On weak non-equivalence of wavelet–like systems in L_1†

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Abstract

We show that in $L_1(\mathbb{R})$ the Haar wavelet basis is not equivalent to any permutation with any signs of the Strömberg wavelet basis. We also construct a Haar-type system in $L_1[0,1]$ which is not equivalent to any subsequence with signs of the classical Haar basis.

1. *Introduction*

Equivalence of bases is a rich subject in the theory of Banach (and other) spaces, see e.g. [**3**] for the review of the subject. In this paper we introduce a new kind of equivalence

DEFINITION 1. *Suppose that $\{x_n\}_{n\in N}$ and $\{y_m\}_{m\in M}$ are two systems of vectors in Banach spaces X and Y respectively. We say that those systems are* weakly equivalent *if there exists an 1-1 and onto map $\pi : N \to M$ and a sequence of signs $\{\eta_n\}_{n\in N}$ and constants $0 < c \le C < \infty$ such that for every (finitely non-zero) sequence of scalars $\{a_n\}_{n\in N}$ we have*

$$c\left\|\sum_{n\in N} a_n y_{\pi(n)}\right\| \le \left\|\sum_{n\in N} a_n \eta_n x_n\right\| \le C\left\|\sum_{n\in N} a_n y_{\pi(n)}\right\|. \qquad (1\cdot 1)$$

Despite apparent nonsymetry this weak equivalence is really an equivalence relation. If systems $\{x_n\}_{n\in N}$ and $\{y_m\}_{m\in M}$ are unconditional bases this is simply permutative equivalence. If $N = M$, π is the identity and $\eta_n = 1$ we get the equivalence. For more information about those notions see e.g. [**3**]. The need for such a concept arose in the subject of m–term approximation [**8**] and [**12**]. This is a study of approximation from

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sets

$$\Sigma_n = \left\{ \sum_{n\in\Lambda} a_n x_n \ : \quad \Lambda \subset N \text{ has } n \text{ elements, } a_n\text{'s are arbitrary scalars} \right\}. \tag{1·2}$$

It is clear that if $\{x_n\}_{n\in N}$ and $\{y_m\}_{m\in M}$ are weakly equivalent the corresponding sets Σ_n have the same properties in spaces X and Y. This was used in the work of V.N. Temlyakov [**9**] when he showed the good properties of the thresholding algorithm with respect to the Haar system in L_p, $1 < p < \infty$ and used known results about equivalence to get the same results for very general wavelet systems. For the importance of this type of results in numerical analysis see [**2**].

Our results deal with weak equivalence in L_1 norm. It is generally the case that results which hold for $1 < p < \infty$ fail for $p = 1$. In particular it is well known that the good wavelet bases, in particular Strömberg system, are equivalent in $L_p(\mathbb{R})$ with $1 < p < \infty$ to the Haar wavelet basis (see e.g. [**10**], [**11**]). It is also known that many of those results fail for $L_1(\mathbb{R})$. It was proved by Sjölin [**6**] that the Franklin system and the Haar system are not equivalent in $L_1[0, 1]$. Analogous result was shown by P. Bechler for Strömberg and other wavelet system [**1**]. In Theorem 2 we show that Haar wavelet basis and Strömberg wavelet basis are not weakly equivalent in $L_1(\mathbb{R})$. Theorem 3 gives an example of a Haar-type system which is not weakly equivalent in $L_1[0, 1]$ norm to any subsystem of the Haar system. It was shown by A. Kamont [**4**] that any Haar-type system in $L_p[0, 1]$ for $1 < p < \infty$ is equivalent to a subsequence of the Haar system.

2. *Computation of some integrals*

Strömberg wavelet was introduced in [**7**] as an analog on $\mathbb{R}$ of the classical Franklin system. Let us recall the definition of the Strömberg wavelet see e.g. [**7**], [**10**]. Let

$$A_1 = \{\frac{k}{2} \text{ with } k = 1, 0, -1, -2, \dots\} \cup \{k \text{ with } k = 1, 2, \dots\}$$

and

$$A_0 = \{\frac{k}{2} \text{ with } k = 0, -1, -2, \dots\} \cup \{k \text{ with } k = 1, 2, \dots\}.$$

By $\mathcal{S}(A_i)$ we denote the set of all square integrable, continuous, piecewise–linear functions on $\mathbb{R}$ with nodes in A_i. *The Strömberg wavelet* is the function $S \in \mathcal{S}(A_1)$ which is orthogonal to $\mathcal{S}(A_0)$. To fix the normalisation we choose $S(1) = 1$. The explicite values of the Strömberg wavelet are as follows:

$$S(k) = (\sqrt{3} - 2)^{k-1} \text{ for } k = 1, 2, 3, \dots \tag{2·1}$$

$$S(\frac{1}{2}) = -(\sqrt{3} + \frac{1}{2}) \tag{2·2}$$

$$S(-\frac{k}{2}) = (2\sqrt{3} - 2)(\sqrt{3} - 2)^k \text{ for } k = 0, 1, 2, \dots \tag{2·3}$$

We introduce the function $F(x)$ as a continuous, piecewise linear function on $\mathbb{R}$ given by the conditions

$$F(x) = \begin{cases} 0 & \text{if } x \le 0 \text{ or } x \ge 1 \\ 1 & \text{if } x = \frac{1}{2} \\ -\frac{1}{2} & \text{if } x = \frac{1}{4} \text{ or } x = \frac{3}{4}. \end{cases} \tag{2·4}$$

One easily checks that F is orthogonal to all functions linear on the interval $[0,1]$.

For a dyadic interval $I = (\frac{k}{2^j}, \frac{k+1}{2^j})$ with $k, j \in \mathbb{Z}$ we will denote

$$S_I(x) = S_{j,k}(x) = 2^j S(2^j x - k).$$

It is well known that $(S_{j,k})_{j,k\in\mathbb{Z}}$ is an orthogonal basis in $L_2(\mathbb{R})$ and one easily checks that $\|S_{j,k}\|_1 = \|S\|_1$ for all $j,k \in \mathbb{Z}$, so we normalise dilates and translates of S in L_1.

We will denote $F_I(x) = F_{j,k}(x) = F(2^j x - k)$, so dilates and translates of F are normalised in L_∞.

We need to calculate scalar products $\int_{\mathbb{R}} F_I(x) S_J(x)\, dx$. Let us start with the following

LEMMA 1. *Let $a < b$ and let f and g be linear functions on $[a,b]$. Then*

$$\int_a^b f(x)g(x)\, dx = \frac{b-a}{3}\big(f(a)g(a) + f(b)g(b) + \frac{1}{2}f(a)g(b) + \frac{1}{2}f(b)g(a)\big). \tag{2·5}$$

Proof. Straightforward calculation. □

PROPOSITION 1. *If $j < 0$ or $j \geq 2$ the integral $\int_{-\infty}^{\infty} FS_{j,k} = 0$ for every $k \in \mathbb{Z}$. In the remaining cases we have:*

$$\int_{-\infty}^{\infty} FS_{0,0} = \int_{-\infty}^{\infty} FS_{[0,1]} = -\frac{1}{2\sqrt{3}} \tag{2·6}$$

$$\int_{-\infty}^{\infty} FS_{0,k} = \int_{-\infty}^{\infty} FS_{[k,k+1]} = 0 \textit{ for } k = -1, -2, \dots \tag{2·7}$$

$$\int_{-\infty}^{\infty} FS_{0,k} = \int_{-\infty}^{\infty} FS_{[k,k+1]} = \frac{1}{2}(7 - 4\sqrt{3})^{k-1}(5 - 3\sqrt{3}) \textit{ for } k = 1, 2, \dots \tag{2·8}$$

$$\int_{-\infty}^{\infty} FS_{1,k} = \int_{-\infty}^{\infty} FS_{[\frac{k}{2}, \frac{k+1}{2}]} = 0 \textit{ for } k = 2, 3, \dots \tag{2·9}$$

$$\int_{-\infty}^{\infty} FS_{1,-k} = \int_{-\infty}^{\infty} FS_{[\frac{-k}{2}, \frac{-k+1}{2}]} = \frac{1}{2}(\sqrt{3} - 2)^k \textit{ for } k = 1, 2, \dots \tag{2·10}$$

REMARK: In the above Proposition we omit cases (1,0) and (1,1) because we will not use them. Also note that in (2·8) all the integrals are negative while in (2·10) integrals change signs.

Proof. All claims that $\int_{-\infty}^{\infty} FS_{j,k} = 0$ follow by standard change of variables either from the fact that F is orthogonal to linear functions on $[0,1]$ or from the definition of Strömberg wavelet. The remaining claims are a matter of routine but slightly tedious calculations. Since the exact numerical values are important to us we will provide at least an outline.

To get (2·6) let $g(x)$ be a linear function such that $g(1) = S(1)$ and $g(\frac{1}{2}) = S(\frac{1}{2})$. Then we easily get that $S - g$ is continuous, equals 0 on $[\frac{1}{2}, 1]$, is linear on $[0, \frac{1}{2}]$ and $(S-g)(0) = 4\sqrt{3}$. So $(S-g)(\frac{1}{4}) = 2\sqrt{3}$. Because F is orthogonal to linear functions we get $\int_0^1 S(x)F(x)\, dx = \int_0^1 (S-g)(x)F(x)\, dx$. Using Lemma 1 twice we get

$$\int_0^{\frac{1}{4}} F(x)(S-g)(x)\, dx = -\frac{1}{6}\sqrt{3}$$

$$\int_{\frac{1}{4}}^{\frac{1}{2}} F(x)(S-g)(x)\, dx = 0.$$

Summing this we get (2·6).

To get (2·8) we note from (2·3) that

$$\int_0^1 f(x)S(x-k)\,dx = (\sqrt{3}-2)^{2k-2}\int_0^1 F(x)S(x-1)\,dx \qquad (2{\cdot}11)$$

so we need to compute one integral only. Let $g(x)$ be a linear function such that

$$g(1) = S(0) = 2\sqrt{3}-2 \quad \text{and} \quad g(\frac{1}{2}) = S(-\frac{1}{2}) = (2\sqrt{3}-2)(\sqrt{3}-2). \qquad (2{\cdot}12)$$

As in the previous case we have $\int_0^1 F(x)S(x-1)\,dx = \int_0^1 F(x)\big(S(x-1)-g(x)\big)\,dx$, so using Lemma 1 twice we get

$$\int_0^{\frac{1}{4}} F(x)\big(S(x-1)-g(x)\big)\,dx = -(2\sqrt{3}-2)\frac{1}{4}(2-\sqrt{3})$$
$$\int_{\frac{1}{4}}^{\frac{1}{2}} F(x)\big(S(x-1)-g(x)\big)\,dx = 0.$$

Simplifying slightly and using (2·11) we get (2·8).

To get (2·10) we write explicitely

$$\int_0^1 F(x)S_{1,-k}(x)\,dx = 2\int_0^1 F(x)S(2x+k)\,dx$$
$$= 2(\sqrt{3}-2)^{k-1}\int_0^1 F(x)S(2x+1)\,dx.$$

From (2·1) and (2·3) and using $f(x) = F(1-x)$ we see that up to a factor $(2\sqrt{3}-2)$ the last integral above reduces to the integral $\int_0^1 F(x)S(x-1)\,dx$ considered above. This gives (2·10). □

3. *Strömberg system*

Let $\mathcal{D}$ denotes the set of all dyadic subintervals of $\mathbb{R}$. It is partially ordered by inclusion. For a set $\Lambda \subset \mathcal{D}$ and an interval $I \in \Lambda$ we introduce the following notation

$$\max(I,\Lambda) = \{J \in \Lambda \;:\; J \subset I,\; J \neq I \text{ and if } J \subset K \subset I \text{ with } K \in \Lambda$$
$$\text{then either } K = J \text{ or } K = I\}.$$

If the set Λ is clear we will write $\max(I)$ instead of $\max(I,\Lambda)$. We will call elements of $\max(I)$ *sons* of I in Λ; if the set Λ is clear simply sons of I. We will divide intervals from Λ into three types as follows

(i) $I \in \Lambda$ is of type I if $\max(I)$ is empty
(ii) $I \in \Lambda$ is of type II if $\max(I)$ consists of exactly one element
(iii) $I \in \Lambda$ is of type III if $\max(I)$ consists of more than one element.

$\Lambda_1,\ \Lambda_2,\ \Lambda_3$ will denote the set of intervals from Λ of the corresponding type. Also Λ' will denote the set of all intervals in Λ which are not sons of any interval from Λ.

Let us start with the following

LEMMA 2. *If Λ is finite, then*
(i) $|\Lambda'| \leq |\Lambda_1|$
(ii) $|\Lambda_3| < |\Lambda_1|$
(iii) $\sum_{I\in\Lambda_3} |\max(I)| < 2|\Lambda_1|$

Proof. Claim 1. is clear since each element from Λ' contains at least one element from Λ_1. To see 2. and 3. let us look at $M = \Lambda_1 \cup \Lambda_3$. We observe that M has no intervals of type II and that Λ_1 consists of intervals of type I and Λ_3 consists of intervals of type III (*all with respect to* M). Also for $I \in \Lambda_3$ the number of sons in M and in Λ is the same. To get 2. we use induction on the cardinality of M. If we have M with $N+1$ elements we fix an interval $I \in M$ which is not a son of any interval from M (in M) and apply the inductive hypothesis to each of the sets $M_K = \{J \in M : J \subset K\}$ where $K \in \max(I)$ and $M \setminus \left(\bigcup_{K\in\max(I)} M_K \cup \{I\}\right)$. Now 3. follows from 2. because

$$\sum_{I\in\Lambda_3} |\max(I)| < |M| = |\Lambda_1| + |\Lambda_3| < 2|\Lambda_1|.$$

□

For $I \in \Lambda_2$ let $K(I)$ be the (unique) element of $\max(I)$. Let us state the following obvious observation.

Lemma 3. *Sets $I \setminus K(I)$ for $I \in \Lambda_2$ are disjoint and sets $I \in \Lambda_1$ are disjoint.*

First of our main results is the following

Theorem 1. *There exists a constant $c > 0$ such that for all finite $\Lambda \subset \mathcal{D}$*

$$\Big\|\sum_{I\in\Lambda} S_I\Big\|_1 \geq c|\Lambda|.$$

Using the definition introduced in [**5**] we may say that the Strömberg wavelet basis is democratic in $L_1(\mathbb{R})$.

Proof. In what follows we fix the set Λ and we will denote $G = \sum_{J\in\Lambda} S_J(x)$. Let $a > 0$ denote a small constant to be fixed later. We will split the proof into three cases:
Case 1: $|\Lambda_1| \geq a|\Lambda|$.
Then for $I \in \Lambda_1$ we look at

$$\int_{\mathbb{R}} \big(\sum_{J\in\Lambda} S_J(x)\big)F_I(x)\,dx = \sum_{J\in\Lambda}\int_{\mathbb{R}} S_J(x)F_I(x)\,dx. \tag{3·1}$$

By a change of variables we can assume that $I = [0,1]$. Then from Proposition 1 we see that we get nonzero integrals only from J's (if any) of length 1 and lengths $\frac{1}{2}$. $J = I$ gives us negative integral $-\frac{1}{2\sqrt{3}}$ and from (2·7) and (2·8 we infer that

$$\sum_{J\in\Lambda\,:\,|J|=1}\int_{\mathbb{R}} S_J(x)F_I(x)\,dx \leq -\frac{1}{2\sqrt{3}}. \tag{3·2}$$

In the $\sum_{J\in\Lambda\,:\,|J|=\frac{1}{2}}\int_{\mathbb{R}} S_J(x)F_I(x)\,dx$ we may have some positive and some negative summanda but from (2·10) we see that positive summands add up to at most

$$\frac{1}{2}\sum_{k=1}^{\infty}(\sqrt{3}-2)^{2k} = \frac{7-4\sqrt{3}}{8\sqrt{3}-12} = \frac{1}{12}(2\sqrt{3}-3). \tag{3·3}$$

Comparing (3·2) with (3·3) we see that

$$\int_{\mathbb{R}} \big(\sum_{J\in\Lambda} S_J(x)\big)F_I(x)\,dx \leq -\frac{1}{2\sqrt{3}} + \frac{1}{12}(2\sqrt{3}-3) = -\frac{1}{4}. \tag{3·4}$$

Since intervals in Λ_1 are disjoint the function $H = \sum_{I\in\Lambda_1} F_I$ is uniformly bounded by 1 and from(3·1) and (3·4) we infer that

$$\left|\int_{\mathbb{R}} \big(\sum_{J\in\Lambda} S_J(x)\big)H(x)\,dx\right| \geq \frac{1}{4}|\Lambda_1| \geq \frac{a}{4}|\Lambda|$$

which implies

$$\left\|\sum_{J\in\Lambda} S_J(x)\right\|_1 \geq \frac{a}{4}|\Lambda|. \tag{3·5}$$

Case 2: The number of intervals in Λ_2 such that $K(I)$ is not adjacent to the left end of I is greater than $a|\Lambda|$.

Denote this set as Λ_2^r.

In such a case let $J(I)$ denotes the dyadic interval such that $|K(I)| = |J(I)|$ just to the left of $K(I)$. Clearly $J(I) \subset I$. We define $H(x) = \sum_{I\in\Lambda_2^r} F_{J(I)}(x)$. From Lemma 3 we infer that all intervals $J(I)$ in this sum are disjoint, so $\|H\|_\infty = 1$. Like in the previous case we look first at the sum

$$\sum_{J\in\Lambda}\int_{\mathbb{R}} S_J(x)F_{J(I)}(x)\,dx = \int_{\mathbb{R}} G(x)F_{J(I)}(x)\,dx. \tag{3·6}$$

By the change of variables we can assume that $|J(I)| = 1$ and we see from Proposition 1 that we get nonzero integrals only from J's (if any) of length 1 and lengths $\frac{1}{2}$. $J = K(I)$ gives us negative integral $\frac{1}{2}(5-3\sqrt{3})$ and from (2·7) we infer that

$$\sum_{J\in\Lambda\,:\,|J|=1}\int_{\mathbb{R}} S_J(x)F_{J(I)}(x)\,dx \leq \frac{1}{2}(5-3\sqrt{3}). \tag{3·7}$$

The $\sum_{J\in\Lambda\,:\,|J|=\frac{1}{2}}\int_{\mathbb{R}} S_J(x)F_{J(I)}(x)\,dx$ may have some positive and some negative summanda but from (2·10) we see that positive summands add up to at most

$$\frac{1}{2}\sum_{k=1}^{\infty}(\sqrt{3}-2)^{2k} = \frac{7-4\sqrt{3}}{8\sqrt{3}-12} = \frac{1}{12}(2\sqrt{3}-3). \tag{3·8}$$

Comparing (3·7) with (3·8) we get

$$\int_{\mathbb{R}} G(x)F_{J(I)}(x)\,dx \leq \frac{1}{2}(5-3\sqrt{3}) + \frac{1}{12}(2\sqrt{3}-3) = 2\frac{1}{4} - \frac{4}{3}\sqrt{3} < -0.0594.$$

From this we infer that

$$\|G\|_1 \geq 0.0594a|\Lambda|. \tag{3·9}$$

Case 3: Neither Case 1 nor Case 2 holds.

Since Case 1. does not hold we infer from Lemma 2 that $|\Lambda_2| \geq (1-2a)|\Lambda|$. Let Λ_2^l be the set of all intervals I from Λ_2 such that $K(I)$ is adjacent to the left endpoint of I. Since Case 2. does not hold we get $|\Lambda_2^l| \geq (1-3a)|\Lambda|$. Let $\overline{\Lambda}$ denote the set of those intervals I from Λ_2^l which are of the form $K(J)$ for some $J \in \Lambda_2^l$. For an interval $I \in \Lambda_2^l \setminus \overline{\Lambda}$ we have the following possibilities:

- $I \in \Lambda'$, but by Lemma 2 there are at most $a|\Lambda|$ such intervals.
- $I \in \max(J)$ for some $J \in \Lambda_3$ but from Lemma 2 there are at most $2a|\Lambda|$ such intervals.
- $I = K(J)$ for some $J \in \Lambda_2 \setminus \Lambda_2^l$ but there are at most $a|\Lambda|$ such intervals.

So we get $|\overline{\Lambda}| \geq (1-7a)|\Lambda|$. Let us iterate this procedure once more and define the set Ω to be the set of all intervals $I \in \Lambda_2^l$ which are sons of some $J \in \Lambda_2^l$ (so far $I \in \overline{\Lambda}$) and J is also a son of some $S \in \Lambda_2^l$. As above we get $|\Omega| \geq (1-11a)|\Lambda|$.

For $I \in \Lambda_2^l$ we set $U(I) = (\frac{k+2}{2^j}, \frac{k+4}{2^j})$ where $K(I) = (\frac{k}{2^j}, \frac{k+1}{2^j})$. Since $\frac{k}{2^j}$ is the left end of the interval $I \supset K(I)$ we see that k is even and so $U(I) \in \mathcal{D}$.

Lemma 4. *Intervals $U(I)$ with $I \in \overline{\Lambda}$ are disjoint.*

Proof. Suppose that for two intervals $I, J \in \overline{\Lambda}$ we have $U(J)$ is strictly contained in $U(I)$. So $|U(J)| < |U(I)|$ and also

$$|K(J)| < |K(I)|. \tag{3·10}$$

Let $I = K(S)$ and $J = K(S_1)$ for some $S, S_1 \in \Lambda_2$. From the definition of $U(.)$ we infer that $U(I) \subset S$ and $U(J) \subset S_1$. Constructing $K(J)$ from $U(J)$ and using those inclusions we see that $K(J) \subset S$ so since $S \in \Lambda_2$ we get from (3·10) that $J \subset K(I)$. If $J = K(I)$ we see from the construction that $U(J)$ is to the left of $U(I)$. If $J \neq K(I)$ then $S_1 \subset K(I)$ so also $U(J) \subset S_1 \subset K(I)$ which is also to the left of $U(I)$. Those contradictions prove the Lemma. $\square$

As in the previous cases we look at

$$\sum_{J\in\Lambda} \int_{\mathbb{R}} S_J(x) F_{U(I)}(x)\, dx = \int_{\mathbb{R}} G(x) F_{U(I)}(x)\, dx. \tag{3·11}$$

for a fixed $I \in \Omega$. By the change of variables we can assume that $|U(I)| = 1$, so from Proposition 1 we see that we need only to consider J's with $|J| = \frac{1}{2}$ and $|J| = 1$. From (2·10) we have

$$\int_{\mathbb{R}} S_{K(I)}(x) F_{U(I)}(x)\, dx = \frac{1}{2}(\sqrt{3}-2)^2 \sim 0.035898 \tag{3·12}$$

For other intervals J such that $|J| = |K(I)| = \frac{1}{2}$ we see from (2·10) that the sum of all negative integrals is greater than

$$\begin{aligned} \frac{1}{2}\sum_{k=1}^{\infty}(\sqrt{3}-2)^{2k+1} &= \frac{1}{2}(\sqrt{3}-2)^3 \frac{1}{1-(\sqrt{3}-2)^2} \\ &= -\frac{1}{12}(7\sqrt{3}-12) \sim -0.01036. \end{aligned} \tag{3·13}$$

The intervals with $|J| = 1$ will give only negative integrals but since $I \in \Omega$ at least two closest to $U(I)$ from the right are not in Λ so the sum of all integrals corresponding to J's with $|J| = 1$ is greater than

$$\begin{aligned} \frac{1}{2}(5-3\sqrt{3})\sum_{k=2}^{\infty}(7-4\sqrt{3})^k &= \frac{1}{2}(5-3\sqrt{3})(7-4\sqrt{3})^2 \frac{1}{4\sqrt{3}-6} \\ &= \frac{1}{12}(265\sqrt{3}-459) \sim -0.0035446 \end{aligned} \tag{3·14}$$

From (3·12), (3·13) and (3·14) we get

$$\int_{\mathbb{R}} G(x) F_{U(I)}(x)\, dx \geq \frac{1}{4}(78\sqrt{3}-135) \sim 0.024990. \tag{3·15}$$

As in the previous cases we define $H(x) = \sum_{I\in\Omega} F_{U(I)}(x)$ and summing (3·15) we get

$\int_{\mathbb{R}} G(x)H(x)\,dx \geq c|\Omega|$. From Lemma 4 we infer that $\|H\|_\infty = 1$ so we get

$$\|G\|_1 \geq c|\Omega| \geq c(1-11a)|\Lambda|. \tag{3·16}$$

□

LEMMA 5. *Let $\eta_I = \pm 1$ for $I \in \mathcal{D}$ be given. There are sets $\Lambda_1 \subset \Lambda_2 \subset \cdots \subset \mathcal{D}$ such that each Λ_n has exactly n elements and*

$$\left\| \sum_{I\in\Lambda_n} \eta_I h_I \right\|_1 \leq 2.$$

Proof. We can assume that $\eta_{[0,1]} = 1$. Let us start with $I(1) = [0,1]$ and $I(2) = [0, \frac{1}{2}]$ and we put $\Lambda_1 = \{h_{I(1)}\}$ and $\Lambda_2 = \{h_{I(1)}, h_{I(2)}\}$. We easily see that $h_{I(1)} + \eta_{I(2)}h_{I(2)}$ equals $2^2 - 1$ on a dyadic interval $I(3) \subset I(2)$ of length 2^{-2} and has absolute value 1 on the rest of $[0,1]$. Clearly

$$h_{I(1)} + \eta_{I(2)}h_{I(2)} + \eta_{I(3)}h_{I(3)}$$

equals $2^3 - 1$ on an interval $I(4) \subset I(3)$ of length 2^{-3} and has absolute value equal to 1 on the rest of $[0,1]$. We continue in this way to get desired sets. □

Now we are ready for our second result.

THEOREM 2. *The Haar wavelet system $(h_I)_{I\in D}$ is not weakly equivalent in $L_1(\mathbb{R})$ to a subsequence of the Strömberg wavelet system $(S_I)_{I\in D}$.*

Proof. If not there exist a sequence of signs $\eta_I = \pm 1$ for $I \in D$ and a 1-1 map $\pi : \mathcal{D} \to \mathcal{D}$ such that

$$\left\| \sum_{I\in\mathcal{D}} a_I \eta_I h_I \right\|_1 \sim \left\| \sum_{I\in\mathcal{D}} a_I S_{\pi(I)} \right\|_1 \tag{3·17}$$

for all (finite) sequence of scalars a_I. Now we use Lemma 5 for this sequence η_I to get sets Λ_n. From Lemma 5, (3·17) and Theorem 1 we get

$$2 \geq \left\| \sum_{I\in\Lambda_n} \eta_I h_I \right\|_1 \sim \left\| \sum_{I\in\pi(\Lambda_n)} S_I \right\|_1 \geq cn \tag{3·18}$$

which is clearly impossible for all n. □

4. *An example of a Haar type system nonequivalent to any subsequence of the Haar system*

In this section we will work on the interval $[0,1]$. Let us recall the definition of a Haar-type system. We will normalise it in $L_1[0,1]$. Let us put $t_0 = 0,\ t_1 = 1$ and $\Delta_0^{(1)} = [0,1)$. We choose an arbitrary internal point $t_2 \in \Delta_0^{(1)}$ and we define $\Delta_1^{(1)} = [0,t_2),\ \Delta_1^{(2)} = [t_2,1)$. In general for arbitrary $i = 1,2,\ldots$ and $j = 1,2,\ldots 2^i$ we take arbitrary internal point $t_{2^i+j} \in \Delta_i^{(j)} = [a,b)$ and we define $\Delta_{i+1}^{(2j-1)} = [a, t_{2^i+j}),\ \Delta_{i+1}^{(2j)} = [t_{2^i+j}, b)$. The only requirement we impose in this construction is that

$$\lim_{i\to+\infty} \max_{1\leq j\leq 2^i} |\Delta_i^{(j)}| = 0. \tag{4·1}$$

This means that the set of points $\{t_i\}$ is dense in $[0,1]$. We define $h_1 = h_0^{(0)} \equiv 1$ and for $j = 1, 2, \ldots 2^i$; $i = 0, 1, 2, \ldots$ we define

$$h_{2^i+j}(x) = h_i^{(j)}(x) = \begin{cases} \frac{1}{2|\Delta_{i+1}^{(2j-1)}|} & \text{for} \quad x \in \Delta_{i+1}^{(2j-1)} \\ \\ \frac{-1}{2|\Delta_{i+1}^{(2j)}|} & \text{for} \quad x \in \Delta_{i+1}^{(2j)} \\ \\ 0 & \text{otherwise.} \end{cases} \tag{4·2}$$

It is clear that each function h_n has mean zero, is normalised in $L_1[0,1]$ and that the system $\{h_n\}_{n=1}^{\infty}$ is orthogonal. When $\{t_i\} = \{0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \ldots\}$ we get the Haar system on $[0,1]$ which we denote as $\{\chi_n\}_{n=1}^{\infty}$. Note that this system consists of all functions from the Haar wavelet system $(h_I)_{I\in\mathcal{D}}$ such that $I \subset [0,1]$ and also contains the constant function. Under the density assumption (4·1) the Haar-type system is a basis in the space $L_1[0,1]$.

A. Kamont [**4**] has shown that in $L_p[0,1]$ with $1 < p < \infty$ every Haar-type system is equivalent to a subsequence of the Haar system. Our aim here is to show that this does not hold for $p = 1$. We have the following

THEOREM 3. *There exists a Haar-type system $\{h_n\}_{n=1}^{\infty}$ which is not weakly equivalent to any subsequence of the Haar system.*

Explicitely we will show that for every sequence of signs $\epsilon_n = \pm 1$, $n = 1, 2, \ldots$ and any sequence of distinct natural numbers $\{\sigma(n)\}_{n=1}^{\infty}$ the system $\{h_n\}$ is not equivalent in $L^1[0,1]$ to $\{\epsilon_{\sigma(n)}\chi_{\sigma(n)}\}$. Suppose we are given a sequence of signs $\{\epsilon_n\}_{n=1}^{\infty}$. For every finite set of natural numbers S we define a polynomial in the Haar system Id_S by the formula

$$Id_S = \sum_{i\in S} \epsilon_i \chi_i. \tag{4·3}$$

LEMMA 6. *For any finite disjoint subsets P and Q of natural numbers and any sequence of signs $\epsilon_i = \pm$*

$$\|Id_P\| \le 2\|Id_P + \frac{1}{2} Id_Q\|. \tag{4·4}$$

In the following $\|.\|$ will denote the norm in $L_1[0,1]$ and $\|f\|_I = \int_I |f|$.

Proof. Let us write elements of the set $P \cup Q$ in an increasing order:

$$P \cup Q = \{k_1, k_2, ..., k_m\} \qquad (m = |P| + |Q|).$$

To simplify the notation we will write $P_i = S_{k_i}(Id_P)$ and $Q_i = S_{k_i}(Id_Q)$, where $S_n(f)$ is the n-th partial sum of f in the Haar system. Using induction on i we will show that

$$\frac{\|P_i\|}{\|P_i + \frac{1}{2}Q_i\|} \le 2. \tag{4·5}$$

For $i = 1$ when $k_1 \in Q$ we have

$$\frac{\|P_1\|}{\|P_1 + \frac{1}{2}Q_1\|} = 0 \tag{4·6}$$

and when $k_1 \in P$

$$\frac{\|P_1\|}{\|P_1 + \frac{1}{2}Q_1\|} = 1. \tag{4·7}$$

Now we assume that for some $1 \le i \le m-1$ inequality (4·5) holds and we will show the analogous inequality for $i+1$. If $k_{i+1} \in Q$, using the monotonicity of the Haar system we get

$$\frac{\|P_{i+1}\|}{\|P_{i+1} + \frac{1}{2}Q_{i+1}\|} = \frac{\|P_i\|}{\|P_i + \frac{1}{2}Q_{i+1}\|} \le \frac{\|P_i\|}{\|P_i + \frac{1}{2}Q_i\|} \le 2.$$

So let us assume $k_{i+1} \in P$. We define numbers n and s by the condition $\mathrm{supp}\chi_{k_{i+1}} = [\frac{s}{2^n}, \frac{s+1}{2^n}] = I$. Functions P_i and $P_i + \frac{1}{2}Q_i$ are constant on I. Let us denote those values by H and L respectively. Clearly

$$H = 2^{n-1}\delta_{n-1} + 2^{n-2}\delta_{n-2} + \ldots + 2\delta_1 + \delta_0 + \delta_0^{(0)}$$

and

$$L = H + \frac{2^{n-1}\eta_{n-1} + 2^{n-2}\eta_{n-2} + \ldots + 2\eta_1 + \eta_0 + \eta_0^{(0)}}{2}$$

where $\delta_0^{(0)}, \delta_0, \ldots, \delta_{n-1}, \eta_0^{(0)}, \eta_0, \ldots, \eta_{n-1}$ takes values $-1, 0, +1$, and moreover $\delta_0^{(0)} \cdot \eta_0^{(0)} = 0$ and $\delta_i \cdot \eta_i = 0$ for all $0 \le i \le n-1$ because sets P and Q are disjoint. From this follows

$$|H| \le 2^{n-1} + 2^{n-2} + \ldots + 2^1 + 1 + 1 = 2^n \tag{4·8}$$

and

$$\begin{aligned} 2|L| &= |2H + 2^{n-1}\eta_{n-1} + 2^{n-2}\eta_{n-2} + \ldots + 2\eta_1 + \eta_0 + \eta_0^{(0)}| \\ &\le |H| + |2^{n-1}(\delta_{n-1} + \eta_{n-1}) + 2^{n-2}(\delta_{n-2} + \eta_{n-2}) + \ldots \\ &\quad +2(\delta_1 + \eta_1) + (\delta_0 + \eta_0) + (\delta_0^{(0)} + \eta_0^{(0)})| \\ &\le |H| + 2^n \end{aligned}$$

so we get

$$|L| \le \frac{|H| + 2^n}{2}. \tag{4·9}$$

From(4·8) and (4·9) we infer that

$$|L| \le 2^n. \tag{4·10}$$

From (4·8) we get

$$\|P_{i+1}\|_I = \frac{1}{2^{n+1}}\Big(|H + 2^n| + |H - 2^n|\Big) = 1,$$

and from (4·10) we get

$$\|P_{i+1} + \frac{1}{2}Q_{i+1}\|_I = \frac{1}{2^{n+1}}\Big(|L + 2^n| + |L - 2^n|\Big) = 1.$$

From the two last inequalities we infer

$$\|P_{i+1}\| = \|P_i\| - \|P_i\|_I + \|P_{i+1}\|_I = \|P_i\| + 1 - \frac{|H|}{2^n} \tag{4·11}$$

and

$$\begin{aligned} \|P_{i+1} + \frac{1}{2}Q_{i+1}\| &= \|P_i + \frac{1}{2}Q_i\| - \|P_i + \frac{1}{2}Q_i\|_I + \|P_{i+1} + \frac{1}{2}Q_{i+1}\|_I \\ &= \|P_i + \frac{1}{2}Q_i\| + 1 - \frac{|L|}{2^n} \end{aligned} \tag{4·12}$$

From (4·5), (4·9), (4·11) and (4·12) we get

$$\begin{aligned} \|P_{i+1}\| = \|P_i\| + 1 - \frac{|H|}{2^n} &\leq 2\|P_i + \frac{1}{2}Q_i\| + 1 - \frac{|H|}{2^n} \\ &= 2\|P_i + \frac{1}{2}Q_i\| + 2 - 2\frac{|L|}{2^n} - 1 + \frac{2|L| - |H|}{2^n} \\ &\leq 2\|P_{i+1} + \frac{1}{2}Q_{i+1}\|, \end{aligned}$$

so we get the inductive claim. To finish the proof of the Lemma one has to observe that for $i = m$ we have $P_m = S_{k_m}(Id_P) = Id_P$ and $Q_m = S_{k_m}(Id_Q) = Id_Q$. $\square$

Now we are ready to prove Theorem 3. We take *any* Haar-type system $\{h_n\}_{n=1}^{\infty}$, such that

$$h_{2n}^{(1)}(x) = \begin{cases} \left(\frac{8}{3}\right)^n & \text{for } 0 \leq x < \frac{1}{2}\left(\frac{3}{8}\right)^n \\ -\left(\frac{8}{3}\right)^n, & \text{for } \frac{1}{2}\left(\frac{3}{8}\right)^n \leq x < \left(\frac{3}{8}\right)^n \end{cases} \tag{4·13}$$

and

$$h_{2n+1}^{(1)}(x) = \begin{cases} \frac{4}{3}\left(\frac{8}{3}\right)^n & \text{for } 0 \leq x < \left(\frac{3}{8}\right)^{n+1} \\ -4\left(\frac{8}{3}\right)^n, & \text{for } \left(\frac{3}{8}\right)^{n+1} \leq x < \frac{1}{2}\left(\frac{3}{8}\right)^n. \end{cases} \tag{4·14}$$

Assume that the system $\{h_n\}$ is equivalent to $\{\epsilon_{\sigma(n)}\chi_{\sigma(n)}\}$, with the coefficient of equivalence $C > 0$. For every natural number d we define the function

$$F_d = h_0^{(0)} + \sum_{i=1}^{d}\left(h_{2i}^{(1)} + \frac{1}{2}h_{2i+1}^{(1)}\right).$$

Clearly

$$F_d(x) = \begin{cases} \left(\frac{8}{3}\right)^{2d+1} & 0 \leq x < \left(\frac{3}{8}\right)^{2d+1} \\ \\ 0, & \frac{1}{2}\left(\frac{3}{8}\right)^{2d+1} \leq x \leq 1, \end{cases}$$

so we have

$$\|F_d\| = 1. \tag{4·15}$$

Using induction on d we can show that

$$\|h_0^{(0)} + \sum_{i=1}^{d} h_{2i}^{(1)}\| \geq \frac{d}{8}. \tag{4·16}$$

The assumed equivalence equivalence of $\{h_n\}$ with $\{\epsilon_{\sigma(n)}\chi_{\sigma(n)}\}$ would transfer $h_0^{(0)} + \sum_{i=1}^{d} h_{2i}^{(1)}$ onto Id_P for appropriate set P (see (4·3)) and $\sum_{i=1}^{d} h_{2i+1}^{(1)}$ onto Id_Q for appropriate set Q, clearly $P \cap Q = \emptyset$.

Using (4·15), (4·16) and the equivalence of $\{h_n\}$ with $\{\epsilon_{\sigma(n)}\chi_{\sigma(n)}\}$ we get

$$\|Id_P + \frac{Id_Q}{2}\| \leq C\|F_d\| = C$$

and

$$\|Id_P\| \geq \frac{1}{C}\|h_1 + \sum_{i=1}^{d} h_{2^{2i}+1}\| \geq \frac{d}{8C}.$$

So we have

$$\frac{\|Id_P\|}{\|Id_P + \frac{1}{2}Id_Q\|} \geq \frac{d}{8C^2}.$$

But for large d this inequality will contradict Lemma 6. So systems $\{h_n\}$ and $\{\epsilon_{\sigma(n)}\chi_{\sigma(n)}\}$ are not equivalent and Theorem 3 follows.

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