Greedy algorithm for general systems [*†]

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Abstract

We consider biorthogonal systems in quasi-Banach spaces such that the greedy algorithm converges for each $x \in X$ (quasi-greedy systems). We construct quasi-greedy conditional bases in a wide range of Banach spaces. We also compare the greedy algorithm for the multidimensional Haar system with the optimal m-term approximation for this system. This substantiates a conjecture by Temlyakov.

1 Introduction

We consider a general quasi-Banach space X with the norm $\|.\|$ such that for all $x, y \in X$ we have $\|x + y\| \leq \alpha(\|x\| + \|y\|)$. The letter α will always in this paper denote this constant. We fix a biorthogonal system $(x_n, x_n^*)_{n\in F}$ in X such that $\overline{\text{span}}(x_n)_{n\in F} = X$ and $\inf_{n\in F} \|x_n\| > 0$ and $\sup_{n\in F} \|x_n^*\| < \infty$. This implies that for each $x \in X$ we have $\lim_{n\to\infty} x_n^*(x) = 0$. For each $x \in X$ and $m = 1, 2, \ldots$ we define

$$\mathcal{G}_m(x) = \sum_{n\in A} x_n^*(x)x_n \tag{1}$$

*Key words: quasi-greedy basis, conditional basis, Haar system

†1991 AMS Mathematics Subject Classification: 41A61, 46B15, 41A25

‡This research was partially supported by the grant 2 PO3A 00113 from Komitet Badań Naukowych and was completed while he visited Erwin Schrödinger Institute in Wien.

where $A \subset F$ is a set of cardinality m such that $|x_n^*(x)| \geq |x_k^*(x)|$ whenever $n \in A$ and $k \notin A$. The above set may not be uniquely defined but if this happens we take any such set. The operator $\mathcal{G}_m(x)$ is a non-linear and discontinuous operator.

This simple theoretical algorithm is a model for a procedure which is widely used in numerical applications. It also raises many interesting questions in functional analysis. A more detailed description of the connections with purely numerical questions and results for some concrete systems the reader can find in [9]–[11].

In this paper we will use standard Banach space notation as explained in detail in [12] or [6]. The basic reference for simple facts we are going to use about quasi-Banach space is [3].

DEFINITION 1. *A biorthogonal system $(x_n, x_n^*)_{n\in F}$ is called a quasi-greedy system if for each $x \in X$ the sequence $\mathcal{G}_m(x)$ converges to x in norm. If this system is a basis we will use the phrase quasi-greedy basis.*

Clearly every unconditional basis is a quasi-greedy basis. Let us recall the definition of the best m–term approximation. For $x \in X$ and $m = 0, 1, \ldots$ we put

$$\sigma_m(x) = \inf\{\|x - \sum_{n\in A} a_n x_n\| \,:\, |A| \leq m\}. \tag{2}$$

DEFINITION 2. *A basis $(x_n, x_n^*)_{n\in F}$ is called greedy if there exists a constant C such that for every $x \in X$ we have $\|x - \mathcal{G}_m(x)\| \leq C\sigma(x)$.*

After the research reported in this paper was practically completed I received the preprint [4] where the above terminology was introduced, so I decided to follow this terminology in this note. It is shown in [4] that each greedy basis is unconditional and an example of a conditional quasi–greedy basis is given.

Tha author would like to express his gratitude to prof. Aleksander Pełczyński for many helpful conversations about the research reported in this paper.

2 Quasi-greedy systems

Let us start with some general results. The following Theorem gives some natural equivalent conditions for the quasi-greedy systems.

Theorem 1. *The following conditions are equivalent:*

1. *The system $(x_n, x_n^*)_{n\in F}$ is quasi-greedy.*
2. *For each $x \in X$ the series $\sum_{n=1}^{\infty} x^*_{\sigma(n)}(x)x_{\sigma(n)}$ converges to x where σ is such an ordering of F that $(|x^*_{\sigma(n)}(x)|)_{n=1}^{\infty}$ is a decreasing sequence*
3. *there exists a constant C such that for any $x \in X$ and $m = 1, 2, \ldots$ we have $\|\mathcal{G}_m(x)\| \leq C\|x\|$.*

This Theorem is basically a uniform boundedness result. However, since operators $\mathcal{G}_m$ are non-linear and discontinuous we have to give an independent proof.

Proof. Clearly 1. $\iff$ 2.

3. $\Rightarrow$ 1. Since the convergence is clear for x's with finite expansion in the biorthogonal system, let us assume that x has an infinite expansion. Take $x_0 = \sum_{n\in A} a_n x_n$ such that $\|x - x_0\| < \epsilon$ where A is a finite set and $a_n \neq 0$ for $n \in A$. If we take m big enough we can ensure that $\mathcal{G}_m(x - x_0) = \sum_{n\in B} x_n^*(x - x_0)x_n$ with $B \supset A$ and $\mathcal{G}_m(x) = \sum_{n\in B} x_n^*(x)x_n$. Then

$$\begin{aligned} \|x - \mathcal{G}_m(x)\| &\leq \alpha(\|x - x_0\| + \|x_0 - \mathcal{G}_m(x)\|) \\ &\leq \alpha(\epsilon + \|\mathcal{G}_m(x_0 - x)\|) \leq \alpha(C+1)\epsilon. \end{aligned}$$

This gives 1.

1. $\Longrightarrow$ 3. Let us start with the following Lemma.

Lemma 1. *If 3. does not hold then for each constant K and each finite set $A \subset F$ there exists a finite set $B \subset F$ disjoint with A and a vector $x = \sum_{n\in B} a_n x_n$ such that $\|x\| = 1$ and $\|\mathcal{G}_m(x)\| \geq K$ for some m.*

Proof. Let us fix M to be the maximum of norms of (linear) projections $P_\Omega(x) = \sum_{n\in\Omega} x_n^*(x)x_n$ where $\Omega \subset A$. Let us start with a vector x^1 such that $\|x^1\| = 1$ and $\|\mathcal{G}_m(x^1)\| \geq K_1$. For $x^2 = x^1 - \sum_{n\in A} x_n^*(x^1)x_n$ we have $\|x^2\| \leq \alpha(M+1)$ and $\mathcal{G}_m(x^1) = \mathcal{G}_k(x^2) + P_\Omega(x^1)$ for some $k \leq m$ and $\Omega \subset A$. Thus $\|\mathcal{G}_k(x^2)\| \geq \frac{K_1}{\alpha} - M$ and for $x^3 = x^2 \cdot \|x^2\|^{-1}$ we have $\|\mathcal{G}_k(x^3)\| \geq (K_1/\alpha - M)/(1+M)\alpha$. Let us put

$$\delta = \inf\{|x_n^*(\mathcal{G}_k(x^3))| \ : \ x_n^*(\mathcal{G}_k(x^3)) \neq 0\}$$

and take a finite set B_1 such that for $n \notin B_1$ we have $|x_n^*(x^3)| \leq \delta/2$. Let us take η very small with respect to $|B_1|$ and $|A|$ and find x^4 with finite expansion such that $\|x^3 - x^4\| < \eta$. If η is small enough we can modify all coefficients of x^4 from B_1 and A so that resulting x^5 will have k biggest coefficients the same as x^3 and $\|x^4 - x^5\| < \delta$. Moreover x^5 will have the form $x^5 = \sum_{n\in B} x_n^*(x^5)x_n$ with B finite and disjoint with A. Since $\|x^5\| \leq \alpha(\|x^4\| + \delta) \leq \alpha(\alpha(1+\eta)+\delta) \leq C\alpha$ and $\mathcal{G}_k(x^5) = \mathcal{G}_k(x^3)$, for $x = x^5 \cdot \|x^5\|^{-1}$ we get $\|\mathcal{G}_k(x)\| \geq (\frac{K_1}{\alpha} - M)(\alpha + \alpha M)^{-1}(C\alpha)^{-1}$ which can be made $\geq K$ if we take K_1 big enough. □

Using this Lemma we can apply the standard gliding hump argument to get a sequence of vectors $y_n = \sum_{k\in B_n} a_k x_k$ with sets B_n disjoint and $\|y_n\| = 1$, a decreasing sequence of positive numbers $\epsilon_n \leq 2^{-n}$ such that if $x_k^*(y_n) \neq 0$ then $|x_k^*(y_n)| \geq \epsilon_n$ and a sequence of integers m_n such that $\|\mathcal{G}_{m_n}(y_n)\| \geq 2^n \prod_{j=1}^{n-1} \epsilon_j^{-1}$. Now we put $x = \sum_{n=1}^{\infty} \big(\prod_{j=1}^{n-1}\epsilon_j\big) y_n$. This series is clearly convergent in X. If we write $x \sim \sum_{n\in F} b_n x_n$ we infer that

$$\inf\{|b_n| \,:\, n \in \bigcup_{s=1}^{j} B_s \text{ and } b_n \neq 0\} \geq \prod_{s=1}^{j} \epsilon_s \geq \max\{|b_n| \,:\, n \notin \bigcup_{s=1}^{j} B_s\}.$$

This implies that for $k = \sum_{s=1}^{j-1} |B_s| + m_j$ we have

$$\mathcal{G}_k(x) = \sum_{n\leq j} \big(\prod_{s=1}^{n-1} \epsilon_s\big) y_n + \mathcal{G}_{m_j}\big(\prod_{s=1}^{j} \epsilon_s\big) y_{j+1}$$

so $\|G_k(x)\| \geq \alpha^{-1}\big(\prod_{s=1}^{n-1}\epsilon_s\big)\|\mathcal{G}_{m_j} y_{j+1}\| - C \geq 2^{j+1}/\alpha - C$. Thus $\mathcal{G}_m(x)$ does not converge to x. □

Now let us introduce the following definition:

DEFINITION 3. *A system $(x_n, x_n^*)_{n\in F}$ is called unconditional for constant coefficients if there exist constants C and $c > 0$ such that for each finite $A \subset F$ and each sequence of signs $(\varepsilon_n)_{n\in A} = \pm 1$ we have*

$$\|\sum_{n\in A} x_n\| \leq \|\sum_{n\in A} \varepsilon_n x_n\| \leq C\|\sum_{n\in A} x_n\|. \tag{3}$$

This Definition is justified by the following observation

Proposition 2. *Every quasi-greedy system is unconditional for constant coefficients.*

Proof. For a given sequence of signs $(\varepsilon_n)_{n\in A}$ let us define the set $A_1 = \{n \in A : \varepsilon_n = 1\}$. For each $\delta > 0$ we apply Theorem 1 and we get

$$\|\sum_{n\in A_1} x_n\| \leq C\|\sum_{n\in A_1} x_n + \sum_{n\in A\setminus A_1} (1-\delta)x_n\|.$$

Since this is true for each $\delta > 0$ we easily obtain the right hand side inequality in (3). The other inequality follows by analogous argument. □

Our definitions of quasi-greedy system and of the operator $\mathcal{G}_m$ depend on the normalisation of the system considered. This, however, is not essential, namely we have

Proposition 3. *Suppose that $(x_n, x_n^*)_{n\in F}$ is a quasi-greedy system like we are discussing. Let $(\lambda_n)_{n\in F}$ be a sequence of numbers such that $0 < a =: \inf_{n\in F}|\lambda_n| \leq b =: \sup_{n\in F}|\lambda_n| < \infty$. Then the system $(\lambda_n x_n, x_n^*/\lambda_n)_{n\in F}$ is also quasi-greedy.*

Proof. By homogeneity we can and will assume that $b = 1$. Let $\mathcal{G}_m^1$ be the greedy approximation operator corresponding to the system $(\lambda_n x_n, x_n^*/\lambda_n)_{n\in F}$. Let us fix $x \in X$ and a natural number m. Explicitely we have $\mathcal{G}_m^1(x) = \sum_{n\in A} x_n^*(x)x_n$ where $A \subset F$ is a set of cardinality m such that $|x_n^*(x)/\lambda_n| \geq |x_s^*(x)/\lambda_s|$ whenever $n \in A$ and $s \notin A$. Let us write $\eta = \inf_{n\in A}|x_n^*(x)|$ and let $V = \{n \in F : |x_n^*(x) \geq \eta\}$ and let $U = \{n \in F : |x_n^*(x)| \geq \eta/a\}$. We put $|V| = k$ and $|U| = l$. Clearly $U \subset V$ so $l \leq k$. Using those notations we can write

$$\mathcal{G}_m^1(x) = \mathcal{G}_k(x) - \sum_{s\in B} x_s^*(x)x_s = \mathcal{G}_l(x) + \big(\mathcal{G}_k(x) - \mathcal{G}_l(x) - \sum_{s\in B} x_s^*(x)x_s\big) \quad (4)$$

where B is a certain subset of $V \setminus U$ and so we know that for $s \in B$ we have

$$\eta \leq |x_s^*(x)| \leq \eta/a. \quad (5)$$

Clearly $\mathcal{G}_k(x) - \mathcal{G}_l(x) = \sum_{n\in V\setminus U} x_n^*(x)x_n$. From Proposition 2, using the extreme point argument we infer that for each set $D \subset F$ and each sequence $(a_n)_{n\in D}$ with $|a_n| \leq 1$ we have $\|\sum_{n\in D} a_n x_n\| \leq C\|\sum_{n\in D} x_n\|$. Thus we infer from (5) that

$$\begin{aligned}\|\mathcal{G}_k(x) - \mathcal{G}_l(x) - \sum_{s\in B} x_n^*(x)x_n\| &\leq C\|\sum_{n\in V\setminus U} (\eta/a)x_n\| \\ &\leq C^2/a\|\sum_{n\in V\setminus U} x_n^*(x)x_n\| \qquad (6) \\ &\leq C^2/a\|\mathcal{G}_k(x) - \mathcal{G}_l(x)\|.\end{aligned}$$

Comparing (4) and (6) we infer that $\|\mathcal{G}^1_m(x)\| \leq C'\|x\|$ so by Theorem 1 the system $(\lambda_n x_n, x^*_n/\lambda_n)_{n\in F}$ is a quasi-greedy system. $\square$

REMARK *Proposition 2 allows us to show that the trigonometric system in $L_p(\mathbb{T})$ with $1 \leq p \leq \infty$ is quasi-greedy only if $p = 2$, because only then it is unconditional for constant coefficients. To see this observe that for $1 < p \leq \infty$ we have $\|\sum_{n=1}^N e^{int}\|_p^p \sim N^{p-1}$ and for $p = 1$ we have $\|\sum_{n=1}^N e^{int}\| \sim \log(N+1)$. On the other hand for $1 \leq p < \infty$ the average over all signs $\pm$ of $\|\sum_{n=1}^N \pm e^{int}\|_p^p$ can be written as $\int_0^1 \|\sum_{n=1}^N r_n(s)e^{int}\|_p^p\, ds$ where $r_n(s)$ are classical Rademacher functions. It is well known in the theory of type and cotype of Banach spaces (see [12]) and easily follows from the Khintchine's inequality that $\int_0^1 \|\sum_{n=1}^N r_n(s)e^{int}\|_p^p\, ds \sim N^{p/2}$ so the trigonometric system in $L_p(\mathbb{T})$ ($1 \leq p < \infty$) can be quasi-greedy only when $p - 1 = p/2$ i.e. when $p = 2$. In the case $p = \infty$ we can invoke the Rudin-Shapiro polynomials i.e. polynomials $\phi_N = \sum_{n=1}^N \pm e^{int}$ such that $\|\phi_N\|_\infty \leq C\sqrt{N}$.*

Now we will discuss examples of conditional quasi-greedy bases. Let us recall

DEFINITION 4. *A biorthogonal system (resp. basis) $(x_n, x^*_n)_{n\in F}$ is a $p-$besselian system, $0 < p < \infty$ if there exists a constant C such that for each $x \in X$ we have*

$$\Big(\sum_{n\in F} |x^*_n(x)|^p\Big)^{1/p} \leq C\|x\|.$$

A 2–besselian system (resp. basis) will be called besselian.

Theorem 2. *Suppose X is a quasi-Banach space with besselian basis $(x_n, x^*_n)_{n=1}^\infty$. The space $X \oplus \ell_2$ has a quasi-greedy basis. If the basis $(x_n, x^*_n)_{n=1}^\infty$ is conditional we get a conditional quasi-greedy basis in $X \oplus \ell_2$.*

Proof. Let us recall about Olevskii matrices (cf. [7]). For $k = 1, 2, \ldots$ we define $2^k \times 2^k$ matrices $A^k = (a^{(k)}_{ij})_{i,j=1}^{2^k}$ by the following formulas

$$a^{(k)}_{i1} = 2^{-k/2} \text{ for } i = 1, 2, \ldots, 2^k$$

and for $j = 2^s + \nu$ with $1 \leq \nu \leq 2^s$ and $s = 1, 2, \ldots, k-1$ we put

$$a^{(k)}_{ij} = \begin{cases} 2^{(s-k)/2} & \text{for } (\nu-1)2^{k-s} < i \leq (2\nu-1)2^{k-s-1} \\ -2^{(s-k)/2} & \text{for } (2\nu-1)2^{k-s-1} < i \leq \nu 2^{k-s} \\ 0 & \text{otherwise} \end{cases}$$

One easily checks that A^k are orthonormal matrices and there exists a constant C_p such that for all i, k we have

$$\sum_j |a_{ij}^{(k)}|^p \leq C_p. \tag{7}$$

Note that A^k is a matrix which maps an orthonormal Haar-like system in $\mathbb{R}^{2^k}$ onto the unit vector basis. We put $N_k = 2^{10^k}$ and define S_k so that $S_1 = N_1 - 1$ and $S_{k+1} - S_k = N_k - 1$. Let $(e_r)_{r=1}^\infty$ denote the unit vector basis in ℓ_2. Let us denote by $(g_s)_{s=1}^\infty \subset X \oplus \ell_2$ the following basis

$$x_1, e_1, \ldots, e_{S_1}, x_2, e_{S_1+1}, \ldots, e_{S_2}, x_3, e_{S_2+1}, \ldots, e_{S_3}, x_4, \ldots.$$

To each block $\{x_k, e_{S_{k-1}+1}, \ldots, e_{S_k}\}$ we apply the matrix A^{10^k} to get a new system

$$\psi_i^k = \frac{u_k}{\sqrt{N_k}} + \sum_{j=2}^{N_k} a_{ij}^{10^k} r_{S_{k-1}+j}. \tag{8}$$

The system $\psi_1^1, \ldots, \psi_{N_1}^1, \psi_1^2, \ldots, \psi_{N_2}^2, \ldots,$ ordered in this fashion will be denoted by $(\psi_j)_{j=1}^\infty$. It is clear that $0 < \inf_j \|\psi_j\| \leq \sup_j \|\psi_j\| < \infty$ and that $(\psi_j)_{j=1}^\infty$ is a complete biorthogonal system in $X \oplus \ell_2$ with the biorthogonal functionals given by the formula

$$\psi_i^{k*} = \frac{x_n^*}{\sqrt{N_k}} + \sum_{j=2}^{N_k} a_{ij}^{10^k} e_{S_{k-1}+j}^*.$$

It is also a basis in $X \oplus \ell_2$.

Since the system $(g_j)_{j=1}^\infty$ is a basis it suffices to check that for each k the system $(\psi_i^k)_{i=1}^{N_k}$ have uniformly bounded basis constant. But on each subspace $\mathrm{span}\{x_k, e_{S_{k-1}+1}, \ldots, r_{S_k}\}$ the $\ell_2^{N_k}$ norm and the norm in $X \oplus \ell_2$ are uniformly equivalent, so any orthonormal basis in this finite dimensional space has uniformly bounded basis constant in $X \oplus \ell_2$. If $(x_n)_{n=1}^\infty$ was a conditional basis then $(\psi_j)_{j=1}^\infty$ is also conditional because $(x_n)_{n=1}^\infty$ is a block basis of $(\psi_j)_{j=1}^\infty$.

Thus we still have to show:

If $f = \sum_{j=1}^\infty a_j \psi_j \in X \oplus \ell_2$ with $\|f\| = 1$, and σ is a permutation such that $|a_{\sigma(j)}|$ is a decreasing sequence then the series $\sum_{j=1}^\infty a_{\sigma(j)} \psi_{\sigma(j)}$ converges in $X \oplus \ell_2$.

First observe that $(g_s)_{s=1}^\infty$ is a besselian basis in $X \oplus \ell_2$, so we can define an operator $I : X \oplus \ell_2 \longrightarrow \ell_2$ as $I(\sum_{s=1}^\infty a_s g_s) = (a_s)_{s=1}^\infty$. Since $(\psi_j)_{j=1}^\infty$ is obtained from $(g_s)_{s=1}^\infty$ by the action of a unitary matrix we infer that $(\psi_j)_{j=1}^\infty$ is also besselian and $(I\psi_j)_{j=1}^\infty$ is an orthonormal basis in ℓ_2. Let P denote the natural projection from $X \oplus \ell_2$ onto ℓ_2. Note that $I|\{0\} \oplus \ell_2$ is an isometric embedding. Let Q denotes the orthogonal projection onto $I(\{0\} \oplus \ell_2)$.

Let us write f as a double sum $f = \sum_{k=1}^\infty \sum_{i=1}^{N_k} b_i^k \psi_i^k$. Since $(\psi_j)_{j=1}^\infty$ is besselian we get

$$\sum_{k=1}^\infty \sum_{i=1}^{N_k} |b_i^k|^2 \leq C. \tag{9}$$

This implies that the series $\sum_{k=1}^\infty \sum_{i=1}^{N_k} b_i^k I(\psi_i^k)$ converges in ℓ_2 in any order so also $\sum_{k=1}^\infty \sum_{i=1}^{N_k} b_i^k QI(\psi_i^k)$ converges in any order. This implies that also the series $\sum_{k=1}^\infty \sum_{i=1}^{N_k} b_i^k P\psi_i^k$ converges in $X \oplus \ell_2$ in any order. Thus we have to study the convergence of the series $\sum_{k=1}^\infty \sum_{i=1}^{N_k} b_i^k (I-P)\psi_i^k = \sum_{k=1}^\infty \sum_{i=1}^{N_k} b_i^k \frac{x_k}{\sqrt{N_k}}$ in the decreasing order of $|b_i^k|$. Let us denote

$$\begin{aligned}
\Lambda_k &= \{i : N_k^{-1} < |b_i^k| < N_k^{-1/10}\} \\
\Lambda_k' &= \{i : |b_i^k| \leq N_k^{-1}\} \\
\Lambda_k'' &= \{i : |b_i^k| \geq N_k^{-1/10}\}.
\end{aligned}$$

Note that for each k we have $\sum_{i \in \Lambda_k'} |b_i^k|/\sqrt{N_k} \leq N_k \cdot 1/N_k \cdot 1/\sqrt{N_k} \leq N_k^{-1/2}$. Thus the series $\sum_{k=1}^\infty \sum_{i \in \Lambda_k'} \frac{b_i^k}{\sqrt{N_k}} x_k$ is absolutely convergent. It follows from (9) that for each k $\sum_{i=1}^{N_k} |b_i^k|^2 \leq C$ so $C \geq \sum_{i \in \Lambda_k''} |b_i^k|^2 \geq |\Lambda_k''| N_k^{-1/5}$ which gives $|\Lambda_k''| \leq C N_k^{1/5}$. From this we get

$$\sum_{k=1}^\infty \sum_{i \in \Lambda_k''} \frac{|b_i^k|}{\sqrt{N_k}} \leq \sum_{k=1}^\infty |\Lambda_k''| \frac{1}{\sqrt{N_k}} \leq \sum_{k=1}^\infty N_k^{-3/10} < \infty$$

so the series $\sum_{k=1}^\infty \sum_{i \in \Lambda_k''} \frac{b_i^k}{\sqrt{N_k}} x_k$ is absolutely convergent. Since $N_{k+1}^{-1/10} \leq N_k^{-1}$ we see that the decreasing permutation of the series $\sum_{k=1}^\infty \sum_{i \in \Lambda_k} b_i^k \psi_i^k$ has to take place inside each Λ_k. But in this case (since (ψ_k) is a basis in $X \oplus \ell_2$) the series over k converges in $X \oplus \ell_2$. From Schwartz inequality and (9) we get

$$\sum_{i \in \Lambda_k} \frac{|b_i^k|}{\sqrt{N_k}} \leq \big(\sum_{i \in \Lambda_k} |b_i^k|^2\big)^{1/2} \big(|\Lambda_k| N_k^{-1}\big)^{1/2} = o(1)$$

so we see that the series $\sum_{k=1}^{\infty}\sum_{i\in\Lambda_k}\frac{b_i^k}{\sqrt{N_k}}x_k$ converges when rearranged in the decreasing order of coefficients (b_i^k). $\square$

This proof is a modification of an argument for the main result in [5].

Let us note some corollaries from the above construction.

Corollary 4. *A separable, infinite dimensional Hilbert space has a quasi-greedy conditional basis.*

Proof. It is known (cf. e.g. [7]) that Hilbert space has a conditional basselian basis, so writing $\ell_2 = \ell_2 \oplus \ell_2$ we get the claim. $\square$

Corollary 5. *The space ℓ_p for $1 < p < \infty$ has a conditional quasi-greedy basis.*

Proof. It is well known (cf. [8]) that ℓ_p is isomorphic to $\left(\sum_{n=1}^{\infty}\ell_2^n\right)_p$. Let us fix $(\psi_j)_{j=1}^{\infty}$, a conditional greedy basis in ℓ_2 which exists by Corollary 4. Then ℓ_2^n is isometric to $\mathrm{span}(\psi_j)_{j=1}^n$ so the obvious basis in $\left(\sum_{n=1}^{\infty}\mathrm{span}(\psi_j)_{j=1}^n\right)_p$ is a conditional greedy basis in ℓ_p. $\square$

Corollary 6. *If X has an unconditional basis and contains a complemented subspace isomorphic to ℓ_p with $1 < p < \infty$ then X has a conditional greedy basis.*

Proof. We write $X \sim Y \oplus \ell_p \sim Y \oplus \ell_p \oplus \ell_p \sim X \oplus \ell_p$ so taking the unconditional basis in the first summand and a conditional greedy basis in ℓ_p (cf. Corollary 5) we get a conditional greedy basis in X. $\square$

Note that the above Corollary gives the existence of conditional greedy bases in $L_p[0,1]$ with $1 < p < \infty$ and also in H_1.

The following Theorem shows that quasi-greedy bases in a Hilbert space are rather close to unconditional bases.

Theorem 3. *Let $(x_n)_{n=1}^{\infty}$ be a normalized quasi-greedy basis in a Hilbert space H. Then there exist constants $0 < c \le C < \infty$ such that for each $x = \sum_{n=1}^{\infty} a_n x_n$ we have*

$$c\|(a_n)\|_{2,\infty} \le \|x\|_2 \le C\|(a_n)\|_{2,1} \tag{10}$$

where $\|.\|_{2,\infty}$ and $\|.\|_{2,1}$ are natural Lorentz sequence space norms. In particular such basis is p-besselian for each $p < 2$.

Proof. First observe that applying the Khintchine's inequality (or type and cotype 2 of the Hilbert space) we infer from Proposition 2 that for each finite set of indices A and each choice of signs we have $\|\sum_{n\in A} \pm x_n\| \sim \sqrt{|A|}$. Now let us denote $n_k = |\{n : |a_n| \geq 2^{-k}\}|$. Reordering the series $\sum_n a_n x_n$ so that $|a_n| \searrow 0$ we have

$$\begin{aligned} \|\sum_n a_n x_n\| &\leq 2\sum_k 2^{-k}\|\sum_{s=1}^{n_k} x_s\| \leq C\sum_k 2^{-k}\sqrt{n_k} \\ &\leq C\sum_{n=1} \frac{1}{\sqrt{n}}|a_n|. \end{aligned}$$

To prove the other inequality observe that from the Abel transform we obtain that if $\sum_{n=1}^{\infty} y_n$ converges in a Banach space X and $\sup_N \|\sum_{n=1}^{N} y_n\| = C$ and $\alpha_n \searrow 0$ then the series $\sum_{n=1}^{\infty} \alpha_n y_n$ converges and $\sup_N \|\sum_{n=1}^{\infty} \alpha_n y_n\| \leq C\alpha_1$. Now we consider the series $\sum_{n=1}^{\infty} a_n x_n$ and assume that $|a_n| \searrow 0$. Since the basis is quasi-greedy this series converges and $\sup_N \|\sum_{n=1}^{N} a_n x_n\| \leq C\|x\|$, so for each N we have $\sup_k \|\sum_{s=1}^{k} a_{N+1-s} x_{N+1-s}\| \leq 2C\|x\|$. Applying our observation to the sequence $\alpha_k = |a_N||a_{N+1-k}|^{-1}$ we get

$$\left\|\sum_{s=1}^{N} \frac{a_{N+1-s}|a_N|}{|a_{N+1-s}|} x_{N+1-s}\right\| \leq 2C\|x\|.$$

From this, using the unconditionality for constant coefficients we get

$$|a_N|\|\sum_{s=1}^{N} x_s\| \leq C\|x\|$$

which gives $\sup_n |a_n|\sqrt{n} \leq C\|x\|$ which completes the proof.

$\square$

3 Optimality

Suppose now that X is a quasi-Banach space with an unconditional basis (x_n, x_n^*) and let us assume that $\inf_{n\in F} \|x_n\| > 0$ so $\sup_{n\in F} \|x_n^*\| < \infty$. An unconditional basis is called a lattice basis if $\|\sum_n a_n x_n\| \leq \|\sum_n b_n x_n\|$ whenever $|a_n| \leq |b_n|$ for all n's. If we have an unconditional basis we can always introduce an equivalent lattice norm by

$$|||x||| = \sup\{\|\sum_n a_n x_n^*(x) x_n\| \,:\, |a_n| \leq 1\}.$$

With this norm we have $\|x\| \leq |||x||| \leq C\|x\|$ and $|||x_n||| = \|x_n\|$.

Proposition 7. *Let X be a quasi-Banach space with lattice basis (x_n, x_n^*). For each x and each $m = 1, 2, \dots$ there exists an element $T_m(x)$ of best m-term approximation i.e. $T_m(x) = \sum_{n \in A} a_n x_n$ with $|A| = m$ and $\|x - T_m(x)\| = \sigma_m(x)$.*

Proof. Let $x_k = \sum_{n \in A_k} a_n^k x_n$ with $|A_k| = m$ be such that $\|x - x_k\| \to \sigma_m(x)$. Using a standard diagonal procedure we can assume that for each n $\lim_{k\to\infty} a_n^k = a_n$. Clearly a_n's are not zero for at most m indices n. Write $x^\infty = \sum_n a_n x_n = \sum_{n \in A} a_n x_n$ where $|A| = m$. If we take B a finite set $B \supset A$ then

$$\|x - x_k\| \geq \|P_B x - P_B x_k\| \to \|P_B x - x^\infty\|.$$

Thus for each such set B we have $\sigma_m(x) \geq \|P_B x - x^\infty\| = \|P_B(x - x^\infty)\|$. Take a sequence of B's exhausting the whole index set we obtain $\sigma_m(x) \geq \|x - x^\infty\|$ so we can put $T_m(x) = x^\infty$. □

Let us recall the following quantities essentially considered in [9]:

$$\begin{aligned} e_m &= \sup_{x \in X} \frac{\|x - \mathcal{G}_m(x)\|}{\sigma_m(x)} \quad (\text{ with } \frac{0}{0} = 1) \\ \mu_m &= \sup_{k \leq m} \frac{\sup\{\|\sum_{n \in A} x_n\| \,:\, |A| = k\}}{\inf\{\|\sum_{n \in A} x_n\| \,:\, |A| = k\}} \end{aligned}$$

The importance of those quantities is clear. The sequence e_m estimates the error between the greedy algorithm $\mathcal{G}_m$ and the best possible m-term approximation. The quantity μ_m measures some sort of asymetry of the basis. The important fact is that they are closely connected.

Theorem 4. *Let (x_n, x_n^*) be a lattice basis in a quasi-Banach space X. Then for each $m = 1, 2, \dots$ we have*

$$\frac{1}{2\alpha}\mu_m \leq e_m \leq 2\alpha\mu_m. \tag{11}$$

The proof of this Theorem follows the ideas from [9].

Proof. Let us fix m and $x = \sum_n a_n x_n \in X$. Let $T_m(x) = \sum_{n\in A} b_n x_n$ be the best m-term approximation. Let $\mathcal{G}_m(x) = \sum_{n\in B} a_n x_n$. Note that

$$\begin{aligned} \|x - P_A x\| &= \|x - T_m(x) + P_A T_m(x) - P_A x\| = \|(Id - P_A)(x - T_m(x)\| \\ &\leq \|x - T_m(x)\| = \sigma_m(x) \end{aligned} \tag{12}$$

Thus we can take $T_m(x) = P_A(x)$. In order to estimate $\|x - \mathcal{G}_m(x)\|$ write

$$\begin{aligned} x - \mathcal{G}_m(x) &= x - P_A x + P_A x - P_B x = (x - P_A x) + P_{A\setminus B} x - P_{B\setminus A} x \\ &= P_{F\setminus B}(x - P_A x) + P_{A\setminus B} x \end{aligned}$$

so $\|x - \mathcal{G}_m(x)\| \leq \alpha\big(\|x - T_m(x)\| + \|P_{A\setminus B}x\|\big) \leq \alpha\big(\sigma_m(x) + \|P_{A\setminus B}x\|\big)$. Note now that $\max\{|x_n^*(x)| : n \in A \setminus B\} := c \leq \min\{|x_n^*(x)| : n \in B \setminus A\}$ and also $|A \setminus B| = |B \setminus A| \leq m$. This implies that $\|P_{A\setminus B}x\| \leq c\|\sum_{n\in A\setminus B} x_n\|$ and $\|P_{B\setminus A}x\| \geq c\|\sum_{n\in B\setminus A} x_n\|$. Thus estimating c from the second inequality and substituting it into the first we get

$$\|P_{A\setminus B}x\| \leq \frac{\|P_{B\setminus A}x\|}{\|\sum_{n\in B\setminus A} x_n\|} \cdot \|P_{A\setminus B}x\| \leq \mu_m \|P_{B\setminus A}x\| \leq \mu_m \sigma_m(x) \tag{13}$$

so we get

$$\|x - \mathcal{G}_m(x)\| \leq \alpha\sigma_m(x)(1 + \mu_m) \leq 2\alpha\mu_m\sigma_m(x).$$

In order to prove the other inequality we will need the following

Lemma 8. *For each m there exist disjoint sets A and B with $|A| = |B| \leq m$ such that $\|\sum_{n\in A} x_n\| \|\sum_{n\in B} x_n\|^{-1} \geq (2\alpha)^{-1}\mu_m$.*

Proof. If $\mu_m \leq 2\alpha$ the claim is obvious. Otherwise take sets A and B with $|A| = |B| \leq m$ such that $\|\sum_{n\in A} x_n\| \|\sum_{n\in B} x_n\|^{-1} > \max(2\alpha, \mu_m - \epsilon)$. For simplicity write

$$a = \|\sum_{n\in A} x_n\| \qquad b = \|\sum_{n\in B} x_n\|$$
$$a_1 = \|\sum_{n\in A\cap B} x_n\| \qquad a_2 = \|\sum_{n\in A\setminus B} x_n\|$$

With this notation we have $2 < \frac{1}{\alpha}\frac{a}{b} \leq \frac{1}{\alpha}\frac{a}{a_1}$ so $a_1 < \frac{1}{2\alpha}a$. This implies

$$\frac{a}{b} \leq \frac{\alpha(a_1 + a_2)}{b} = \alpha\frac{a_1}{b} + \alpha\frac{a_2}{b} < \frac{a}{2b} + \alpha\frac{a_2}{b}$$

so $\frac{a_2}{b} > \frac{1}{2\alpha}\frac{a}{b}$. Thus it suffices to replace A by any set of proper cardinality which contains $A \setminus B$ and is disjoint with B. □

Now let us take sets as in the Lemma and denote $|A| = |B| = k \leq m$. Let $C \supset A$ be a set of cardinality m disjoint with B. Consider

$$x := (1+\epsilon)\sum_{n\in B} x_n + (1+\epsilon/2)\sum_{n\in C\setminus A} x_n + \sum_{n\in A} x_n. \tag{14}$$

Then $\mathcal{G}_m(x) = x - \sum_{n\in A} x_n$ so $\|x - \mathcal{G}_m(x)\| = \|\sum_{n\in A} x_n\|$. From (12) we see that

$$\sigma_m(x) = \min\{\|P_S x\| \,:\, S \subset B\cup C \text{ and } |S| = k\} \leq \|P_B x\| \leq (1+\epsilon)\|\sum_{n\in B} x_n\|.$$

This and Lemma 8 give

$$e_m \geq \frac{\|\sum_{n\in A} x_n\|}{\sigma_m(x)} \geq \frac{\|\sum_{n\in A} x_n\|}{(1+\epsilon)\|\sum_{n\in B} x_n\|} \geq \frac{1}{(1+\epsilon)2\alpha}\mu_m.$$

Since ϵ was arbitrary we get the claim. □

Remark *Actually one can show that for x defined in (14) we have $\sigma_m(x) \sim \|\sum_{n\in B} x_n\|$. Namely $\|P_S x\| \geq \frac{1}{1+\epsilon}\|\sum_{n\in S} x_n\|$ and since $\mu_m \geq \|\sum_{n\in A} x_n\|\|\sum_{n\in S} x_n\|^{-1}$ from Lemma 8 we get*

$$\|\sum_{n\in S} x_n\| \geq \mu_m^{-1}\|\sum_{n\in A} x_n\| \geq \frac{1}{2\alpha}\frac{\|\sum_{n\in B} x_n\|}{\|\sum_{n\in A} x_n\|}\|\sum_{n\in A} x_n\| \geq \frac{1}{2\alpha}\|\sum_{n\in B} x_n\|$$

so $\sigma_m(x) \geq \frac{1}{2(1+\epsilon)\alpha}\|\sum_{n\in B} x_n\|$.

Remark *Observe that we need to have μ_m defined as $\sup_{k\leq m}$ in order to have the above estimate. As an example take $\ell_\infty^n \oplus \ell_1$ with the natural basis. Let $(f_j)_{j=1}^n$ be the basis in ℓ_∞^n and $(e_k)_{k=1}^\infty$ the basis in ℓ_1. For $m > n$ and $x := 2\sum_{j=1}^n f_j + \sum_{k=1}^m e_k$ we have $\sigma_m(x) = 2$ and $\mathcal{G}_m(x) = 2\sum_{j=1}^n f_j + \sum_{k=1}^{m-n} e_k$ so $\|x - \mathcal{G}_m(x)\| = n$ which gives $e_m \geq n$. Also $\mu_m = n$. But*

$$\xi_m := \frac{\sup\{\|\sum_{n\in A} x_n\|,\colon\ |A| = m\}}{\inf\{\|\sum_{n\in A} x_n\|,\colon\ |A| = m\}} = \frac{m}{m-n}$$

so no estimate of the form $e_m \leq C\xi_m$ is valid for all m unless $C \geq n$. But n can be arbitrary.

For general biorthogonal systems we have the following result.

Theorem 5. *Suppose $(x_n, x_n^*)_{n\in F}$ is a complete biorthogonal system in a quasi-Banach space X with $\|x_n\| = 1$ for $n \in F$. Assume that for some $0 < c \leq C$ and $0 < p \leq q \leq \infty$ we have*

$$c\Big(\sum_{n\in F} |x_n^*(x)|^q\Big)^{1/q} \leq \|x\| \leq C\Big(\sum_{n\in F} |x_n^*(x)|^p\Big)^{1/p}. \tag{15}$$

Then $e_m \leq K m^{\frac{1}{p}-\frac{1}{q}}$ where K depends only on α, C and c.

The proof is similar to the proof of Theorem 4 and Theorem 2.1 from [11].

Proof. Let us fix an $x \in X$ and $m = 1, 2, \ldots$. For any given $\epsilon > 0$ we fix almost best m–term approximation i.e. $T_m = \sum_{n\in A} b_n x_n$ such that $\|x - T_m\| \leq \sigma_m(x) + \epsilon$. First note that for any finite subset $V \subset F$ we have

$$\|P_V x\| \leq C\Big(\sum_{n\in V} |x_n^*(x)|^p\Big)^{1/p} \leq C|V|^{\frac{1}{p}-\frac{1}{q}}\Big(\sum_{n\in V} |x_n^*(x)|^q\Big)^{1/q} \leq \frac{C}{c}|V|^{\frac{1}{p}-\frac{1}{q}}\|x\|. \tag{16}$$

Let $\mathcal{G}_m(x) = \sum_{n\in B} a_n x_n$. We write

$$\|x - \mathcal{G}_m(x)\| = \|x - P_A x + P_A x - \mathcal{G}_m(x)\| \tag{17}$$
$$\leq \alpha(\|x - P_A x\| + \|P_A x - \mathcal{G}_m(x)\|). \tag{18}$$

The first summand is estimated as

$$\|x - P_A x\| \leq \alpha(\|x - T_m\| + \|P_A(x - T_m)\|) \leq \alpha(\sigma_m(x) + \epsilon + \|P_A\|(\sigma_m(x) + \epsilon))$$

so

$$\|x - P_A x\| \leq \alpha\big(\sigma_m(x) + \epsilon + \tfrac{C}{c}|A|^{\frac{1}{p}-\frac{1}{q}}(\sigma_m(x) + \epsilon)\big). \tag{19}$$

The second summand we write as

$$\|P_A(x) - P_B x\| \leq \alpha\big(\|P_{A\setminus B} x\| + \|P_{B\setminus A} x\|\big)$$

and obtain

$$\|P_{B\setminus A} x\| = \|P_{B\setminus A}(x - T_m)\| \leq \tfrac{C}{c}|B \setminus A|^{\frac{1}{p}-\frac{1}{q}}(\sigma_m(x) + \epsilon). \tag{20}$$

To estimate the other summand we note that $|B\setminus A| = |A\setminus B|$ and $|x_n^*(x)| \geq |x_s^*(x)|$ whenever $n \in B\setminus A$ and $s \in A\setminus B$. Thus

$$\begin{aligned}
\|P_{A\setminus B}x\| &\leq C\Big(\sum_{n\in A\setminus B} |x_n^*(x)|^p\Big)^{1/p} \leq \Big(\sum_{n\in B\setminus A} |x_n^*(x)|^p\Big)^{1/p} \\
&\leq C|B\setminus A|^{\frac{1}{p}-\frac{1}{q}}\Big(\sum_{n\in B\setminus A} |x_n^*(x)|^q\Big)^{1/q} \\
&= C|B\setminus A|^{\frac{1}{p}-\frac{1}{q}}\Big(\sum_{n\in B\setminus A} |x_n^*(x-T_m)|^q\Big)^{1/q} \\
&\leq \tfrac{C}{c}|B\setminus A|^{\frac{1}{p}-\frac{1}{q}}\|x-T_m\| \leq \tfrac{C}{c}|B\setminus A|^{\frac{1}{p}-\frac{1}{q}}(\sigma_m(x)+\epsilon).
\end{aligned} \tag{21}$$

Since ϵ was arbitrary and $|B\setminus A| \leq m = |A|$ from (19), (20) and (21) we obtain $\|x - \mathcal{G}_m(x)\| \leq K(C,c,\alpha)\sigma_m(x)\cdot m^{\frac{1}{p}-\frac{1}{q}}$. □

Remark *Using Theorem 3 and arguing like in the above proof we can get that for each quasi-greedy basis in a Hilbert space we have $e_m \leq C\ln(m+1)$.*

Now we will list some immediate consequences of Theorem 5.

Corollaries:

a) If $(x_n)_{n=1}^\infty$ is a complete, uniformly bounded orthonormal system (in particular the trigonometric system) then in $L_p[0,1]$ with $1\leq p\leq\infty$ we have $e_m \leq Km^{|\frac{1}{2}-\frac{1}{p}|}$. This follows immediately from F. Riesz inequality which says that for $2\leq p\leq\infty$ we have

$$\Big(\sum_{n=1}^\infty |\langle x_n, f\rangle|^2\Big)^{1/2} \leq \|f\|_p \leq M\Big(\sum_{n=1}^\infty |\langle x_n, f\rangle|^{p'}\Big)^{1/p'}$$

where $\frac{1}{p}+\frac{1}{p'}=1$, and also from the dual inequality valid for $1\leq p\leq 2$. This is an optimal inequality as was shown for the trigonometric system in [11] Remark 2. For $p>1$ it also follows from Remark 2.

b) Since for any semi-normalized biorthogonal system (x_n, x_n^*) in a Banach space X we have

$$c\sup_n |x_n^*(x)| \leq \|x\| \leq C\sum_n |x_n^*(x)|$$

we infer that for each such system we have $e_m \leq Cm$. This estimate is also optimal, even for unconditional bases. One easily checks that for the natural unconditional basis in $\ell_1 \oplus c_0$ one has $e_m \geq m$.

c) In each super-reflexive space, in particular in $L_p[0,1]$ with $1 < p < \infty$, each semi-normalized basis (x_n, x_n^*) satisfies equation (15) for some $1 < q \leq p < \infty$ (see [1]). So we obtain that for each semi-normalized basis in a super-reflexive space we have $e_m \leq Km^\beta$ with $\beta < 1$.

d) If (x_n, x_n^*) is a semi-normalized unconditional basis in L_p with $1 < p < \infty$, then for $p \geq 2$ it satisfies

$$c\Big(\sum_{n\in F} |x_n^*(x)|^p\Big)^{1/p} \leq \|x\|_p \leq C\Big(\sum_{n\in F} |x_n^*(x)|^2\Big)^{1/2}.$$

and the dual inequality for $1 < p \leq 2$. Thus for an unconditional basis in L_p we have $e_m \leq Km^{|\frac{1}{2}-\frac{1}{p}|}$. Also this estimate is optimal. To see it consider L_p as being isomorphic to $\ell_2 \oplus L_p$ and take the natural basis in ℓ_2 and the Haar basis in L_p.

4 Multiple Haar system

In this section we will discuss the efficiency of the greedy algorithm with respect to the multi-dimensional Haar wavelet. For more detailed exposition of general background sketched below the reader may consult [13]. We will argue in the context of the square function for $0 < p < \infty$. To start we define

$$H(t) = \begin{cases} 1 & \text{if } t \in [0, \frac{1}{2}), \\ -1 & \text{if } t \in [\frac{1}{2}, 1), \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

For a dyadic interval $I = [k2^{-n}, (k+1)2^{-n})$ we put $h_I(t) = 2^{n/p}H(2^n t - k)$. For a dyadic rectangle in $J = I_1 \times \cdots \times I_d \subset \mathbb{R}^d$ we put

$$h_J^{(d)}(t) = h_{I_1}(t_1) \cdot \cdots \cdot h_{I_d}(t_d). \tag{23}$$

The set of all dyadic intervals in $\mathbb{R}$ will be denoted by $\mathcal{D}(1)$ and the set of all dyadic rectangles in $\mathbb{R}^d$ will be denoted by $\mathcal{D}(d)$. The system $(h_i^{(d)})_{I\in\mathcal{D}(d)}$

is a complete orthogonal system in $L_2(\mathbb{R}^d)$ and is normalized in $L_p(\mathbb{R}^d)$. Note that formally the definition of this system depends on p but since p will be fixed in our future arguments we will not indicate this dependence explicitely.

A function $f = \sum_{I\in\mathcal{D}(d)} a_I h_I^{(d)}$ is in $H_p(\mathbb{R}^d)$ if the norm

$$\|f\| = \Big(\int_{\mathbb{R}^d} \big(\sum_{I\in\mathcal{D}(d)} |a_I h_I^{(d)}(t)|^2\big)^{p/2}\, dt\Big)^{1/p} \tag{24}$$

is finite. It is known by the Littlewood–Paley theory that for $1 < p < \infty$ this norm is equivalent to the usual L_p norm and we have $H_p(\mathbb{R}^d) = L_p(\mathbb{R}^d)$. For $0 < p \leq 1$ we get the dyadic H_p–space.

The main result of this section is the following

Theorem 6. *For $0 < p < \infty$ and $d = 1, 2, \ldots$ for the system $(h_I^{(d)})_{I\in\mathcal{D}(d)}$ in $H_p(\mathbb{R}^d)$ we have*

$$e_m \sim (\log m)^{(d-1)|\frac{1}{2}-\frac{1}{p}|}. \tag{25}$$

This result substantiates the conjecture formulated (for $p > 1$) in [9]. Our argument is a modification of the argument from [2]. Let us start with a Lemma.

Lemma 9. *For $0 < p < \infty$ and any finite subset $B \subset \mathcal{D}(1)$ we have*

$$2^{-\frac{1}{p}}|B|^{\frac{1}{p}} \leq \|\sum_{I\in B} h_I\|.$$

Proof. Let us denote $2^{M(t)} = \max_{I\in B} |h_I(t)|^p$. From the definition of the Haar system we infer that $2^{M(t)} \geq \frac{1}{2}\sum_{I\in B} |h_I(t)|^p$ so

$$\|\sum_{I\in B} h_I\| \geq \Big(\int_{\mathbb{R}} 2^{M(t)} dt\Big)^{1/p} \geq \Big(\frac{1}{2}\int_{\mathbb{R}} \sum_{I\in B} |h_I(t)^p\, dt\Big)^{1/p} = 2^{-\frac{1}{p}}|B|^{1/p}.$$

$\square$

Proposition 10. *For any $d = 1, 2, \ldots,$ any finite $B \subset \mathcal{D}(d)$, $|B| = m$ and any numbers $(a_I)_{I\in B}$ we have*

a) if $0 < p \leq 2$ then

$$(\log m)^{(\frac{1}{2}-\frac{1}{p})d}\Big(\sum_{I\in B} |a_I|^p\Big)^{1/p} \leq \|\sum_{I\in B} a_I h_I^{(d)}\| \leq \Big(\sum_{I\in B} |a_I|^p\Big)^{1/p} \tag{26}$$

b) if $2 \le p < \infty$ then

$$\Big(\sum_{I\in B}|a_I|^p\Big)^{1/p} \le \|\!|\sum_{I\in B} a_I h_I^{(d)}\|\!| \le (\log m)^{(\frac{1}{p}-\frac{1}{2})d}\Big(\sum_{I\in B}|a_I|^p\Big)^{1/p} \tag{27}$$

Proof. The right hand side inequality in (26) is easy (it is actually the type of H_p). We simply apply the Hölder's inequality with exponent $\frac{2}{p}\ge 1$ to the inside sum and we get

$$\Big(\int_{\mathbb{R}^d}\big(\sum_{I\in B}|a_I h_I^{(d)}(t)|^2\big)^{p/2}dt\Big)^{1/p} \le \Big(\int_{\mathbb{R}^d}\sum_{I\in B}|a_I h_I^{(d)}(t)|^p\,dt\Big)^{1/p} = \Big(\sum_{I\in B}|a_I|^p\Big)^{1/p}.$$

Now let $d=1$ and $0<p\le 2$. Let $\sigma:\{1,2,\ldots,|B|\}\longrightarrow B$ be such that $|a_{\sigma(i)}|$ is a decreasing sequence. Fix s such that $2^{s-1}<m\le 2^s$ and put $f_k=\big(\sum_{j=2^{k-1}+1}^{2^k}|a_{\sigma(j)}h_{\sigma(j)}|^2\big)^{1/2}$. Then

$$\begin{aligned}
\|\!|\sum_{I\in B}a_I h_I\|\!| &= \Big(\int_{\mathbb{R}}\big(\sum_{k=0}^{s}f_k^2(t)\big)^{p/2}\,dt\Big)^{1/p} = \Big(\int_{\mathbb{R}}\big(\sum_{k=0}^{s}(f_k^p(t))^{2/p}\big)^{p/2}\,dt\Big)^{1/p}\\
&\ge \Big(\big(\sum_{k=0}^{s}\big(\int_{\mathbb{R}}f_k^p(t)\,dt\big)^{2/p}\big)^{p/2}\Big)^{1/p}\\
&= \Big(\sum_{k=0}^{s}\|\!|\sum_{j=2^{k-1}+1}^{2^k}a_{\sigma(j)}h_{\sigma(j)}\|\!|^2\Big)^{1/2}\\
&\ge \Big(\sum_{k=0}^{s}\|\!|\sum_{j=2^{k-1}+1}^{2^k}a_{\sigma(2^k)}h_{\sigma(j)}\|\!|^2\Big)^{1/2}\\
&\quad\text{and from Lemma 9}\\
&\ge \Big(\sum_{k=0}^{s}2^{\frac{2(k-1)}{p}}|a_{\sigma(2^k)}|^2\Big)^{1/2}.
\end{aligned}$$

Since

$$\sum_{I\in B}|a_I|^p = \sum_{j=1}^{|B|}|a_{\sigma(j)}|^p \le \sum_{k=0}^{s}2^k|a_{\sigma(2^k)}|^p \le s^{1-\frac{p}{2}}\Big(\sum_{k=0}^{s}2^{\frac{2k}{p}}|a_{\sigma(2^k)}|^2\Big)^{p/2}$$

we get

$$\|\!|\sum_{I\in B}a_I h_I\|\!| \ge 2^{-\frac{1}{p}}(\log m)^{-(1-\frac{p}{2})\frac{1}{p}}\Big(\sum_{I\in B}|a_I|^p\Big)^{1/p} = 2^{-\frac{1}{p}}(\log m)^{\frac{1}{2}-\frac{1}{p}}\Big(\sum_{I\in B}|a_I|^p\Big)^{1/p}.$$

Now we will prove the left hand side inequality in (26) by induction on d. Suppose we have (26) valid for $d-1$. Given a finite set $B \subset \mathcal{D}(d)$ we write each $I \in B$ as $I = J \times K$ with $J \in \mathcal{D}(1)$ and $K \in \mathcal{D}(d-1)$ and then $h_I^{(d)}(t) = h_J(t_1) \cdot h_K^{(d-1)}(\xi)$ where $\xi = (t_2, \ldots, t_d)$. Now we estimate

$$\begin{aligned} \|\sum_{I\in B} a_I h_I^{(d)}\| &= \Big(\int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \big(\sum_{I\in B} |a_I h_J(t_1)|^2 \, |h_K^{(d-1)}(\xi)|^2\big)^{p/2} \, dt_1 \, d\xi\Big)^{1/p} \\ &= \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}^{d-1}} \big(\sum_K \big(\sum_J |a_I h_J(t_1)|^2\big) |h_K^{(d-1)}(\xi)|^2\big)^{p/2} d\xi\Big) dt_1\Big)^{1/p} \end{aligned} \tag{28}$$

For each t_1 we apply the inductive hypothesis (note that the number of different K's is at most $|B|$) and we continue the estimates

$$\begin{aligned} &\geq C(d-1,p)(\log|B|)^{(d-1)(\frac{1}{2}-\frac{1}{p})}\Big(\int_{\mathbb{R}} \sum_K \big(\sum_J |a_I h_J(t_1)|^2\big)^{p/2} \, dt_1\Big)^{1/p} \\ &\geq C(d-1,p)(\log|B|)^{(d-1)(\frac{1}{2}-\frac{1}{p})}\Big(\sum_K \int_{\mathbb{R}} \big(\sum_J |a_I h_J(t_1)|^2\big)^{p/2} dt_1\Big)^{1/p} \end{aligned} \tag{29}$$

Now we apply the estimate (26) for $d = 1$ and we continue as

$$\begin{aligned} &\geq C(d-1,p)(\log|B|)^{(d-1)(\frac{1}{2}-\frac{1}{p})}\big(\sum_K \sum_J |a_I|^p\big)^{1/p} C(1,p)(\log|B|)^{(\frac{1}{2}-\frac{1}{p})} \\ &= C(d,p)(\log|B|)^{d(\frac{1}{2}-\frac{1}{p})}\big(\sum_{I\in B} |a_I|^p\big)^{1/p}. \end{aligned} \tag{30}$$

The inequality (27) follows by duality from (26) for $1 < p \leq 2$. □

Proposition 11. *For every finite set $B \subset \mathcal{D}(d)$ we have*

a) if $0 < p \leq 2$ then

$$C(d,p)(\log|B|)^{(\frac{1}{2}-\frac{1}{p})(d-1)}|B|^{1/p} \leq \|\sum_{I\in B} h_I^{(d)}\| \leq |B|^{1/p} \tag{31}$$

b) if $2 \leq p < \infty$ then

$$|B|^{1/p} \leq \|\sum_{I\in B} h_I^{(d)}\| \leq C(d,p)(\log|B|)^{(\frac{1}{2}-\frac{1}{p})(d-1)}|B|^{1/p} \tag{32}$$

Proof. As in the previous Proposition inequality (32) follows by duality from (31). Note also that (31) for $d = 1$ is Lemma 9. For $d > 1$ we proceed like in the proof of Proposition 10. We write each $I \in B$ as $J \times K$ and estimate $\| \sum_{I \in B} h_I^{(d)} \|$ exactly like in (28) and (29). Since $a_I = 1$ instead of (26) for $d = 1$ we apply Lemma 9 and we obtain

$$\| \sum_{I \in B} h_I^{(d)} \| \geq C(d-1, p)(\log |B|)^{(\frac{1}{2} - \frac{1}{p})(d-1)} 2^{-\frac{1}{p}} |B|^{1/p}.$$

□

Proof of Theorem 6. The estimate $e_m \leq C(\log m)^{|\frac{1}{2} - \frac{1}{p}|(d-1)}$ follows immediately from Theorem 4 and Proposition 11. The estimate from below was proved in [9].

Theorem 6 covers and extends main results about the Haar system proved in [9] and [10]. In particular it gives a new proof that the Haar wavelet is a greedy basis in $L_p(\mathbb{R})$. One can note that the Haar system is not the only such basis in L_p for $2 < p < \infty$. Let us recall the definition of the Rosenthal space (cf. [6] p.169). Fix $0 < \beta \leq 1$ and define a norm on sequences $(a_n)_{n=1}^{\infty}$ as

$$\|(a_n)\|^{\beta} = \big(\sum_{n=1}^{\infty} |a_n|^p \big)^{1/p} + \big(\sum_{n=1}^{\infty} |a_n w_n^{\beta}|^2 \big)^{1/2}$$

with $w_n^{\beta} = n^{-\beta(p-2)/2p}$. It is known that for all β's we get the same space X (called Rosenthal space) and that $X \oplus L_p$ is isomorphic to L_p. However for different β's we get different unconditional bases in X. If $(e_n)_{n=1}^{\infty}$ denotes the unit vector basis then for each finite set $A \subset \mathbb{N}$ we have

$$\begin{aligned} \| \sum_{n \in A} e_n \|^{\beta} &= |A|^{1/p} + \big(\sum_{n \in A} |w_n^{\beta}|^2 \big)^{1/2} \\ &\leq |A|^{1/p} + \big(\sum_{n=1}^{|A|} |w_n^{\beta}|^2 \big)^{1/2} \\ &\leq |A|^{1/p} + C|A|^{\frac{1-\beta}{2} - \frac{\beta}{p}} \end{aligned}$$

For $\beta = 1$ we get $\| \sum_{n \in A} e_n \|^1 \sim |A|^{1/p}$ so this basis and the Haar basis give an unconditional greedy basis in L_p which is not equivalent to the Haar basis.

For $0 < \beta < 1$ we get $\| \sum_{n=1}^{N} e_n \|^{\beta} \sim N^{\frac{1-\beta}{2} + \frac{\beta}{p}}$ so for this basis in X and the Haar basis in L_p we get an unconditional basis in L_p with

$$\mu_m \sim m^{\frac{1-\beta}{2} + \frac{\beta}{p}} m^{-1/p} \sim m^{(1-\beta)(\frac{1}{2} - \frac{1}{p})}$$

so we get all possible power type behaviours of e_m (c.f. Corollary d) after Theorem 5).

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