar system is not quasi-greedy.

Remark 6.4. The 'dual' version of Theorem 6.1 is false. The L_∞-normalized Haar system is a basis for its closed linear span in L_∞. However, $(h_1^{n_k})$ is equivalent to the summing basis of c_0 for every subsequence (n_k). It is easy to see that the summing basis is not quasi-greedy.

For our final result, let us recall the definition of the *best m-term approximation.* For $x \in X$ and $m = 0, 1, \dots$ we set

$$\sigma_m(x) = \inf\left\{\|x - \Sigma_{n\in A} a_n x_n\| : |A| \le m\right\}.$$

Then the error of the greedy algorithm as compared to the error in the best m-term approximation is measured by the following quantity [23]:

$$e_m = \sup_{x\in X} \frac{\|x - \mathcal{G}_m(x)\|}{\sigma_m(x)} \qquad (\text{with } \frac{0}{0} = 1).$$

Theorem 6.5. *Given an unbounded increasing sequence $(f(n))$, with $f(1) \ge 73$, there exists a lacunary Haar system (x_n) in L_1 such that $e_n \le f(n)$ for all $n \in \mathbb{N}$.*

The proof of Theorem 6.5 uses the following result which is a special case of [23, Th. 5]. We refer the reader to [23] for the proof.

Theorem B. *Let $(g(n))$ be a positive increasing sequence. Suppose that (x_n) is a normalized basis for X such that*

$$\frac{1}{g(|A|)} \sum_{n\in A} |a_n| \le \left\|\sum_{n\in A} a_n x_n\right\| \le \sum_{n\in A} |a_n|,$$

for all finite $A \subset \mathbb{N}$ and scalars (a_n). Then $e_n \le 3g(n) + 1$ for all n.

Proof of Theorem 6.5. Let $g(n) = (f(n) - 1)/72$. Then $(g(n))$ is an unbounded increasing sequence with $g(1) \ge 1$. By Proposition 5.3 there exists a lacunary Haar system $(x_n)_{n=1}^\infty$ which is a $g(n)$-approximate ℓ_1 system. By Proposition 2.5 ,

$$\frac{1}{24g(|A|)} \sum_{n\in A} |a_n| \le \left\|\sum_{n\in A} a_n x_n\right\| \le \sum_{n\in A} |a_n|.$$

Thus, by Theorem B, $e_n \le 72g(n) + 1 = f(n)$. ☐

Remark 6.6. Theorem 6.5 is essentially best possible since (e_n) is a bounded sequence only if (x_n) is an unconditional basis [15].

Acknowledgement. We thank P.-K. Lin and Th. Schlumprecht for their personal communications, L. Szèkely for bringing [3] to our notice, and S. Konyagin and V. Temlyakov for giving us [15].

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