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Greedy Type Bases in Banach Spaces

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The aim of this survey is to present recent results related to m-term approximation with respect to biorthogonal systems in Banach spaces. Special attention is focused on greedy approximation. The results of many authors over the last few years have demonstrated that this is the area where there is a fruitful interaction between abstract, Banach space approach and concrete questions of approximation theory. We will work in the framework of Banach spaces, but extensions to quasi-Banach spaces are possible, see e.g. [21].

1. General Framework

Let X be a Banach space. We will consider only Banach spaces over the real scalars $\mathbb{R}$. All our results remain valid for complex scalars but some constants may be different. For a general introduction to Banach spaces the reader may consult e.g. [24]. A countable system of vectors $\Phi = (x_n, x_n^*)_{n\in A} \subset X \times X^*$ is called a *biorthogonal system* if for $n, m \in A$ we have

$$x_n^*(x_m) = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases} \tag{1.1}$$

Always in this paper we will assume that

$$\begin{aligned} 0 < \inf_{n\in A} \|x_n\| &\leq \sup_{n\in A} \|x_n\| < \infty, && (1.2)\\ 0 < \inf_{n\in A} \|x_n^*\| &\leq \sup_{n\in A} \|x_n^*\| < \infty, && (1.3)\\ \overline{\operatorname{span}\{x_n\}_{n\in A}} &= X. && (1.4) \end{aligned}$$

A biorthogonal system satisfying (1.2)–(1.4) will be called *natural*. We will usually assume that $\|x_n\| = 1$ for all $n \in A$. If (1.4) holds, then the functionals $(x_n^*)_{n\in A}$ are uniquely determined by the set $\{x_n\}_{n\in A}$, so sometimes we will

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abuse the language and speak about $(x_n)_{n\in A}$ being a biorthogonal system. If the above assumptions (1.1)–(1.4) are satisfied, then for each $x \in X$ we can assign a formal series

$$\sum_{n\in A} x_n^*(x)x_n. \tag{1.5}$$

We do not assume anything about the convergence of this series, but we should note that element $x \in X$ is determined by the values $\big(x_n^*(x)\big)_{n\in A}$ (i.e., by the above series) and that $x_n^*(x) \to 0$ for every $x \in X$. For a subset $B \subset A$ we denote $P_B(x) = \sum_{j\in B} x_j^*(x)x_j$ whenever this makes sense. Clearly it is well defined for finite B.

For each $m = 0, 1, 2, \ldots$ we define Σ_m as

$$\Sigma_m = \Big\{ y = \sum_{j\in B} a_j x_j \; : \; B \subset A, \; |B| = m, \; a_j\text{'s are scalars} \Big\},$$

so Σ_m is the set of all linear combinations of length m of elements $(x_n)_{n\in A}$. For $x \in X$ we define the best m-term approximation (with respect to the system Φ) of x as

$$\sigma_m\{\Phi\}(x) = \sigma_m(x) = \inf\left\{\|x - y\| \; : \; y \in \Sigma_m\right\}$$

for $m = 0, 1, 2, \ldots$. If the system Φ is clear from the context we will suppress it from the above notation.

It follows from (1.4) that for each $x \in X$ we have $\lim_{m\to\infty} \sigma_m(x) = 0$.

Generally speaking our task is to provide an "algorithm" which for each x and $m = 0, 1, 2, \ldots$ in terms of series (1.5) gives an element $y_m \in \Sigma_m$ such that $\|x - y_m\| \le C\sigma_m(x)$, where C is an absolute constant. The most obvious and in some sense natural attempt to define such an "algorithm" is to put

$$\mathcal{G}_m\{\Phi\}(x) = \mathcal{G}_m(x) = \sum_{j\in B} x_j^*(x)x_j$$

where the set $B \subset A$ is chosen in such a way that $|B| = m$ and $|x_j^*(x)| \ge |x_k^*(x)|$ whenever $j \in B$ and $k \notin B$.

Let us make some comments about the operators $\mathcal{G}_m(x)$.

1. It may happen that for some x and m the element $\mathcal{G}_m(x)$ (i.e., the set B) is not uniquely defined by the above conditions. In such case we take any set B.

2. The operator $\mathcal{G}_m$ is not linear (even if appropriate sets *are* uniquely defined).

3. The operator $\mathcal{G}_m$ is discontinuous. To see it let us fix two subsets $B, C \subset A$ such that $B \cap C = \emptyset$ and $|B| = |C| = m$. We define two sequences of

vectors

$$\begin{aligned} z_n &= \frac{n+1}{n}\sum_{j\in B} x_j + \sum_{k\in C} x_k, \\ y_n &= \sum_{j\in B} x_j + \frac{n+1}{n}\sum_{k\in C} x_k. \end{aligned}$$

Clearly both z_n and y_n converge to $\sum_{j\in B\cup C} x_j$, but

$$\begin{aligned} \mathcal{G}_m(x_n) &= \frac{n+1}{n}\sum_{j\in B} x_j \to \sum_{j\in B} x_j \\ &\text{and} \\ \mathcal{G}_m(y_n) &= \frac{n+1}{n}\sum_{k\in C} x_k \to \sum_{k\in C} x_k. \end{aligned}$$

Modifying the above example one can show that the operator $\mathcal{G}_m$ is continuous at the point $x \in X$ if and only if the set B used in the definition of $\mathcal{G}_m(x)$ is uniquely defined.

Now let us introduce few definitions which are fundamental for our considerations.

Definition 1. *A natural biorthogonal system* Φ *is called a* greedy basis *if there exists a constant* C *such that for all* $x \in X$ *and* $m = 0, 1, 2, \ldots$ *we have*

$$\|x - \mathcal{G}_m\{\Phi\}(x)\| \leq C\sigma_m\{\Phi\}(x).$$

The smallest such constant C will be called the *greedy constant* of Φ.

Definition 2. *A natural biorthogonal system* Φ *is called a* quasi-greedy basis *if for every* $x \in X$ *the norm limit* $\lim_{m\to\infty} \mathcal{G}_m\{\Phi\}(x)$ *exists (and equals* x*).*

Clearly every greedy basis is quasi-greedy. Those concepts were formally defined in [14], but they were implicit in earlier work of Temlyakov [17]–[19].

Now we will provide characterizations of the above concepts. To state them we will need more definitions. Let us start with the following concept, which is well known in Banach space theory, cf. [23], [24]:

Definition 3. *A biorthogonal system* $\Phi = (x_n, x_n^*)_{n\in A}$ *is* unconditional *if there exists a constant* K *such that for all* $x \in X$ *and finite sets* $B \subset A$ *we have* $\|P_B(x)\| \leq K\|x\|$.

The smallest such constant K will be called *unconditional constant* of the system Φ. It is well known that this definition is equivalent to requiring that $\|P_B\| \leq K$ for all (not necessarily finite) $B \subset A$. One easily checks that for an unconditional system $\{x_n, x_n^*\}_{n\in A}$ and each $x \in X$ the series $\sum_{n\in A} x_n^*(x)x_n$ converges unconditionally (in particular in any order) to x. This implies that each unconditional system is quasi-greedy. Quite often in the sequel we will assume that the unconditional system has unconditional constant equal to 1. This is not an important restriction since given an unconditional system Φ in X we can introduce a new norm

$$|||x||| = \sup_{|\lambda_n|\le 1} \| \sum_{n\in A} \lambda_n x_n^*(x)x_n\|.$$

It is clear by a standart extreme point argument that this is an equivalent norm on X, more precisely $\|x\| \le |||x||| \le 2K\|x\|$ for $x \in X$ and Φ has unconditional constant 1 in $(X||| \cdot |||)$.

Remark 1. There is certain terminological confusion in this area. Generally the term greedy or quasi-greedy *basis* is used. However in Banach space theory the term basis usually stands for Schauder basis, i.e., a biorthogonal system satisfying (1.4) such that the set A equals $\mathbb{N}$ and for each $x \in X$ the series (1.5) converges to x. Thus, being a basis depends on the order while our Definitions 1 and 2 do not. Actually the above remarks show that unconditional system is an unconditional basis, so (see Theorem 1) a greedy system is automatically a basis. This explains the terminology established in Definition 1. For quasi-greedy systems, the term *quasi-greedy basis* will indicate that the system is quasi greedy and a basis in some (preferably natural) order.

Definition 4 ([14]). *A biorthogonal system Φ is called* democratic *if there exists a constant D such that for any two finite subsets $P, Q \subset A$ with $|P| = |Q|$ we have*

$$\| \sum_{n\in P} x_n\| \le D \| \sum_{n\in Q} x_n\|.$$

The smallest such constant D will be called a *democratic constant* of Φ. Clearly, it follows from Definition 4 that the norm $\| \sum_{j\in B} x_j\|$ is essentially a function of $|B|$, not of the set B itself.

All the above concepts are, up to a certain extent, independent of the normalization of the system. Namely, we have:

If $(\lambda_n)_{n\in A}$ is a sequence of numbers such that

$$0 < \inf_{n\in A} |\lambda_n| \le \sup_{n\in A} |\lambda_n| < \infty$$

and $\Phi = (x_n, x_n^*)_{n\in A}$ is a system which satisfies any of the Definitions 1–4, then the system $(\lambda_n x_n, \lambda_n^{-1} x_n^*)_{n\in A}$ satisfies the same definition.

For unconditional and democratic this is routine, for quasi-greedy it was proved in [21, Proposition 3] while for greedy it follows from Theorem 1 below.

Using the above concepts we can formulate the following important

Theorem 1 ([14]). *If the natural biorthogonal system Φ is greedy with the greedy constant $\leq C$, then it is unconditional with unconditional constant $\leq C$ and democratic with the democratic constant $\leq C^2$. Conversely, if it is unconditional with constant K and democratic with constant D, then it is greedy with greedy constant $\leq K + K^3 D$.*

Proof. Let us assume that $\Phi = (x_n, x_n^*)_{n\in A}$ has greedy constant C. Let us fix a finite set $B \subset A$ of cardinality m, $x \in X$ and a number $N > \sup_{n\in A} |x_n^*(x)|$. We put $y = x - P_B x + N\sum_{j\in B} x_j$. Clearly $\sigma_m(y) \leq \|x\|$ and $\mathcal{G}_m(y) = N\sum_{j\in B} x_j$. Thus

$$\|x - P_B x\| = \|y - \mathcal{G}_m(y)\| \leq C\sigma_m(y) \leq C\|x\|. \tag{1.6}$$

Now if $S \subset A$ is an arbitrary set we take a sequence of finite sets $B_1 \subset B_2 \subset \ldots$ such that $\bigcup B_j = S$. For each $x = \sum a_n x_n$, where the sum is finite, from (1.6) we have

$$\|x - P_S x\| = \lim_{n\to\infty} \|x - P_{B_n} x\| \leq C\|x\|.$$

Since such finite sums are dense in X we get $\|x - P_S x\| \leq C\|x\|$ for all $S \subset A$.

To show that Φ is democratic let us fix two subsets $B, C \subset A$ with $|B| = |C| = m$. Choose a third subset $D \subset A$ such that $|D| = m$ and $B \cap D = \emptyset = C \cap D$. For

$$x = (1+\epsilon)\sum_{n\in B} x_n + \sum_{n\in D} x_n$$

we have

$$\sigma_m(x) \leq (1+\epsilon)\|\sum_{n\in B} x_n\|$$

and

$$\|\sum_{n\in D} x_n\| = \|x - \mathcal{G}_m(x)\| \leq C\sigma_m(x) \leq C(1+\epsilon)\|\sum_{n\in B} x_n\|.$$

Analogously we get

$$\|\sum_{n\in C} x_n\| \leq C(1+\epsilon)\|\sum_{n\in D} x_n\|$$

and the conclusion follows.

Now we will prove the converse. Fix $x \in X$ and $m = 1, 2, \ldots$. Choose $p_m = \sum_{n\in B} a_n x_n$ with $|B| = m$ and $\|x - p_m\| \leq \sigma_m(x) + \epsilon$. Clearly

$$\mathcal{G}_m(x) = \sum_{n\in C} x_n^*(x) x_n = P_C x$$

for appropriate $C \subset A$ with $|C| = m$. We write

$$\|x - \mathcal{G}_m(x)\| = \|x - P_C x + P_B x - P_B x\| = \|x - P_B x + P_{B\setminus C} x - P_{C\setminus B} x\|. \tag{1.7}$$

Using unconditionality we get

$$\begin{aligned} \|x - P_B x - P_{C\setminus B} x\| &= \|x - P_{B\cup C} x\| = \|P_{A\setminus(B\cup C)}(x - p_m)\| \\ &\leq K(\sigma_m(x) + \epsilon), \end{aligned} \tag{1.8}$$

and analogously

$$\|P_{C\setminus B} x\| \leq K(\sigma_m(x) + \epsilon).$$

From the definition of $\mathcal{G}_m$ we infer that

$$\alpha = \min_{j\in C\setminus B} |x_j^*(x)| \geq \max_{j\in B\setminus C} |x_j^*(x)| = \beta,$$

so from unconditionality we get

$$K\|P_{C\setminus B} x\| \geq \alpha \| \sum_{j\in C\setminus B} x_j \| \tag{1.9}$$

and

$$\|P_{B\setminus C} x\| \geq K\beta \| \sum_{j\in B\setminus C} x_j \|. \tag{1.10}$$

Since $|B \setminus C| = |C \setminus B|$ from (1.9) and (1.10) and democracy we get

$$\|P_{B\setminus C} x\| \leq K^2 D \|P_{C\setminus B} x\|. \tag{1.11}$$

From (1.7), (1.8) and (1.11) we get (ϵ is arbitrary)

$$\|x - \mathcal{G}_m(x)\| \leq (K + K^3 D)\sigma_m(x).$$

□

Remark 2. The above proof is taken from [14]. However the arguments except the proof that greedy implies unconditional, were already in previous papers [18] and [21].

If we disregard constants Theorem 1 says that a system is greedy if and only if it is unconditional and democratic. The isometric situation is not so clear. Naturally from Theorem 1 follows that a system with greedy constant 1 have both unconditional and democratic constant equal to 1. However this is not a characterization of system with greedy constant 1. Let E denotes $\mathbb{R}^2$ with the norm whose unit ball is a regular octagon with vertices $(\pm 2^{-1/2}, \pm 2^{-1/2})$, $(\pm 1, 0)$ and $(0, \pm 1)$ and let F denotes $\mathbb{R}^2$ with the usual euclidean norm. For

$X = (E \oplus F)_2$ the unit vectors have unconditional and democratic constants equal to 1. The greedy constant of this basis is > 1. To see it consider vectors $x = \big((1, .5), (1 + \epsilon, .5)\big)$. Since $\|(1, .5)\|_E > \|(1, .5)\|_F$ we infer that for small $\epsilon > 0$

$$\Big\|\big((0, .5), (1 + \epsilon, .5)\big)\Big\|_X < \Big\|\big((1, .5), (0, .5)\big)\Big\|_X$$

which gives

$$\sigma_1(x) < \|x - \mathcal{G}_1(x)\|_X.$$

One can remark (and it is an easy two dimensional exercise) that if $(x_n)_{n\in A}$ is a system in a Hilbert space with greedy constant 1, then it is orthogonal and $\|x_n\| = \|x_m\|$ for all $n, m \in A$. This suggest

Problem 1. *Characterize system with greedy constant* 1.

Theorem 2 ([21]). *A natural biorthogonal system Φ is quasi-greedy if and only if there exists a constant C such that for every $x \in X$ and $m = 1, 2, \ldots$ we have*

$$\|\mathcal{G}_m\{\Phi\}(x)\| \leq C\|x\|.$$

The smallest constant C in the above theorem will be called *quasi-greedy constant* of the system Φ. Theorem 2 is a version of the uniform boundedness principle. The proof in [21] is by direct construction.

Problem 2. *It would be nice to have a category type proof.*

In order to discuss various properties of biorthogonal systems related to "greediness" let us introduce the following quantities:

$$\begin{aligned}
\varphi(m) &= \sup\{\|\sum_{n\in A} x_n\| : |A| \leq m\},\\
\psi(m) &= \inf\{\|\sum_{n\in A} x_n\| : |A| \geq m\},\\
e_m &= \sup_{x\in X,\, x\neq 0} \frac{\|x - \mathcal{G}_m(x)\|}{\sigma_m(x)},\\
\mu_m &= \sup_{k\leq m} \frac{\sup\{\|\sum_{n\in B} x_n\| : |B| = k\}}{\inf\{\|\sum_{n\in B} x_n\| : |B| = k\}}.
\end{aligned}$$

2. Examples of Systems

In this report we are mainly interested in concrete spaces and concrete systems. However we will also present results dealing with general (abstract) systems in general Banach spaces.

The most natural scale of spaces is the scale of L_p spaces, $1 \le p \le \infty$, but we may also consider Hardy space H_1, or the space VMO of function of vanishing mean oscillation, or the space BV of functions of Bounded Variation. As for the systems we will be mainly interested in wavelet type systems, especially the Haar system or similar, and trigonometric or Walsh system. Let us recall the definition and notations connected with the Haar system. We start with the function

$$h(t) = \begin{cases} 1, & \text{if } 0 \le t < 1/2, \\ -1, & \text{if } 1/2 \le t < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Clearly $\operatorname{supp} h = [0,1)$. For a pair $(j,k) \in \mathbb{Z}^2$ we define the function $h_{j,k}(t) = h(2^j t - k)$. The support of $h_{j,k}$ is the dyadic interval $I = I(j,k) = [k2^{-j}, (k+1)2^{-j})$. Quite often we will index Haar functions by dyadic intervals I and write h_I instead of $h_{j,k}$. The set of *all* dyadic subintervals of $\mathbb{R}$ will be denoted by $\mathfrak{D}$. It is well known and easy to check that the system $\{h_{j,k}\}_{(j,k)\in\mathbb{Z}^2} = \{h_I\}_{I\in\mathfrak{D}}$ is a complete orthogonal system in $L_2(\mathbb{R})$. When we consider the Haar system in a specified function space X on $\mathbb{R}$ we will consider the normalized system $h_I/\|h_I\|_X$. In function spaces in $\mathbb{R}^d$ we will consider two Haar systems:

1. *The tensored Haar system* defined as follows: If $\mathbf{J} = I_1 \times \cdots \times I_d$ where $I_1, \ldots, I_d \in \mathfrak{D}$, then we put $h_{\mathbf{J}}(t_1, \ldots, t_d) = h_{I_1}(t_1) \cdot \cdots \cdot h_{I_d}(t_d)$. One easily checks that the system $\{h_{\mathbf{J}}\}_{\mathbf{J}\in\mathfrak{D}^d}$ is a complete, orthogonal system in $L_2(\mathbb{R}^d)$. We will consider this system normalized in L_p with $1 \le p \le \infty$ and we will denote it by $\mathfrak{h}^p_d$, i.e., $\mathfrak{h}^p_d = \{h^p_{\mathbf{J}}\}_{\mathbf{J}\in\mathfrak{D}^d}$ where $h^p_{\mathbf{J}} = \|h_{\mathbf{J}}\|_p^{-1} h_{\mathbf{J}}$.

   The feature of this system is that supports of the functions are dyadic parallelograms with arbitrary sides.

2. *The Haar system* defined as follows: Let us denote by $h^1(t)$ the function $h(t)$ defined in (2.1). Let us denote $h^0(t) = \mathbf{1}_{[0,1]}$. For a fixed $d = 1, 2, \ldots$ let $\mathfrak{E}$ denotes the set of sequences $\epsilon = (\epsilon_1, \ldots, \epsilon_d)$ such that $\epsilon_j = 0$ or 1 and $\sum_{j=1}^d \epsilon_j > 0$. For $\epsilon \in \mathfrak{E}$, $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$ we define a function on $\mathbb{R}^d$ by the formula

   $$h^\epsilon_{j,k}(t_1, \ldots, t_d) = 2^{jd/2} \prod_{j=1}^d h^{\epsilon_j}(2^j x - k_j). \tag{2.2}$$

   It is easily checked that the system $(h^\epsilon_{j,k})$ where ϵ varies over $\mathfrak{E}$, j varies over $\mathbb{Z}$ and k varies over $\mathbb{Z}^d$ is a complete orthonormal system in $L_2(\mathbb{R}^d)$. The full set of indices of this system, i.e., $\mathfrak{E} \times \mathbb{Z} \times \mathbb{Z}^d$ will be denoted by $\mathfrak{I}(d)$. The Haar system normalized in $L_p(\mathbb{R}^d)$ with $1 \le p \le \infty$ will be denoted by $\mathfrak{H}^p_d$, i.e., $\mathfrak{H}^p_d = \{H^p_\alpha\}_{\alpha\in\mathfrak{I}(d)}$ where for $\alpha = (\epsilon, j, k) \in \mathfrak{I}(d)$ we have $H^p_\alpha = \|h^\epsilon_{j,k}\|_p^{-1} h^\epsilon_{j,k}$.

The feature of this system is that supports of the functions are all dyadic cubes. We can restrict the Haar system $\mathfrak{H}_d^p$ to the unit cube $[0,1]^d$. We simply consider all Haar functions whose support is contained in $[0,1]^d$ plus the constant function. In this way we get a system in $L_p[0,1]^d$.

Analogous procedures can be applied to general wavelet bases. Suppose that $\varphi^0(t)$ is a scaling function of a multiresolution on $\mathbb{R}$ and $\varphi^1(t)$ is an associated wavelet in $L_2(\mathbb{R})$. We assume that both φ^0 and φ^1 have sufficient decay to ensure that $\varphi^0, \varphi^1 \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$. The function $\mathbf{1}_{[0,1]}$ is the simplest example of a scaling function and the function $h(t)$ is the simplest example of a wavelet. Formula (2.2) with h replaced by φ defines a *wavelet basis* on $\mathbb{R}^d$. We may also define a tensored wavelet basis as defined in 1. above, but we will not discuss it in this survey. The detailed information on wavelets can be found e.g. in [23] and the detailed description of the above procedures is in Section 5.1.

3. Estimates for Unconditional Bases

In this section we will concentrate on estimates dealing with approximation properties of unconditional bases. This restriction is justified by the fact that very often unconditional bases provide right tools for approximation cf. e.g. [10]. Since in each space with an unconditional basis we can introduce an equivalent norm in which this basis will have unconditional constant equal to 1, we will always assume this in this section.

It follows from Theorem 1 that for an unconditional system the sequence e_m is bounded if and only if the sequence μ_m is bounded. Actually, more is true.

Theorem 3. *If Φ is a natural biorthogonal system with unconditional constant* 1*, then* $\frac{1}{2}\mu_m\{\Phi\} \leq e_m\{\Phi\} \leq 2\mu_m\{\Phi\}$.

This result we quote from [21] but it was already well established for concrete systems in [18]. More elaborate results of this type are presented in [16].

Let us also state explicitly an easy but interesting fact which is established in the first four lines of the proof of this theorem as presented in [21].

Proposition 1. *Let $\Phi = (x_n, x_n^*)_{n\in A}$ be a natural biorthogonal system with unconditional constant* 1*. Then for each $x \in X$ and each $m = 1, 2, \ldots$ there exists a subset $B \subset A$ of cardinality m such that*

$$\|x - \sum_{n\in B} x_n^*(x)x_n\| = \sigma_m(x).$$

The question of existence of *best* m-term approximation for a given natural system is an interesting one. It was discussed (even in a more general setting) by B. Baishanski in [1]. A more detailed study in our context can be found in [25].

Theorem 4. *Let Φ be a natural biorthogonal system with unconditional constant 1. Suppose that $s(m)$ is a function such that for some $c > 0$*

$$\psi(s(m)) \geq c\varphi(m) \qquad \text{for } m = 1, 2, \dots. \tag{3.1}$$

Then

$$\|x - \mathcal{G}_{m+s(m)}(x)\| \leq C\sigma_m(x)$$

for some constant C and $m = 1, 2, \dots$.

Proof. Let us fix $x \in X$ with $\|x\| = 1$ and $m = 1, 2, \dots$. Using Proposition 1, let us fix a subset $B \subset A$ of cardinality m such that

$$\sigma_m(x) = \|x - P_B(x)\|,$$

and $C \subset A$ a subset of cardinality $s(m) + m$ such that $\mathcal{G}_{s(m)+m}(x) = P_C(x)$. Using unconditionality of the system we get

$$\begin{aligned} \|x - P_B(x)\| &\geq \max\{\|x - P_{B\cup C}(x)\|, \|P_{C\setminus B}(x)\|\}, \\ \|x - P_C(x)\| &\leq \|x - P_{B\cup C}(x)\| + \|P_{B\setminus C}(x). \end{aligned}$$

Let $\xi = \inf_{j\in C} |x_j^*(x)|$. Then from unconditionality we get

$$\|P_{C\setminus B}(x)\| \geq \|\xi \sum_{j\in C\setminus B} x_j\| \geq \xi\psi(s(m)). \tag{3.2}$$

Since for $j \in B \setminus C$ we have $|x_j^*(x)| \leq \xi$, we get

$$\|P_{B\setminus C}(x)\| \leq \xi\| \sum_{j\in B\setminus C} x_j\| \leq \xi\varphi(m). \tag{3.3}$$

From (3.2), (3.3) and (3.1) we get

$$\|P_{B\setminus C}(x)\| \leq C\|P_{C\setminus B}(x)\|$$

so

$$\begin{aligned} \|x - \mathcal{G}_{s(m)+m}(x)\| &= \|x - P_C(x)\| \leq C\big(\|x - P_{B\cup C}(x)\| + \|P_{C\setminus B}(x)\|\big) \\ &\leq 2C\|x - P_B(x)\| \leq 2C\sigma_m(x). \end{aligned}$$

□

Let $\Phi = (x_n, x_n^*)_{n\in A}$ be a biorthogonal system. The system $(x_n^*, x_n)_{n\in A}$ considered as a system in X^* (we identify x_n's with elements of X^{**}) may not satisfy (1.4). However, if we consider it as a system in $\overline{\text{span}\{x_n^*\}_{n\in A}} \subset X^*$, then it will satisfy all our assumptions, we will denote it by Φ^*. If Φ is unconditional then so is Φ^*.

Theorem 5. *Let Φ be a natural biorthogonal system with unconditional constant* 1*. Then*

$$\mu_m\{\Phi^*\} \leq 2\log m\,\mu_m\{\Phi\}$$

for $m = 2, 3, \ldots$.

Proof. Let us fix m, $k \leq m$, and a set $B \subset A$ of cardinality k. We have

$$\|\sum_{j\in B} x_j^*\| \geq k\|\sum_{j\in B} x_j\|^{-1} \geq \frac{k}{\varphi(k)}. \tag{3.4}$$

On the other hand we can find $x \in X$ with $\|x\| = 1$ such that

$$\|\sum_{j\in B} x_j^*\| \leq 2\sum_{j\in B} |x_j^*(x)| \leq 2\sum_{j=1}^{k} \varphi(j)^{-1}. \tag{3.5}$$

Thus from (3.4) and (3.5), using the fact that $\frac{\varphi(k)}{k}$ is decreasing, we get

$$\begin{aligned}
\mu_m\{\Phi^*\} &\leq 2\sup_{k\leq m}\frac{1}{k}\sum_{j=1}^{k}\frac{\varphi(k)}{\psi(j)} \leq 2\sup_{k\leq m}\sum_{j=1}^{k}\frac{1}{j}\frac{\varphi(j)}{\psi(j)} \qquad (3.6)\\
&\leq 2\log m\sup_{j\leq m}\frac{\varphi(j)}{\psi(j)} \leq 2\log m\mu_m\{\Phi\}.
\end{aligned}$$

□

Theorems 4 and 5 are formally new, but the arguments are easy modifications of arguments from [7]. Similar results are also stated without proof in [15] and atributed to A. Kamont and V.Temlyakov.

Corollary 1. *Suppose that Φ is a greedy basis and that $\varphi(m) \sim n^\alpha$ with $0 < \alpha < 1$. Then Φ^* is also greedy.*

Proof. From Theorem 1 we know that Φ is unconditional, so we can renorm it to be 1-unconditional. Also, because Φ is greedy we have $\varphi(m) \sim \psi(m)$. We repeat the proof of Theorem 5 but in (3.6) we explicitly calculate as follows:

$$\mu_m\{\Phi^*\} \leq 2C\sup_{k\leq m}\frac{1}{k}\sum_{j=1}^{k}\frac{k^\alpha}{j^\alpha} \leq \text{const.}$$

so Φ^* is greedy. □

This is a special case of Theorem 5.1 from [7].

Let us recall that it was proved in [11] that each unconditional basis in L_p, $1 < p < \infty$, has a subsequence equivalent to the unit vectors basis in ℓ_p, so for each greedy basis Φ in L_p we have $\varphi\{\Phi\}(m) \sim m^{1/p}$. Thus we get

Corollary 2. *If Φ is a greedy basis in L_p, $1 < p < \infty$, then Φ^* is a greedy basis in L_q, $1/p + 1/q = 1$.*

Let us remark, although this does not belong to this Section, that for the quasi-greedy basis in ℓ_1, constructed in [9], the dual basis is not unconditional for constant coefficients, so is not quasi-greedy. On the other hand Corollary 4.5 and Theorem 5.4 from [7] show that the dual of a quasi-greedy basis in a Hilbert space is also quasi-greedy. Otherwise not much is known about duality for quasi-greedy bases.

Problem 3. *Investigate the duality for quasi-greedy bases.*

4. Examples of Quasi-greedy Systems

It is clear from our assumptions that each unconditional system is quasi-greedy. This allows us to observe that for quasi-greedy system the greedy approximation can be very inefficient. For the natural basis in $\ell_1 \oplus c_0$, which is clearly unconditional, we have $e_m \sim m$.

For quasi-greedy bases we have the following proposition which shows that they are rather close to unconditional systems. We say that they are unconditional for constant coefficients, as formulated in the following

Proposition 2 ([21]). *If Φ has a quasi-greedy constant C, then for every (finite) subset $B \subset A$ and every sequence of signs $\epsilon = (\epsilon_j)_{j\in B}$ we have*

$$\frac{1}{2C}\|\sum_{j\in B} x_j\| \le \|\sum_{j\in B} \epsilon_j x_j\| \le 2C\|\sum_{j\in B} x_j\|. \tag{4.1}$$

Proof. The estimate (4.1) easily follows from the estimate

$$\|\sum_{j\in B} x_j\| \le C\|\sum_{j\in C} x_j\|$$

for $B \subset C \subset A$. This follows from Theorem 2 applied to

$$x_\delta = (1+\delta)\sum_{j\in B} x_j + \sum_{j\in C\setminus B} x_j$$

and taking the limit as $\delta \searrow 1$. □

One should remark that Proposition 2 immediately shows that systems like Walsh or trigonometric are quasi-greedy in L_p only for $p = 2$. To see this for

$p < \infty$ suppose that say trigonometric system satisfies (4.1). Then taking the average over signs we get

$$\left(\int_0^1 \|\sum_{j=1}^N r_j(t)e^{ijs}\|_p^p dt\right)^{1/p} \sim \|\sum_{j=1}^N e^{ijs}\|_p.$$

The symbol r_j in the above denotes the Rademacher system. The right hand side (which is the L_p norm of the Dirichlet kernel) is of order $N^{1-\frac{1}{p}}$ if $p > 1$ and of order $\log N$ when $p = 1$. Changing the order of integration and using the Kchintchine inequality we see that the left hand side is of order $\sqrt{N}$. To decide the case $p = \infty$ we may recall that the well-known Rudin-Shapiro polynomials are of the form $p_N(s) = \sum_{j=1}^N \pm e^{ijs}$ for appropriate signs but $\|p_N\|_\infty \sim \sqrt{N}$ while the L_∞ norm of the Dirichlet kernel is clearly equal to N. This violates (4.1). Those results were proven in [17] and [3], see also [21].

In view of the above observations it is natural to look for examples of conditional quasi-greedy bases, especially in spaces which do not have unconditional bases. Some examples were given in [14] but the general treatment was presented in [21] and recently generalized in [6]. In both papers the approach is quite abstract and uses the existence of good complemented subspaces. A very general result (Corollary 7.3 from [6]) is as follows

Theorem 6 ([6]). *If X has a basis and contains a complemented subspace S with a symmetric basis, where S is not isomorphic to c_0, then X has a quasi-greedy basis.*

On the other hand it is proved in [6] that spaces like $C[0,1]$ or the disc algebra A do not have quasi-greedy bases.

Let me comment on one very special case of Theorem 6: *The space $L_1[0,1]$ has a quasi-greedy basis.* Since it is known that $L_1[0,1]$ does not have unconditional (in particular greedy) this is a good basis. On the other hand it is not one of classical systems. In particular the Haar basis (and other wavelet type bases) are not quasi-greedy in $L_1(\mathbb{R})$. To see it note that for $I_n = [0, 2^{-n}]$, $n = 1, 2, \ldots, N$, we have $\|\sum_{n=1}^N H^1_{I_n}\|_1 \sim$ const. while $\|\sum_{n=1}^N (-1)^n H^1_{I_n}\|_1 \sim \log N$, so (4.1) is violated.

Problem 4. *It would be very interesting to have a more analytical construction of a quasi-greedy basis in $L_1[0,1]$ which would allow to investigate analytical properties of this basis.*

5. Haar System in L_p

The basic tool to analyze unconditional systems in L_p is the following consequence of Khintchine inequality.

Proposition 3. *If $\Phi = (x_n, x_n^*)_{n\in A}$ is an unconditional system in L_p, $1 < p < \infty$, then the expression*

$$\Big(\int \big(\sum_{n\in A} |x_n^*(x)|^2 |x_n(s)|^2\big)^{p/2}\, ds\Big)^{1/p} \tag{5.1}$$

gives an equivalent norm on L_p.

The above proposition fails for $p = 1$ but if we introduce the norm given by (5.1) for $p = 1$ and the Haar system $\mathfrak{H}_1^1$, then we get a new space denoted as H_1, in which the system $\mathfrak{H}_1^1$ is unconditional. For the concise explanation of those ideas the reader may consult [23, 7.3].

The following theorem was proved in [19] and was a starting point in the investigation of greedy type bases.

Theorem 7 ([19]). *The Haar system $\mathfrak{H}_d^p$ is a greedy basis in $L_p(\mathbb{R}^d)$ for $d = 1, 2, \dots$ and $1 < p < \infty$. The system $\mathfrak{H}_1^1$ is greedy in H_1.*

For the proof we only have to check that those systems are democratic.

Lemma 1. *The Haar system $\mathfrak{H}_d^p$ is democratic in $L_p(\mathbb{R}^d)$ for $d = 1, 2, \dots$ and $1 < p < \infty$ (and also in H_1).*

Proof. Let $B \subset \mathfrak{I}(d)$ be a finite set. Note that if the cube Q is the support of the Haar function H_α^p, then $|H_\alpha^p| = |Q|^{-1/p}\mathbf{1}_Q$. Thus, for each $t \in \mathbb{R}^d$, the non-zero values of the Haar functions $H_\alpha^p(t)$ belong to a geometric progression with a ratio 2^d. Also, for a given $t \in \mathbb{R}^d$ there is at most $2^d - 1$ Haar functions which take a given non-zero value at this point. Thus, if we define $M(t)$ by the condition $2^{M(t)} = \max_{\alpha\in B} |H_\alpha^p(t)|^p$, we immediately see that

$$2^{M(t)} \geq c(d) \sum_{\alpha\in B} |H_\alpha^p(t)|^p$$

for some constant $c(d) > 0$. So

$$\begin{aligned}
\Big(\int \big(\sum_{\alpha\in B} |H_\alpha^p(t)|^2\big)^{p/2}\, dt\Big)^{1/p} &\geq \Big(\int 2^{M(t)}\, dt\Big)^{1/p} \\
&\geq \Big(\int c(d) \sum_{\alpha\in B} |H_\alpha^p(t)|^p\, dt\Big)^{1/p} \\
&= c(d)^{1/p} |B|^{1/p}.
\end{aligned}$$

On the other hand, by the same geometric consideration we see that for each $t \in \mathbb{R}^d$ we have

$$\sum_{\alpha\in B} |H_\alpha^p(t)|^2 \leq C(d) |H_{\alpha_0}^p(t)|^2$$

for some constant $C(d) < \infty$ and $\alpha_0 \in B$ depending on t. Thus

$$\Big(\int \big(\sum_{\alpha\in B} |H^p_\alpha(t)|^2\big)^{p/2}\,dt\Big)^{1/p} \le \Big(\int C(d)\sum_{\alpha\in B} |H^p_\alpha(t)|^p\,dt\Big)^{1/p} \le C(d)^{1/p}|B|^{1/p}.$$

Using Proposition 3 we get the claim. □

The above argument is basically contained in [12].

For the systems $\mathfrak{h}^p_d$ the situation is quite different.

Theorem 8. *For $d = 1, 2, \ldots$ and $1 < p < \infty$ in $L_p(\mathbb{R}^d)$ we have*

$$\varphi\{\mathfrak{h}^p_d\}(m) \sim m^{1/p} \tag{5.2}$$

$$\psi\{\mathfrak{h}^p_d\}(m) \sim m^{1/p}\big(\log m\big)^{(1/2-1/p)(d-1)} \tag{5.3}$$

for $p \le 2$, and

$$\varphi\{\mathfrak{h}^p_d\}(m) \sim m^{1/p}\big(\log m\big)^{(1/2-1/p)(d-1)} \tag{5.4}$$

$$\psi\{\mathfrak{h}^p_d\}(m) \sim m^{1/p} \tag{5.5}$$

for $2 \le p < \infty$. So we have

$$e_m\{\mathfrak{h}^p_d\} \sim (\log m)^{(d-1)|\frac{1}{2}-\frac{1}{p}|}. \tag{5.6}$$

This result was conjectured in [18] and proved in [21] using techniques from [12].

Thus for $d \ge 2$ and $p \ne 2$ the system $\mathfrak{h}^p_d$ is not greedy. Thus the following interesting problem arises.

Problem 5. *Find an algorithm which for a given $x \in L_p(\mathbb{R}^d)$ gives its near best m-term approximation with respect to the system $\mathfrak{h}^p_d$.*

In view of Proposition 1, for a given x one has to find the set of "most essential" coefficients. Observe that from (5.2)–(5.6) one easily calculates that the system $\mathfrak{h}^p_d$ satisfies (3.1) with $s(m) \sim m\big(\log m\big)^{\frac{|p-2|}{2}(d-1)}$ so from Theorem 4 we infer that it suffices to look for those m "essential coefficients" among approximately $m\big(\log m\big)^{\frac{|p-2|}{2}(d-1)}$ biggest coefficients.

Now let us consider the system $\mathfrak{H}^1_1$ in H_1. The dual system $(\mathfrak{H}^1_1)^*$ is the system $\mathfrak{H}^\infty_1$ considered in the space VMO. It was proved in [16] that $e_m\{\mathfrak{H}^\infty_1\} \sim \sqrt{\log m}$ in the space VMO, so we have a natural example of a greedy system whose dual is not greedy. Actually, one can prove that the space VMO does not have any greedy system.

6. Examples of Greedy Bases

It is the obvious observation that being greedy (or quasi-greedy) is an isomorphic property. This means that if $(x_n)_{n\in A}$ is a greedy system in Banach space X and $I : X \to Y$ is a linear isomorphism, then $\big(I(x_n)\big)_{n\in A}$ is a greedy system in Y. There are at least two practically important instances of this fact.

A. If $\mathfrak{B}$ is a good wavelet basis (cf. e.g. [23, Theorem 8.13]), then in L_p, $1 < p < \infty$, it is equivalent to the Haar system $\mathfrak{H}^p$. Thus, all such systems are greedy.

B. It is known (cf. e.g. [23, Chapter 9]) that good wavelet bases in Besov space $B^p_{\alpha,p}(\mathbb{R})$ when properly normalized are equivalent to the unit vector basis in ℓ_p, thus greedy for $1 \le p < \infty$. The same is true for Ciesielski systems on $[0, 1]$.

The above remarks explain the attention we pay to the Haar system $\mathfrak{H}$. Now let us discuss other examples of greedy systems.

Let us start with the most important case of Hilbert space. Clearly every orthonormal basis, and more generally, every Riesz basis is greedy in a Hilbert space. Those are the only greedy systems in the Hilbert space, because they are the only unconditional systems. This easily follows from Proposition 3.

In L_p for $1 < p < \infty$, $p \ne 2$, the situation is not so simple. Kamont [13] investigates generalized Haar systems on $[0, 1]$. Those are the systems obtained as follows:

> The first function is $\mathbf{1}_{[0,1]}$. Next we divide $[0, 1]$ into two subintervals I_l and I_r (nontrivial but generally not equal) and the next function is of the form $a\mathbf{1}_{I_l} + b\mathbf{1}_{I_r}$ and is orthogonal to the previous function. We repeat this process on each of intervals I_l and I_r and continue in this manner.

If we make sure that the lengths of subintervals tend to zero the system will span $L_p[0, 1]$ for $1 \le p < \infty$. One of the main results of [13] says that each generalized Haar system (normalized in $L_p[0, 1]$) is equivalent to a subsequence of $\mathfrak{H}^p_1$, so is greedy.

An example of a basis in L_p for $p > 2$ which is greedy and not equivalent to a subsequence of the Haar system $\mathfrak{H}$ was given in [21]. It follows from Corollary 2 that such an example exists also for $1 < p < 2$.

7. Functions of Bounded Variation

Let $\Omega \subset \mathbb{R}^d$ be an open subset. Let us recall that a function $f \in L_1(\Omega)$ has bounded variation if all its distributional derivatives $\frac{\partial f}{\partial x_j}$ are measures of bounded variation. The space of all such functions equipped with the norm

$$|f|_{\mathrm{BV}(\Omega)} = \sum_{j=1}^{d} \|\frac{\partial f}{\partial x_j}\|$$

is denoted by $BV(\Omega)$.

This function space plays a very important role in geometric measure theory, calculus of variations, image processing and other areas. Recently the problem of m-term approximation with respect to the Haar system $\mathfrak{H}_d^p$ of a function in $BV(\mathbb{R}^d)$ in the L_p norm with $p = d/(d-1)$ was investigated in [2] and [22]. Among other things it was proved there (cf. [22, Theorem 10]) that

$$|\mathcal{G}_m\{\mathfrak{H}_d^p\}(f)|_{BV} \leq C|f|_{BV} \tag{7.1}$$

for all $f \in BV(\mathbb{R}^d)$. One has to note that $BV(\mathbb{R}^d)$ is a non-separable space so it cannot have any countable system satisfying (1.4). However, from Theorem 2 we infer that $\mathfrak{H}_d^p$ is a quasi-greedy system in $\overline{\operatorname{span}\{\mathfrak{H}_d^p\}}$ in the $|\cdot|_{BV}$ norm. This is not a very satisfactory result since $\overline{\operatorname{span}\{\mathfrak{H}_d^p\}}$ is not a very natural space.

A natural separable subspace of $BV(\mathbb{R}^d)$ is the Sobolev space $W_1^1(\mathbb{R}^d)$, i.e., the space of all $f \in BV(\mathbb{R}^d)$ such that $\frac{\partial f}{\partial x_j}$ are absolutely continuous measures for $j = 1, 2, \ldots, d$. A natural and interesting problem which appears in this context is

Problem 6. *Prove that* (7.1) *holds with Haar system replaced by a good, smooth wavelet basis, or only show that a good, smooth wavelet basis is a quasi-greedy basis in* $W_1^1(\mathbb{R}^d)$*.*

Let me remark that it is known that $W_1^1(\mathbb{R}^d)$ does not have unconditional basis, so it does not have a greedy basis. On the other hand, it is an immediate corollary from Theorem 6 that $W_1^1(\mathbb{R}^d)$ has a quasi-greedy basis.

8. Subsequences

It is a natural general mathematical question, although probably without much practical importance, what subsequences of a natural system are greedy or quasi-greedy. In [14] it was observed that a subsequence of a trigonometric system in L_p, $1 \leq p \leq \infty$, is a greedy basis in its linear span if and only if it is unconditional. Thus in L_p, $1 \leq p \leq \infty$, they are equivalent to the unit vector basis in ℓ_2 while for $p = \infty$ they are equivalent to the unit vector basis in ℓ_1. The following seems to be unknown.

Problem 7. *Find a description of quasi-greedy subsequences of the trigonometric system.*

But it is only one of many possible questions of this type. Some quasi-greedy subsequences of the Haar system $\mathfrak{H}_1^1$ in $L_1[0,1]$ were exhibited in [8].

9. Greedy Bases in L_p

From recent work [5] and [22] it became apparent that greedy basis in L_p is a natural substitute for an orthonormal basis in a Hilbert space. Let us explain briefly what we have in mind. In [5] the following general problem is discussed. Let $\mathcal{F}$ be certain Banach space continuously embedded into L_p and let $\mathcal{F}_0$ be its unit ball. For a given basis $B = (b_k)$ in L_p we introduce the quantities

$$\sigma_m\{B\}(\mathcal{F}) = \sup_{x\in\mathcal{F}_0} \sigma_m\{B\}(x).$$

We are looking for a basis B which gives the best order of decay of the quantities $\sigma_m\{B\}(\mathcal{F})$. It is natural to expect that the "best" basis has to have close connection with the class $\mathcal{F}$. We say that a basis B is *aligned* with $\mathcal{F}$ if B is unconditional in $\mathcal{F}$, i.e., if whenever $f \in \mathcal{F}$ equals $\sum_k a_k(f)b_k$, then any $g = \sum a_k(g)b_k$ such that $|a_k(g)| \le |a_k(f)|$ for all k is also in $\mathcal{F}$. The following is proved in [5]: Suppose that $\overline{B} = (\bar{b}_k)$ is a greedy basis in L_p aligned with $\mathcal{F}$. If for some basis B unconditional in L_p we have $\sigma_m\{B\}(\mathcal{F}) = o(n^{-\alpha})$ for some $\alpha > 0$, then also $\sigma_m\{\overline{B}\}(\mathcal{F}) = o(n^{-\alpha})$. For approximation in L_2 this result was proved by Donoho [10].

It is a very nice result but it leaves some open questions.

Problem 8. *It seems likely that greedy aligned basis is "best" in the above sense among all bases in L_p, not only unconditional.*

Problem 9. *Results of [5] do not exclude a situation that for some other unconditional basis B we have $\lim_{m\to\infty} \sigma_m\{B\}(\mathcal{F})\big(\sigma_m\{\overline{B}\}(\mathcal{F})\big)^{-1} = 0$. I do not know any example that it happens and I conjecture that it is impossible.*

In [22] we consider the approximation of functions of bounded variation on $\mathbb{R}^d$ with $d \ge 2$ in $L_p(\mathbb{R}^d)$ where $p = d/(d-1)$ is the critical exponent, by the Haar system. Earlier, the case $d = 2$ which gives $p = 2$ was studied in detail in [2]. Some of those results are discussed in Section 7. The extension from $d = 2$ to $d > 2$ required two ingredients:

1. To prove the boundedness in $BV(\mathbb{R}^d)$ of certain averaging operators;
2. To replace the approximation arguments in Hilbert space setting by approximation in L_p.

In this second ingredient we extensively use the fact that the Haar system is greedy in L_p and various estimates following from this fact.

10. General remarks.

From the point of view of Approximation Theory the subject of greedy type bases is a part of a larger area of non-linear approximation and in particular greedy approximation. The nice survey [4] deals with non-linear approximation in general and shows the greedy bases well in its context. The survey [20] also deals with greedy bases.

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