Characterization of 1-Greedy Bases *

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Abstract

If a basis is greedy, the greedy approximation provides best, up to a constant C, m–term approximation for each element. It is known that the Haar (or good wavelet) basis is a greedy basis in L_p, $1 < p < \infty$, [7]. Unfortunately the constant $C_p > 1$ unless $p = 2$. Our goal is to investigate 1–greedy bases i.e. bases for which greedy approximation provides the best m–term approximation. We show how 1–greedy bases relate to symmetric bases and we give an answer to the problem raised in [9] of finding a characterization of 1-greediness.

1 Introduction

Let X be a (real) Banach space with a semi-normalized basis $(e_n)_{n=1}^{\eta}$ (η finite or infinite). For each $m = 1, 2, \cdots$ we let Σ_m denote the collection of all elements of X which can be expressed as a linear combination of m elements of (e_n):

$$\Sigma_m = \big\{y = \sum_{j \in B} \alpha_j e_j : B \subset \mathbb{N},\ |B| = m,\ \alpha_j' \text{s} \ \text{ scalars}\big\}.$$

Let us notice that, in some cases, it may be possible to write an element from Σ_m in more than one way, and that the space Σ_m is not linear: the sum of

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two elements from Σ_m is generally not in Σ_m, it is in Σ_{2m}. An approximation algorithm is a sequence of maps $T_m : X \to X$, $m \in \mathbb{N}$, so that for each $x \in X$, $T_m(x) \in \Sigma_m$. For $x \in X$, its *best m-term approximation error* (with respect to the given basis) is

$$\sigma_m(x) = \inf_{y \in \Sigma_m} \|x - y\|.$$

The fundamental question is how to construct an approximation algorithm which for every $x \in X$ and each m produces an element $T_m(x) \in \Sigma_m$ so that the error of the approximation of x by $T_m(x)$ be (uniformly) comparable with $\sigma_m(x)$, i.e.

$$\|x - T_m(x)\| \leq C\sigma_m(x)$$

where C is an absolute constant.

The most obvious and in some sense natural attempt to get such an algorithm is to consider the *Greedy Algorithm*, $(\mathcal{G}_m)_{m=1}^{\eta}$, where for each x, $\mathcal{G}_m(x)$ is obtained by taking the largest m coefficients in the series expansion of x. To be precise, if we let $(e_n^*)_{n=1}^{\eta} \subset X^*$ denote the biorthogonal functionals associated to $(e_n)_{n=1}^{\eta}$, for $x \in X$ put

$$\mathcal{G}_m(x) = \sum_{j \in B} e_j^*(x) e_j,$$

where the set $B \subset \mathbb{N}$ is chosen in such a way that $|B| = m$ and $|e_j^*(x)| \geq |e_k^*(x)|$ whenever $j \in B$ and $k \notin B$.
Let us notice that it may happen that for some x and m the set B, hence the element $\mathcal{G}_m(x)$, is not uniquely determined by the previous conditions. In such a case, we pick either of them. Besides, the maps $\mathcal{G}_m$ are neither linear (even when the sets B are uniquely determined) nor continuous.

Following [1], given $x \in X$ we define its *greedy ordering* as the map $\rho : \{1, 2, \dots, \eta\} \to \{1, 2, \dots, \eta\}$ such that $\{j : e_j^*(x) \neq 0\} \subset \rho\big(\{1, 2, \dots, \eta\}\big)$ and so that if $j < k$ then either $|e_{\rho(j)}^*(x)| > |e_{\rho(k)}^*(x)|$ or $|e_{\rho(j)}^*(x)| = |e_{\rho(k)}^*(x)|$ and $\rho(j) < \rho(k)$. With this notation, the m-th greedy approximation of x is now uniquely determined by

$$\mathcal{G}_m(x) = \sum_{j=1}^{m} e_{\rho(j)}^*(x) e_{\rho(j)}.$$

Konyagin and Temlyakov [5] defined a basis to be *greedy* if $\mathcal{G}_m(x)$ is essentially the best m-term approximation to x using the basis vectors:

DEFINITION A basis $(e_n)_{n=1}^{\eta}$ is *greedy* if there is a constant $C \geq 1$ such that for all $x \in X$ and $m \in \mathbb{N}$, we have

$$\|x - \mathcal{G}_m(x)\| \leq C\sigma_m(x).$$

The smallest such constant C will be called the *greedy constant* of (e_n) and, in this case, we will say that (e_n) is C-greedy.

Note that if $C = 1$ then $\|x - \mathcal{G}_m(x)\| = \sigma_m(x)$ for all $x \in X$ and $m = 1, 2, \ldots$, so the greedy algorithm gives the *best* m–term approximation.

They also defined a basis $(e_n)_{n=1}^{\eta}$ to be *democratic* if there is a constant $\Delta > 0$ such that for any two finite subsets A, B of $\mathbb{N}$ with $|A| = |B|$ we have

$$\Big\|\sum_{k\in A} e_k\Big\| \leq \Delta \Big\|\sum_{k\in B} e_k\Big\|,$$

and gave the following characterization of greedy bases:

Theorem 1.1 (Theorem 1 of [5], cf. Theorem 1 of [9]). *If $(e_n)_{n=1}^{\eta}$ is a greedy basis with greedy constant $\leq C$, then (e_n) is unconditional with unconditional basis constant $\leq C$ and democratic with democratic constant $\leq C$. Conversely, if (e_n) is unconditional with unconditional basis constant K and Δ-democratic then (e_n) is greedy with greedy constant $\leq K + K^3\Delta$.*

We will remind the reader the notion of unconditional basis in §2. If we disregard constants, Theorem 1.1 says that a basis is greedy if and only if it is unconditional and democratic. In particular, Theorem 1.1 immediately yields that a 1-greedy basis has both unconditional basis constant and democracy constant equal to 1. However, this is not a characterization of bases with greedy constant 1. In this paper we tackle the problem of finding a characterization for greedy bases with greedy constant equal to 1.

In Section 2 we pay close attention to the unconditional constants of a basis in relation to Theorem 1.1 and show that 1-symmetric bases are 1-greedy.

In Section 3 we introduce a weak symmetry condition for bases that 1-greedy bases enjoy, which we called Property A, and characterize 1-greedy bases in terms of Property A and unconditionality.

In Section 4 we define Property B and characterize 1-symmetric bases as those bases have both Property A and Property B.

Section 5 deals with the problem of renorming equivalently a given Banach space X with a greedy (respectively unconditional or/and democratic) basis

(e_n) in such a way that after renorming we improve the greedy constant of (e_n) (respectively its constant of unconditionality or/and democracy). We prove that for all $\varepsilon > 0$ we can extract a large "lacunary subbasis" of the Haar system in $L_p(0,1)$ which is 1-unconditional and $(1+\varepsilon)$-democratic.

Section 6 is devoted to provide examples (some of which are non-trivial) to ilustrate the concepts that are involved in this article.

We use standard Banach space notation and terminology throughout (see e.g. [8] or [6]). For clarity, however, we single out the following. $|\cdot|$ may denote (depending on the context) either the absolute value of a real number or the cardinality of a finite set. The *convex hull* of a set S (i.e. the set of all convex combinations of points of S) will be denoted by $\mathrm{co}(S)$. Given a sequence $(x_n)_{n=1}^{\eta}$ in X, we say that $(x_n)_{n=1}^{\eta}$ is *semi-normalized* (respectively *normalized*) if there exists a constant $c > 0$ so that $1/c \le \|e_n\| \le c$ (respectively $\|x_n\| = 1$) for all n. The closed linear span of $(x_n)_{n=1}^{\eta}$ is denoted by $[x_n]$. c_{00} will denote the sequence space consisting of sequences with only finitely many nonzero terms. Other concepts from the theory of bases will be introduced as needed.

2 Preliminary Results

To begin let us recall that a basis $(e_n)_{n=1}^{\eta}$ is said to be *unconditional* if there is a constant $K \ge 1$ so that whenever $a_1, \ldots, a_N$, $b_1, \ldots, b_N$ are scalars satisfying $|a_n| \le |b_n|$ for $n = 1, \ldots, N$, then the following inequality holds

$$\Big\| \sum_{n=1}^{N} a_n e_n \Big\| \le K \Big\| \sum_{n=1}^{N} b_n e_n \Big\|. \tag{1}$$

The *unconditional constant* K_u of (e_n) is the least such constant. When this happens we say that (e_n) is K_u-unconditional. If $(e_n)_{n=1}^{\eta}$ is an unconditional basis of X and A is a subset of the integers then there is a bounded linear projection P_A from X onto $[e_k : k \in A]$ defined for each $x = \sum_{k=1}^{\eta} e_k^*(x) e_k$ by

$$P_A(x) = \sum_{k \in A} e_k^*(x) e_k.$$

$\{P_A \,; A \subset \mathbb{N}\}$ are the natural projections associated to the unconditional basis (e_n) and the quantity $K_s = \sup_A \|P_A\| < \infty$ is called the the *suppression* *constant* of the basis (or the *unconditional basis constant*). Let us observe that in general we have $1 \leq K_s \leq K_u \leq 2K_s$ (see, for instance, [6] pag. 380), but there is a situation in which K_s plays the role of K_u in equation (1):

Proposition 2.1 *Let $(e_n)_{n=1}^{\eta}$ be an unconditional basis for a Banach space X. Assume $a_1, \ldots, a_N$, $b_1, \ldots, b_N$ are scalars so that $|a_n| \leq |b_n|$ for all $1 \leq n \leq N$ and, moreover, sgn $(a_n) =$ sgn (b_n) whenever $a_n b_n \neq 0$. Then*

$$\Big\| \sum_{n=1}^{N} a_n e_n \Big\| \leq K_s \Big\| \sum_{n=1}^{N} b_n e_n \Big\|.$$

PROOF: Fix any $N \in \mathbb{N}$ and let $a_1, \ldots, a_N$, $b_1, \ldots, b_N$ be scalars as in the hypothesis. Observe that for each $1 \leq n \leq N$ we have

$$\frac{a_n}{b_n} = \int_0^{\frac{a_n}{b_n}} 1 \, dt,$$

so that we can write

$$\sum_{n=1}^{N} a_n e_n = \sum_{n=1}^{N} \int_0^1 b_n \chi_{(0, \frac{a_n}{b_n})}(t) dt \; e_n = \int_0^1 \Big(\sum_{n=1}^{N} b_n \chi_{(0, \frac{a_n}{b_n})}(t) dt \; e_n \Big) \, dt.$$

Notice that for each $t \in (0,1)$, the unconditionality of the basis yields

$$\Big\| \sum_{n=1}^{N} b_n \chi_{(0, \frac{a_n}{b_n})}(t) e_n \Big\| \leq K_s \Big\| \sum_{n=1}^{N} b_n e_n \Big\| \tag{2}$$

Then, combining the properties of the Bochner integral with equation (2) we obtain

$$\| \sum_{n=1}^{N} a_n e_n \| \leq \int_0^1 \Big\| \sum_{n=1}^{N} b_n \chi_{(0, \frac{a_n}{b_n})}(t) e_n \Big\| \, dt \leq K_s \Big\| \sum_{n=1}^{N} b_n e_n \Big\|.$$

$\square$

It is clear that a K_u-unconditional and Δ-democratic basis $(e_n)_{n=1}^{\eta}$ verifies the inequality

$$\Big\| \sum_{k \in P} \theta_k e_k \Big\| \leq \Gamma \Big\| \sum_{k \in Q} \epsilon_k e_k \Big\| \tag{3}$$

for $\Gamma \leq K_u^2 \Delta$, any two finite sets P and Q of the same cardinality, and any choices of signs $(\theta_k)_{k \in P}$ and $(\epsilon_k)_{k \in Q}$. A basis $(e_n)_{n=1}^{\eta}$ is *superdemocratic* ([5]) if it satisfies (3).

If $(e_n)_{n=1}^{\eta}$ is 1-greedy, by Theorem 1.1, $(e_n)_{n=1}^{\eta}$ is 1-democratic and unconditional with $K_s = 1$. Next we give an example which shows that, nevertheless, K_u need not be 1. First we show a simple fact.

Proposition 2.2 *Let X be a 2-dimensional Banach space with normalized basis (e_1, e_2). If (e_1, e_2) is unconditional with $K_s = 1$ then (e_1, e_2) is 1-greedy.*

PROOF: We need only show that for each $x \in X$ we have

$$\|x - \mathcal{G}_1(x)\| \leq \sigma_1(x).$$

Put $x = \alpha e_1 + \beta e_2$. Clearly we have

$$\sigma_1(x) = \inf_{s,t} \big\{ \|(\alpha - s)e_1 + \beta e_2\|, \|\alpha e_1 + (\beta - t)e_2\| \big\}.$$

Without loss of generality we assume that $|\alpha| \geq |\beta|$. Using the hypothesis we obtain,

$$\|x - \mathcal{G}_1(x)\| = \|\beta e_2\| = \|P_{\{2\}}\big((\alpha - s)e_1 + \beta e_2\big)\| \leq \|(\alpha - s)e_1 + \beta e_2\|$$

and

$$\|x - \mathcal{G}_1(x)\| = |\beta| \leq |\alpha| = \|\alpha e_1\| = \|P_{\{1\}}\big(\alpha e_1 + (\beta - t)e_2\big)\| \leq \|\alpha e_1 + (\beta - t)e_2\|.$$

Thus, $\|x - \mathcal{G}_1(x)\| \leq \sigma_1(x)$ and we are done. $\square$

Example. Put

$$\mathcal{B} = \big\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, xy \geq 0\big\} \cup \big\{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1, xy \leq 0\big\},$$

and let $\|\cdot\|_{\mathcal{B}}$ denote the Minkowski functional of $\mathcal{B}$, i.e., for each $x \in X$

$$\|x\|_{\mathcal{B}} = \inf \big\{t > 0 : \frac{x}{t} \in \mathcal{B}\big\}.$$

$X = (\mathbb{R}^2, \|\cdot\|_{\mathcal{B}})$ is a Banach space and the unit vectors $e_1 = (1, 0)$, $e_2 = (0, 1)$ are a basis for X. It is immediate to check that $\|P_{\{i\}}\| \leq 1$ for $i = 1, 2$, thus by Proposition 2.2, (e_1, e_2) is 1-greedy.

On the other hand, (e_1, e_2) is not 1-superdemocratic since $\|e_1+e_2\| = \sqrt{2}$ whereas $\|e_1 - e_2\| = 2$. Therefore, (e_1, e_2) cannot be 1-unconditional.

One might thing, in view of the previous example, that a basis which is 1-greedy, 1-superdemocratic and such that $\sup_A \|P_A\| \leq 1$ would be 1-unconditional. This is not the case as the next two-dimensional example shows.

Example. Let

$$\begin{aligned} A_1 &= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1, x \geq 0, y \geq 0, x \leq y\}, \\ A_2 &= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1, x \leq 0, y \geq 0, |x| \geq y\}, \\ A_3 &= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1, x \leq 0, y \leq 0, |x| \leq |y|\}, \\ A_4 &= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1, x \geq 0, y \leq 0, x \geq |y|\}, \end{aligned}$$

and $A = A_1 \cup A_2 \cup A_3 \cup A_4$. Now, take $\mathcal{B}$ the convex hull of A and let $\|\cdot\|_{\mathcal{B}}$ denote the Minkowski functional of $\mathcal{B}$. $(X, \|\cdot\|_{\mathcal{B}})$ is a Banach space, of which the vectors $e_1 = (1,0)$ and $e_2 = (0,1)$ are a basis. Clearly, $\|P_{\{i\}}\| \leq 1$ for $i = 1, 2$, hence by Proposition 2.2, (e_1, e_2) is 1-greedy. It is also immediate that $\|\theta_1 e_1 + \theta_2 e_2\| = \|\epsilon_1 e_1 + \epsilon_2 e_2\|$ for any choices of signs $\{\theta_i\}_{i=1}^2$ and $\{\epsilon_i\}_{i=1}^2$, therefore the basis is 1-superdemocratic. Nevertheless, (e_1, e_2) is not 1-unconditional since, for instance, given any $\alpha \in (\frac{\pi}{4}, \frac{\pi}{2})$ the vector $x = (\cos\alpha, \sin\alpha)$ has norm $= 1$ whereas the vector $x' = (\cos\alpha, -\sin\alpha)$ has norm strictly bigger than 1.

There are weaker forms of greediness. For any basis $(e_n)_{n=1}^{\eta}$, let

$$\tilde{\sigma}_m(x) = \inf\left\{\left\|x - \sum_{k\in A} e_k^*(x) e_k\right\| : A \subset \{1, 2, \ldots, \eta\}, |A| \leq m\right\}.$$

A basis $(e_n)_{n=1}^{\eta}$ is *almost greedy* ([1]) if there is a constant C so that for each $x \in X$ and $m = 1, 2, \ldots$ we have

$$\|x - \mathcal{G}_m(x)\| \leq C\tilde{\sigma}_m(x).$$

A basis $(e_n)_{n=1}^{\eta}$ is *quasi-greedy* [5] if for each $x \in X$ the norm limit $\lim_{m\to\infty} \mathcal{G}_m(x)$ exists and equals x. This is equivalent (see [10]) to the condition that for some constant C

$$\sup_m \|\mathcal{G}_m(x)\| \leq C\|x\|.$$

Obviously,

$$\sigma_m(x) \le \tilde{\sigma}_m(x) \le \Big\| x - \sum_{k=1}^{m} e_k^*(x) e_k \Big\| \to 0 \ \text{ as } m \to \infty.$$

The following result appeared in [9]:

Proposition 2.3 *(cf. Proposition 7 of [10]) Let $(e_n)_{n=1}^{\eta}$ be an unconditional basis for a Banach space X with $K_s = 1$. Then, for each $x \in X$ and each $m = 1, 2, \ldots$ there exists $B \subset \mathbb{N}$ of cardinality m such that*

$$\sigma_m(x) = \Big\| x - \sum_{n \in B} e_n^*(x) e_n \Big\|.$$

That is, if $K_s = 1$, then $\sigma_m(x) = \tilde{\sigma}_m(x)$ and the infimum is attained. Therefore we obtain the following immediate consequence that we state for further reference.

Proposition 2.4 *Let $(e_n)_{n=1}^{\eta}$ be a basis of a Banach space X.*

(i) If (e_n) is 1-greedy, then

$$\|x - \mathcal{G}_m(x)\| = \sigma_m(x) = \min\Big\{ \Big\| x - \sum_{k \in A} e_k^*(x) e_k \Big\| : A \subset \{1, 2, \cdots, \eta\}, |A| = m \Big\}.$$

(ii) If (e_n) is unconditional with $K_s = 1$ and

$$\|x - \mathcal{G}_m(x)\| = \min\Big\{ \Big\| x - \sum_{k \in A} e_k^*(x) e_k \Big\| : A \subset \{1, 2, \cdots, \eta\}, |A| = m \Big\}.$$

for each $x \in X$ and every $1 \le m < \eta$, then (e_n) is 1-greedy.

Let us remember that an unconditional basis $(e_n)_{n=1}^{\eta}$ of a Banach space X is *symmetric* if for any permutation σ of $\{1, 2, \cdots, \eta\}$, the basis $(e_{\sigma(n)})_{n=1}^{\eta}$ is *equivalent* to $(e_n)_{n=1}^{\eta}$, i.e., there is a constant C so that for any permutation σ and any choice of scalars $(a_k) \in c_{00}$ we have

$$C^{-1} \Big\| \sum_{n=1}^{\eta} a_n e_n \Big\| \le \Big\| \sum_{n=1}^{\eta} a_n e_{\sigma(n)} \Big\| \le C \Big\| \sum_{n=1}^{\eta} a_n e_n \Big\|.$$

The least constant K such that for all $x = \sum_{n=1}^{\eta} a_n e_n \in X$ the inequality

$$\Big\| \sum_{n=1}^{\eta} \epsilon_n a_n e_{\sigma(n)} \Big\| \leq K \Big\| \sum_{n=1}^{\eta} a_n e_n \Big\|$$

holds for any sequence of signs (ϵ_n) and any permutation σ is called the *symmetric constant* of $(e_n)_{n=1}^{\eta}$. In this case we also say that $(e_n)_{n=1}^{\eta}$ is K-*symmetric.*

A 1-symmetric basis is, in particular, 1-unconditional and 1-democratic. Therefore, by Theorem 1.1, a 1-symmetric basis is greedy with greedy constant ≤ 2. Actually, more can be said:

Theorem 2.5 *If $(e_n)_{n=1}^{\eta}$ is 1-symmetric, then $(e_n)_{n=1}^{\eta}$ is 1-greedy.*

PROOF: Fix $x = \sum_{n=1}^{\eta} e_n^*(x) e_n$ and $1 \leq m < \eta$. Let ρ be the greedy ordering for x and $A = \{\rho(1), \rho(2), \cdots, \rho(m)\}$. Thus, $\mathcal{G}_m(x) = \sum_{n \in A} e_n^*(x) e_n$. We aim to show that

$$\|x - \mathcal{G}_m(x)\| = \min \Big\{ \Big\| x - \sum_{n \in B} e_n^*(x) e_n \Big\| \ : \ B \subset \mathbb{N}, \ |B| = m \Big\}.$$

Given $B \subset \mathbb{N}$ of cardinality m, suppose $A \cap B = \emptyset$. If we take any permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that $\pi(A) = B$ and $\pi(n) = n$ if $n \notin A$, using the 1-symmetry of the basis we have

$$\begin{aligned}
\big\| x - \sum_{n \in B} e_n^*(x) e_n \big\| &= \big\| \sum_{n \in A} e_n^*(x) e_n + \sum_{n \notin A \cup B} e_n^*(x) e_n \big\| \\
&= \big\| \sum_{n \in A} e_n^*(x) e_{\pi(n)} + \sum_{n \notin A \cup B} e_n^*(x) e_n \big\| \\
&\geq \big\| \sum_{n \in B} e_n^*(x) e_n + \sum_{n \notin A \cup B} e_n^*(x) e_n \big\| \\
&= \big\| x - \sum_{n \in A} e_n^*(x) e_n \big\|.
\end{aligned}$$

Let us assume now that $A \cap B \neq \emptyset$. We pick a permutation $\pi : \mathbb{N} \to \mathbb{N}$ so that $\pi(A \setminus B) = B \setminus A$ and $\pi(j) = j$ if $j \notin A \setminus B$. Then, the 1-symmetry of the basis yield

$$\big\| x - \sum_{n \in B} e_n^*(x) e_n \big\| = \big\| \sum_{n \in A \setminus B} e_n^*(x) e_n + \sum_{n \notin A \cup B} e_n^*(x) e_n \big\|$$

$$
\begin{aligned}
&= \Big\| \sum_{n\in A\setminus B} e_n^*(x)e_{\pi(n)} + \sum_{n\notin A\cup B} e_n^*(x)e_n \Big\| \\
&\geq \Big\| \sum_{n\in B\setminus A} e_n^*(x)e_n + \sum_{n\notin A\cup B} e_n^*(x)e_n \Big\| \\
&= \Big\| \sum_{n\notin A} e_n^*(x)e_n \Big\| = \big\| x - \mathcal{G}_m(x) \big\|.
\end{aligned}
$$

□

3 Property A

Let $(e_n)_{n=1}^{\eta}$ be a basis of a Banach space X. Given any $x = \sum_{n=1}^{\eta} e_n^*(x)e_n \in X$, the *support of* x, denoted supp x, consists of those n such that $e_n^*(x) \neq 0$. Let $M(x)$ denote the subset of supp x, where the coordinates of x (in absolute value) are the largest. Obviously the cardinality of $M(x)$ is finite for all $x \in X$. We will say that a $1-1$ map $\pi : \operatorname{supp} x \to \{1, 2, \ldots, \eta\}$ is a *greedy permutation for* x if $\pi(j) = j$ for all $j \in \operatorname{supp} x \setminus M(x)$ and if $j \in M(x)$ then, either $\pi(j) = j$ or $\pi(j) \in \mathbb{N} \setminus \operatorname{supp} x$. That is, a greedy permutation of x puts those coefficients of x whose absolute value is the largest (or some of them) in "gaps" of the support of x, if there are any. If supp $x \neq \mathbb{N}$, we will put $M_\pi^*(x) = \{j \in M(x) : \pi(j) \neq j\}$. $\Pi_G(x)$ will denote the set of all greedy permutations of x.

Definition A basis $(e_n)_{n=1}^{\eta}$ for a Banach space X has *Property A* if for any $x \in X$ we have

$$
\Big\| \sum_{n\in \operatorname{supp} x} e_n^*(x)e_n \Big\| = \Big\| \sum_{n\in \operatorname{supp} x} \theta_{\pi(n)} e_n^*(x)e_{\pi(n)} \Big\|
$$

for all $\pi \in \Pi_G(x)$ and signs (θ_k) such that $\theta_{\pi(n)} = 1$ if $n \notin M_\pi^*(x)$.

Roughly speaking, Property A is a weak symmetry condition for largest coefficients. It allows some symmetry in the norm of a vector provided its support has gaps. When supp $x = \{1, 2, \ldots, \eta\}$, then $\Pi_G(x)$ consists only of the identity permutation and the basis does not allow any symmetry in the norm of x. The opposite extreme case occurs when $x = \alpha \sum_{n\in S} e_k$, with $|\operatorname{supp} x| < \eta$, then $\|x\| = \|\alpha \sum_{n\in P} e_k\|$ for any $P \subset \{1, 2, \ldots, \eta\}$ of cardinality $|\operatorname{supp} x|$. In particular, if a basis $(e_n)_{n=1}^{\eta}$ satisfies Property A then it is 1-democratic.

Theorem 3.1 *A basis $(e_n)_{n=1}^{\eta}$ for a Banach space X is 1-greedy if and only if (e_n) is unconditional with $K_s = 1$ and satisfies Property A.*

PROOF: If (e_n) is 1-greedy then $K_s = 1$ by Theorem 1.1. To see that (e_n) has Property A, fix $x \in X$ and assume that $S = \operatorname{supp} x$ is a proper subset of $\{1, 2, \dots, \eta\}$, otherwise there is nothing to prove. Given π, a greedy permutation of x, and a choice of signs $\theta = (\theta_k)$ such that $\theta_{\pi(n)} = 1$ if $n \notin M_\pi^*(x)$, put $x_{\theta,\pi} = \sum_{n\in S} \theta_{\pi(n)} a_n e_{\pi(n)}$. We want to show that $\|x\| = \|x_{\theta,\pi}\|$. Consider the vector

$$y = x + \sum_{k\in M_\pi^*(x)} \theta_{\pi(k)} a_k e_{\pi(k)},$$

which results from putting as many largest coefficients of x (possibly with different signs) as $|M_\pi^*(x)|$ in gaps of the support of x. Then, on the one hand, if $m = |M_\pi^*(x)|$ we have

$$\mathcal{G}_m(y) = \sum_{k\in M_\pi^*(x)} a_k e_k.$$

Since (e_n) is 1-greedy,

$$\|x_{\theta,\pi}\| = \|y - \mathcal{G}_m(y)\| = \sigma_m(y) \le \Big\| y - \sum_{k\in M_\pi^*(x)} \theta_{\pi(k)} a_k e_{\pi(k)} \Big\| = \|x\|.$$

On the other hand we also have

$$\mathcal{G}_m(y) = \sum_{k\in M_\pi^*(x)} \theta_{\pi(k)} a_k e_{\pi(k)},$$

hence

$$\|x\| = \|y - \mathcal{G}_m(y)\| \le \Big\| y - \sum_{k\in M_\pi^*(x)} a_k e_k \Big\| = \|x_{\theta,\pi}\|.$$

For the converse, since $K_s = 1$, using Proposition 2.4, we will prove that (e_n) is 1-greedy by showing that for each $m \in \mathbb{N}$, $m < \eta$, and any $x \in X$, we have

$$\|x - \mathcal{G}_m(x)\| = \min\Big\{ \big\| x - P_B(x) \big\| : \ B \subset \{1, 2, \cdots, \eta\},\ |B| = m \Big\}.$$

Let ρ be the greedy ordering for x and $A = \{\rho(1), \rho(2), \cdots, \rho(m)\}$. Thus, $\mathcal{G}_m(x) = \sum_{n\in A} e_n^*(x) e_n$. Suppose, first, that B is disjoint with A. Then, if we pick signs $(\theta_n)_{n\in A}$ so that sgn $(\theta_n e^*_{\rho(m)}(x)) =$ sgn $e^*_n(x)$ for all $n \in A$, using Proposition 2.1 we obtain

$$\begin{aligned}\big\|x - P_B(x)\big\| &= \Big\|\sum_{n\in A} e^*_n(x)e_n + \sum_{n\notin A\cup B} e^*_n(x)e_n\Big\| \\ &\geq \Big\|\sum_{n\in A} \theta_n e^*_{\rho(m)}(x)e_n + \sum_{n\notin A\cup B} e^*_n(x)e_n\Big\|\end{aligned}$$

Now pick signs $(\epsilon_n)_{n\in B}$ so that sgn $(\epsilon_n e^*_{\rho(m)}(x)) =$ sgn $e^*_n(x)$ for each $n \in B$. Then Property A gives

$$\Big\|\sum_{n\in A} \theta_n e^*_{\rho(m)}(x)e_n + \sum_{n\notin A\cup B} e^*_n(x)e_n\Big\| \geq \Big\|\sum_{n\in B} \epsilon_n e^*_{\rho(m)}(x)e_n + \sum_{n\notin A\cup B} e^*_n(x)e_n\Big\|,$$

and using Proposition 2.1 again we get

$$\begin{aligned}\Big\|\sum_{n\in B} \epsilon_n e^*_{\rho(m)}(x)e_n + \sum_{n\notin A\cup B} e^*_n(x)e_n\Big\| &\geq \Big\|\sum_{n\in B} e^*_n(x)e_n + \sum_{n\notin A\cup B} e^*_n(x)e_n\Big\| \\ &= \Big\|x - \sum_{n\in A} e^*_n(x)e_n\Big\| \\ &= \|x - \mathcal{G}_m(x)\|.\end{aligned}$$

If $B \cap A \neq \emptyset$, then

$$\begin{aligned}\big\|x - P_B(x)\big\| &= \Big\|\sum_{n\in A\setminus B} e^*_n(x)e_n + \sum_{n\notin A\cup B} e^*_n(x)e_n\Big\| \\ &\overset{(a)}{\geq} \Big\|\sum_{n\in A\setminus B} \theta_n e^*_{\rho(m+1)}(x)e_n + \sum_{n\notin A\cup B} e^*_n(x)e_n\Big\| \\ &\overset{(b)}{=} \Big\|\sum_{n\in B\setminus A} \epsilon_n e^*_{\rho(m+1)}(x)e_n + \sum_{n\notin A\cup B} e^*_n(x)e_n\Big\| \\ &\overset{(c)}{\geq} \Big\|\sum_{n\in B\setminus A} e^*_n(x)e_n + \sum_{n\notin A\cup B} e^*_n(x)e_n\Big\| \\ &= \Big\|\sum_{n\notin A} e^*_n(x)e_n\Big\|,\end{aligned}$$

where $\theta_n = \pm 1$ have been chosen in such a way that sgn $(\theta_n e^*_{\rho(m+1)}(x)) =$ sgn $e^*_n(x)$ for all $n \in A \setminus B$ and we picked $\epsilon_n = \pm 1$ in order to satisfy sgn $(\epsilon_n e^*_{\rho(m+1)}(x)) = \text{sgn } e^*_n(x)$ for all $n \in B \setminus A$. In (a) and (c) we used the fact that $K_s = 1$ and in (b) we used Property A. $\square$

The following result can be proved in an analogous way.

Proposition 3.2 *If $(e_n)_{n=1}^{\eta}$ is a C-greedy basis ($C > 1$) for a Banach space X, then for each $x \in X$ we have*

$$C^{-1}\Big\|\sum_{n=1}^{\eta}\theta_{\pi(n)}e^*_n(x)e_{\pi(n)}\Big\| \leq \Big\|\sum_{n=1}^{\eta}e^*_n(x)e_n\Big\| \leq C\Big\|\sum_{n=1}^{\eta}\theta_{\pi(n)}e^*_n(x)e_{\pi(n)}\Big\|.$$

for any greedy permutation π of x and any choice of sings (θ_k) so that $\theta_{\pi(n)} = 1$ if $n \notin M^*_\pi(x)$.

Proposition 3.3 *Let $(e_n)_{n=1}^{\infty}$ be a basis of a Banach space X. If $(e_n)_{n=1}^{\infty}$ is 1-greedy, then it is 1-superdemocratic.*

Proof: Given $m \in \mathbb{N}$, let A and B be any two subsets of $\mathbb{N}$ of cardinality m. We want to prove that for any choice of signs (ϵ_k) and (θ_k) we have

$$\Big\|\sum_{k\in A}\epsilon_k e_k\Big\| = \Big\|\sum_{k\in B}\theta_k e_k\Big\|.$$

But, if we pick a subset C of integers of cardinality m which is disjoint with both A and B, using Property A twice we obtain

$$\Big\|\sum_{k\in A}\epsilon_k e_k\Big\| = \Big\|\sum_{k\in C}e_k\Big\| = \Big\|\sum_{k\in B}\theta_k e_k\Big\|.$$

$\square$

Let us recall that in section 2 we saw an example of a finite-dimensional Banach space with a 1-greedy basis which is not 1-superdemocratic.

Proposition 3.4 *Suppose $(e_n)_{n=1}^{\infty}$ is a basis for a Banach space $(X, \|\cdot\|_X)$. For $1 \leq p < \infty$, let $Y = X \oplus_p \mathbb{R}$ endowed with the norm*

$$\|(x,\alpha)\|_Y = \big(\|x\|_X^p + |\alpha|^p\big)^{1/p}, \qquad x \in X, \alpha \in \mathbb{R}.$$

Denote $(y_n)_{n=0}^{\infty}$ the natural basis in Y: $\big((0,1),(e_1,0),(e_2,0),\dots\big)$. If (y_n) is 1-greedy then (e_n) is isometrically isomorphic to the canonical ℓ_p-basis.

PROOF: Pick any $N \in \mathbb{N}$ and any linear combination $\sum_{n=1}^{N} \alpha_n e_n$. Without loss of generality we will asume that $|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_N|$. Then, using the fact that (y_n) has Property A, we have

$$\begin{aligned}
\Big\| \sum_{n=1}^{N} \alpha_n e_n \Big\|_X &= \Big\| \Big(\sum_{n=1}^{N} \alpha_n e_n, 0 \Big) \Big\|_Y \\
&= \Big\| \alpha_1 (0,1) + \sum_{n=2}^{N} \alpha_n e_n \Big\|_Y \\
&= \Big\| \Big(\sum_{n=2}^{N} \alpha_n e_n, \alpha_1 \Big) \Big\|_Y \\
&= \big(\| \sum_{n=2}^{N} \alpha_n e_n \|_X^p + |\alpha_1|^p \big)^{1/p}.
\end{aligned}$$

Next we would play the same trick with the norm in X of $\sum_{n=2}^{N} \alpha_n e_n$. After N steps we would obtain

$$\Big\| \sum_{n=1}^{N} \alpha_n e_n \Big\|_X = \big(|\alpha_1|^p + \cdots + |\alpha_N|^p \big)^{1/p}.$$

□

The next two results can be shown in the same fashion and we omit their proof.

Proposition 3.5 *Let X be a Banach space with a basis $(x_n)_{n=1}^{\infty}$ and let $1 \leq p < \infty$. Consider the Banach space $Y = X \oplus_p \ell_p$ with the natural basis $(y_n)_{n=1}^{\infty} = \big((x_1, 0), (0, e_1), (x_2, 0), (0, e_2), \dots\big)$, where (e_n) denotes the unit vector basis of ℓ_p. If (y_n) is 1-greedy then (x_n) is isometrically equivalent to (e_n).*

Proposition 3.6 *Let $(X, \|\cdot\|)$ be a Banach space with a basis $(x_n)_{n=1}^{\infty}$. Consider the space $Y = X \oplus_1 X$ endowed with the norm*

$$\|(x_1, x_2)\|_Y = \|x_1\| + \|x_2\|.$$

The sequence $(y_n)_{n=1}^{\infty} = \big((x_1, 0), (0, x_1), (x_2, 0), (0, x_2), \dots\big)$ is a basis for Y. If (y_n) is 1-greedy then (x_n) is isometrically equivalent to the canonical ℓ_1-basis.

4 Property B

Definition A normalized basis $(e_n)_{n=1}^{\infty}$ of a Banach space X is said to have *Property B* if whenever $\{e_{n_1}, \cdots, e_{n_m}\}$ and $\{e_{k_1}, \cdots, e_{k_m}\}$ are two finite sets of basic elements of the same cardinality satisfying

$$\|\alpha_1 e_{n_1} + \cdots + \alpha_m e_{n_m}\| = \|\epsilon_1 \alpha_1 e_{k_1} + \cdots + \epsilon_m \alpha_m e_{k_m}\|$$

for some scalars $(\alpha_i)_{i=1}^m$ and signs $(\epsilon_k)_{k=1}^m$, then the equality

$$\|\sum_{i=1}^m \alpha_i e_{n_i} + \lambda e_s\| = \|\sum_{i=1}^m \epsilon_i \alpha_i e_{k_i} + \lambda e_s\|,$$

holds for any scalar λ such that $|\lambda| \geq \max_{1 \leq i \leq m} |\alpha_i|$ and any integer $s \notin \{n_1, \cdots, n_m\} \cup \{k_1, \cdots, k_m\}$.

Theorem 4.1 *A basis (e_n) is 1-symmetric if and only if (e_n) has both Property A and Property B.*

Proof: Clearly if (e_n) is 1-symmetric, it is 1-greedy (Theorem 2.5) and has Property B. To prove the converse, we must see that for any integer N and scalars (α_n) we have

$$\Big\|\sum_{n=1}^N \alpha_n e_n\Big\| = \Big\|\sum_{n=1}^N \varepsilon_n \alpha_n e_{\pi(n)}\Big\|, \tag{4}$$

for any $1-1$ map $\pi : \{1, \ldots, N\} \to \mathbb{N}$ and any signs $(\varepsilon_n)_{n=1}^N$.

For the sake of symplicity in the argument, without loss of generality we will assume $|\alpha_1| \geq \cdots \geq |\alpha_N|$. Given $N \in \mathbb{N}$ and π, let us pick an integer $s > \max\{\pi(1), \pi(2), \cdots, \pi(N)\}$. Since $\|\alpha_N e_N\| = \|\varepsilon_N \alpha_N e_{\pi(N)}\|$ and $|\alpha_{N-1}| \geq |\alpha_N|$, using Property B we get

$$\|\alpha_N e_N + \alpha_{N-1} e_s\| = \|\varepsilon_N \alpha_N e_{\pi(N)} + \alpha_{N-1} e_s\|. \tag{5}$$

Now Property A yields, on the one hand

$$\|\alpha_N e_N + \alpha_{N-1} e_s\| = \|\alpha_N e_N + \alpha_{N-1} e_{N-1}\|$$

whereas, on the other hand,

$$\|\varepsilon_N \alpha_N e_{\pi(N)} + \alpha_{N-1} e_s\| = \|\varepsilon_N \alpha_N e_{\pi(N)} + \varepsilon_{N-1}\alpha_{N-1} e_{\pi(N-1)}\|.$$

Combining the last two equalities with equation (5) we get

$$\|\alpha_N e_N + \alpha_{N-1} e_{N-1}\| = \|\varepsilon_N \alpha_N e_{\pi(N)} + \varepsilon_{N-1}\alpha_{N-1} e_{\pi(N-1)}\|.$$

By repeating recurrently $N-1$ times this argument, we obtain (4).

□

5 Renorming

Suppose that (e_n) is a 1-greedy basis for a Banach space $(X, \|\cdot\|)$. By Theorem 1.1, (e_n) is unconditional with $K_s = 1$, and democratic with the democratic constant $= 1$. If we endow X with the equivalent *lattice norm*, defined for $x = \sum_{n=1}^{\infty} a_n e_n \in X$ by

$$\|x\|_\ell = \sup_{\theta_n = \pm 1} \Big\| \sum_{n=1}^{\infty} \theta_n a_n e_n \Big\|, \tag{6}$$

then (e_n) is unconditional in $(X, \|\cdot\|_\ell)$ with $K_{\mathrm{u}} = 1$, but one could expect the democracy constant of (e_n) in the new norm to increase. This is not the case and (e_n) remains 1-superdemocratic in $(X, \|\cdot\|_\ell)$. Indeed, for any $n \in \mathbb{N}$ and any $A \subset \mathbb{N}$ with $|A| = n$, taking into account the 1-superdemocracy of (e_n) in $(X, \|\cdot\|)$, we have

$$\Big\| \sum_{k\in A} \theta_k e_k \Big\|_\ell = \sup_{\epsilon_k = \pm 1} \Big\| \sum_{k\in A} \varepsilon_k \theta_k e_k \Big\| = \Big\| \sum_{k\in A} e_k \Big\|,$$

for any $(\theta_k)_{k\in A}$ signs. Actually we will show that (e_n) still is 1-greedy after renorming X with the norm in (6). In its proof we will use the following elementary lemma.

Lemma 5.1 *Let (e_n) be an unconditional basis for a Banach space X. Then, for each $x = \sum_{n=1}^{\infty} a_n e_n \in X$ there exists a sequence of signs (θ_n) (which depends on x) so that*

$$\|x\|_\ell = \Big\| \sum_{n=1}^{\infty} \theta_n a_n e_n \Big\|.$$

PROOF: It is easy to see that the map from the topological product space $\{-1,1\}^{\mathbb{N}}$ into X which assigns to each sequence of signs (θ_n) the vector $\sum_{n=1}^{\infty} \theta_n a_n e_n$ is continuous. Composing with the norm in X gives us a continuous map from $\{-1,1\}^{\mathbb{N}}$ into $\mathbb{R}$:

$$(\theta_n) \mapsto \Big\| \sum_{n=1}^{\infty} \theta_n a_n e_n \Big\|.$$

By compactness, there is a choice of signs (θ_n) where this map attains its maximum. $\square$

Proposition 5.2 *Let (e_n) be a 1-greedy basis for the Banach space $(X, \|\cdot\|)$. Then (e_n) is (1-unconditional and) 1-greedy in $(X, \|\cdot\|_\ell)$.*

PROOF: Take any $x = \sum_{k=1}^{\infty} a_n e_n \in X$. Without loss of generality we assume that the coefficients of x in absolute value are non-increasing (otherwise we work with the greedy ordering of x). Thus for each $m \in \mathbb{N}$,

$$\|x - \mathcal{G}_m(x)\|_\ell = \Big\| \sum_{n=m+1}^{\infty} a_n e_n \Big\|_\ell = \sup_{\pm 1} \Big\| \sum_{n=m+1}^{\infty} \pm a_n e_n \Big\| = \Big\| \sum_{n=m+1}^{\infty} \theta_n a_n e_n \Big\|, \tag{7}$$

where (θ_n) is the sequence of signs given by the previous lemma. Put

$$y = \sum_{n=1}^{\infty} \theta_n a_n e_n,$$

where $\theta_1 = \theta_2 = \cdots = \theta_m = 1$. Then $\mathcal{G}_m(y) = \sum_{n=1}^{m} a_n e_n$ and, since (e_m) is 1-greedy in $(X, \|\cdot\|)$, we have

$$\Big\| \sum_{n=m+1}^{\infty} \theta_n a_n e_n \Big\| = \|y - \mathcal{G}_m(y)\| \le \sigma_m^{\|\cdot\|}(y). \tag{8}$$

Now, for each set $B \subset \mathbb{N}$ of cardinality m,

$$\sigma_m^{\|\cdot\|}(y) \le \Big\| y - \sum_{k \in B} \theta_k a_k e_k \Big\| \le \Big\| \sum_{n=1}^{\infty} a_n e_n - \sum_{k \in B} a_k e_k \Big\|_\ell,$$

which implies

$$\sigma_m^{\|\cdot\|}(y) \leq \min\Big\{\Big\|\sum_{n=1}^{\infty} a_n e_n - \sum_{k\in B} a_k e_k\Big\|_{\ell} : B \subset \mathbb{N}, |B| = m\Big\} = \sigma_m^{\|\cdot\|_\ell}(x). \quad (9)$$

Combining (7), (8) and (9) we obtain

$$\|x - \mathcal{G}_m(x)\|_{\ell} \leq \sigma_m^{\|\cdot\|_\ell}(x),$$

i.e., (e_n) is 1-greedy in $(X, \|\cdot\|_{\ell})$. □

Analogously, if (e_n) is C-greedy in $(X, \|\cdot\|)$ and we equivalently renorm X with the lattice norm, then one may argue as above to show that, in fact, (e_n) is 1-unconditional and C-greedy in $(X, \|\cdot\|_{\ell})$.

A basic tool to analyze unconditional bases in $L_p(0,1)$ for $1 < p < \infty$ is provided by the following consequence of Khintchine's inequalities.

Proposition 5.3 *Let $1 < p < \infty$. If $(\psi_n)_{n=1}^{\infty}$ is an unconditional basis for $(L_p(0,1), \|\cdot\|_p)$ with biorthogonal functionals (ψ_n^*), then the expression*

$$|||f||| = \Big(\int_0^1 \big(\sum_{n=1}^{\infty} |\psi_n^*(f)|^2 |\psi_n(t)|^2\big)^{p/2}\, dt\Big)^{1/p}, \qquad f \in L_p(0,1),$$

gives a norm on $L_p(0,1)$ which is equivalent to the L_p-norm.

Let $(H_n^{(p)})_{n=1}^{\infty}$ be the Haar basis normalized in $L_p(0,1)$ for $1 \leq p < \infty$: $H_1^{(p)} = 1$ on $[0,1)$ and for $n = 2^k + s$, $k = 0, 1, 2, \cdots$, $s = 1, 2, \cdots, 2^k$,

$$H_n^{(p)}(t) = \begin{cases} 2^{k/p} & \text{if } t \in [\frac{2s-2}{2^{k+1}}, \frac{2s-1}{2^{k+1}}), \\ -2^{k/p} & \text{if } t \in [\frac{2s-1}{2^{k+1}}, \frac{2s}{2^{k+1}}), \\ 0 & \text{otherwise.} \end{cases}$$

For each $f = \sum_{n=1}^{\infty} a_n H_n^{(p)} \in L_p(0,1)$ $(1 < p < \infty)$ put

$$|||f||| = \Big(\int_0^1 \big(\sum_{n=1}^{\infty} |a_n|^2 |H_n^{(p)}(t)|^2\big)^{p/2}\, dt\Big)^{1/p}. \quad (10)$$

Then, as a particular case of the above Proposition, one obtains that equation (10) defines a norm on $L_p(0,1)$ that is equivalent to the L_p-norm. That is:

Proposition 5.4 *For each $1 < p < \infty$ there exists a constant $C(p)$ so that*

$$\Big\| \sum_{n=1}^{\infty} a_n H_n^{(p)} \Big\|_{L_p} \overset{C(p)}{\sim} \Big(\int_0^1 \big(\sum_{n=1}^{\infty} |a_n|^2 |H_n^{(p)}(t)|^2 \big)^{p/2} \, dt \Big)^{1/p}$$

for any sequence $(a_n) \in c_{00}$.

Sometimes it is convenient to describe the normalizeed Haar basis in $L_p(0,1)$ as a sequence of "layers" as follows. Let h_0^0 be the constant function 1. For $n \geq 0$ and $1 \leq k \leq 2^n$ we define h_k^n thus:

$$h_k^n(t) = \begin{cases} 2^{n/p} & \text{if } \; t \in [\frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}}), \\ -2^{n/p} & \text{if } \; t \in [\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}), \\ 0 & \text{otherwise} \end{cases} .$$

Proposition 5.5 *Let $1 < p < \infty$. For each $\varepsilon > 0$ there exists an increasing sequence $(n_i)_{i=1}^{\infty}$ of non-negative integers such that the "lacunary Haar system" $\big((h_j^{n_i})_{j=1}^{2^{n_i}}\big)_{i=0}^{\infty}$ is 1-unconditional, $(1+\varepsilon)$-democratic in $(L_p(0,1), |||\cdot|||)$ and the closed linear span of $\big((h_j^{n_i})_{j=1}^{2^{n_i}}\big)_{i=0}^{\infty}$ in $(L_p(0,1), |||\cdot|||)$ is isomorphic to $(L_p(0,1), \|\cdot\|_p)$.*

Proof: The proof relies basically on an idea that appeared in [4]. Given $\varepsilon > 0$, pick $\alpha \in \mathbb{N}$ such that

$$\frac{2^{\alpha/p}}{(2^{2\alpha/p}-1)^{1/2}} \cdot \frac{2^{\alpha/p}}{(2^{\alpha}-1)^{1/p}} \leq 1 + \varepsilon.$$

Consider the sequence (n_i) defined by $n_0 = 0$ and $n_{i+1} = n_i + \alpha$ and the subbasis of the Haar system

$$\mathcal{S}_\varepsilon = \big((h_j^{n_i})_{j=1}^{2^{n_i}}\big)_{i=0}^{\infty} = (h_0^0, \underbrace{h_1^{n_1}, h_2^{n_1}, \ldots, h_{2^{n_1}}^{n_1}}_{n_1^{\text{th}}-\text{layer}}, \underbrace{h_1^{n_2}, h_2^{n_2}, \ldots, h_{2^{n_2}}^{n_2}}_{n_2^{\text{th}}-\text{layer}}, \ldots).$$

Notice that for each $t \in [0,1)$, the non-zero values of the functions $|h_k^{n_i}(t)|^p$, $i = 1, 2, \ldots$ belong to a geometric progression of ratio 2^α. If A is any finite subset of $\mathcal{S}_\varepsilon$, put

$$M(t) = \max\big\{n_i : \; t \in \operatorname{supp} h_k^{n_i}, h_k^{n_i} \in A\big\},$$

and let $M(t) = -\infty$ if $t \notin \bigcup_{h_k^{n_i} \in A} h_k^{n_i}$. Thus for each $t \in [0,1)$ we see that

$$\sum_{\{h_k^{n_i} \in A\}} |h_k^{n_i}(t)|^p \leq 2^{M(t)} \sum_{i=0}^{\infty} \Big(\frac{1}{2^\alpha}\Big)^i = \frac{2^\alpha}{2^\alpha - 1} \cdot 2^{M(t)},$$

hence

$$2^{M(t)} \geq \frac{2^\alpha - 1}{2^\alpha} \sum_{\{h_k^{n_i} \in A\}} |h_k^{n_i}(t)|^p.$$

Now,

$$\begin{aligned}
\int_0^1 \Big(\sum_{\{h_k^{n_i} \in A\}} |h_k^{n_i}(t)|^2 \Big)^{p/2} dt &\geq \int_0^1 2^{M(t)}\, dt \\
&\geq \frac{2^\alpha - 1}{2^\alpha} \int_0^1 \sum_{\{h_k^{n_i} \in A\}} |h_k^{n_i}(t)|^p \, dt \\
&= \frac{2^\alpha - 1}{2^\alpha} |A|.
\end{aligned}$$

Therefore we obtain

$$\Big|\Big|\Big| \sum_{\{h_k^{n_i} \in A\}} h_k^{n_i} \Big|\Big|\Big| \geq \Big(\frac{2^\alpha - 1}{2^\alpha}\Big)^{1/p} |A|^{1/p}. \tag{11}$$

On the other hand, for each $t \in [0,1)$ we have

$$\sum_{\{h_k^{n_i} \in A\}} |h_k^{n_i}(t)|^2 \leq \big(2^{\frac{M(t)}{p}}\big)^2 \sum_{j=0}^{\infty} \Big(\frac{1}{2^{2\alpha/p}}\Big)^j = 2^{\frac{2M(t)}{p}} \frac{2^{2\alpha/p}}{2^{2\alpha/p} - 1}.$$

Then,

$$\begin{aligned}
\int_0^1 \Big(\sum_{\{h_k^{n_i} \in A\}} |h_k^{n_i}(t)|^2 \Big)^{p/2} &\leq \frac{2^\alpha}{(2^{2\alpha/p} - 1)^{p/2}} \int_0^1 2^{M(t)}\, dt \\
&\leq \frac{2^\alpha}{(2^{2\alpha/p} - 1)^{p/2}} \int_0^1 \sum_{\{h_k^{n_i} \in A\}} |h_k^{n_i}(t)|^p \, dt \\
&= \frac{2^\alpha}{(2^{2\alpha/p} - 1)^{p/2}} |A|.
\end{aligned}$$

Thus we obtain

$$\Big|\Big|\Big| \sum_{\{h_k^{n_i} \in A\}} h_k^{n_i} \Big|\Big|\Big| \le \frac{2^{\alpha/p}}{(2^{2\alpha/p}-1)^{1/2}} |A|^{1/p}. \tag{12}$$

So given any other set $B \subset \mathcal{S}_\varepsilon$ such that $|B| = |A|$, equations (11) and (12) yield

$$\begin{aligned}\Big|\Big|\Big| \sum_{\{h_m^{n_j} \in B\}} h_m^{n_j} \Big|\Big|\Big|^p &\le \frac{2^{\alpha}}{(2^{2\alpha/p}-1)^{p/2}} |B| \\ &\le \frac{2^{\alpha}}{(2^{2\alpha/p}-1)^{p/2}} \frac{2^{\alpha}}{2^{\alpha/p}-1} \Big|\Big|\Big| \sum_{\{h_k^{n_i} \in A\}} h_k^{n_i} \Big|\Big|\Big|^p \\ &\le (1+\varepsilon)^p \Big|\Big|\Big| \sum_{\{h_k^{n_i} \in A\}} h_k^{n_i} \Big|\Big|\Big|^p.\end{aligned}$$

The last statement of the Proposition follows from Gamlen and Gaudet's theorem [3] and from the equivalence of norms given by Proposition 5.4. □

Let us recall that a (real) *quasi-Banach space* X is a complete metrizable vector space whose topology is given by a quasi-norm on X. That is, a map $|||\cdot||| : X \to \mathbb{R}$, $x \mapsto |||x|||$, satisfying

(i) $|||x||| > 0 \quad (x \in X, x \neq 0)$

(ii) $|||\alpha x||| = |\alpha| |||x||| \quad (\alpha \in \mathbb{R}, x \in X)$

(iii) $|||x_1 + x_2||| \le C(|||x_1||| + |||x_2|||) \quad (x_1, x_2 \in X)$,

where C is a constant independent of x_1 and x_2. If $C = 1$, X is a Banach space.

Theorem 5.6 *Let $(X, \|\cdot\|)$ be a Banach space. Suppose that (e_n) is a C-greedy basis ($C > 1$) for X. Then X can be equipped with an equivalent quasi-norm $|||\cdot|||$ so that (e_n) is a 1-greedy basis for the quasi-Banach space $(X, |||\cdot|||)$.*

Proof: For $x = \sum_{n=1}^{\infty} a_n e_n \in X$ put

$$|||x||| = \sup \Big\| \sum_{n=1}^{\infty} \varepsilon_n a_n e_{\pi(n)} \Big\|,$$

the supremum being taken over all possible choices of signs $\varepsilon = (\varepsilon_n)$ and all greedy permutations π of x. Obviously $\|x\| \leq |||x|||$ for all $x \in X$. For each ε and π fixed we have,

$$\begin{aligned}\Big\|\sum_{n=1}^{\infty}\varepsilon_n a_n e_{\pi(n)}\Big\| &\leq \Big\|\sum_{n\in M_\pi^*(x)} a_n e_n\Big\| + \Big\|\sum_{n\notin M_\pi^*(x)} \varepsilon_n a_n e_{\pi(n)}\Big\| \\ &\leq C\Big\|\sum_{n\in M_\pi^*(x)} a_n e_n\Big\| + \Big\|\sum_{n\notin M_\pi^*(x)} \varepsilon_n a_n e_n\Big\| \\ &\leq C^2\Big\|\sum_{n=1}^{\infty} a_n e_n\Big\| + 2C\Big\|\sum_{n=1}^{\infty} a_n e_n\Big\|\end{aligned}$$

Therefore

$$|||x||| \leq (C^2 + 2C)\|x\| \quad \text{for all } x \in X. \tag{13}$$

It is immediate to see that $|||x||| > 0$ if $x \neq 0$ and that $|||\alpha x||| = |\alpha||||x|||$ for all $\alpha \in \mathbb{R}$ and $x \in X$. Finally, combining the convexity of the norm $\|\cdot\|$ with (13), we have

$$|||x+y||| \leq (C^2+2C)\|x+y\| \leq (C^2+2C)(\|x\|+\|y\|) \leq (C^2+2C)(|||x|||+|||y|||),$$

which shows that $|||\cdot|||$ is a quasi-norm on X. $\square$

6 Examples

6.1 1-Unconditional + 1-Democratic does not imply 1-Greedy

Let X be the set of all real sequences $x = (x_1, x_2, \dots) \in \ell_2$ such that

$$\|x\|_1 = \sum_{n=1}^{\infty} \frac{|x_n|}{\sqrt{n}}$$

is finite. Taking into account (we will see below why) that the inequality

$$\frac{1}{2}\sum_{n=1}^{N} \frac{1}{\sqrt{n}} \leq \sqrt{N} \tag{14}$$

holds for all $N \in \mathbb{N}$, we define on X the norm given by:

$$\|x\| = \max\Big\{\|x\|_{\ell_2}, \frac{1}{2}\|x\|_1\Big\}.$$

Then $(X, \|\cdot\|)$ is a Banach space. Let $e_n \in X$, $n = 1, 2, \ldots$, be the vector whose k-th coordinate is 1 if $n = k$ and 0 otherwise. Denote by X_0 the subspace of X generated by $(e_n)_{n=1}^{\infty}$.

It is easy to see that (e_n) is a 1-unconditional basis for X_0.

On the other hand, given any subset $A \subset \mathbb{N}$, we have

$$\Big\|\sum_{k\in A} e_k\Big\|_1 \le \Big\|\sum_{k=1}^{|A|} e_k\Big\|_1 = \sum_{k=1}^{|A|} \frac{1}{\sqrt{k}},$$

which implies, using (14), that

$$\Big\|\sum_{k\in A} e_k\Big\| = \Big\|\sum_{k\in A} e_k\Big\|_{\ell_2} = |A|^{1/2},$$

hence (e_n) is 1-democratic. In fact, (e_n) is 1-superdemocratic.

Let us show that (e_n) does not have Property A. Pick $n \in \mathbb{N}$ such that

$$\frac{1}{2}\Big(1 + \frac{1}{2} + \cdots \frac{1}{n}\Big) > \sqrt{1 + \frac{1}{2} + \cdots \frac{1}{n}}. \tag{15}$$

Then,

$$\|(1, \frac{1}{\sqrt{2}}, \cdots, \frac{1}{\sqrt{n}}, 0, \cdots)\| = \frac{1}{2}\Big(1 + \frac{1}{2} + \cdots \frac{1}{n}\Big),$$

whereas

$$\begin{aligned}\|(0, \frac{1}{\sqrt{2}}, \cdots, \frac{1}{\sqrt{n}}, 1, 0, \cdots)\| &= \max\Big\{\frac{1}{2}\Big(\frac{1}{2} + \cdots \frac{1}{n} + \frac{1}{\sqrt{n+1}}\Big), \sqrt{1 + \frac{1}{2} + \cdots \frac{1}{n}}\Big\} \\ &\neq \|(1, \frac{1}{\sqrt{2}}, \cdots, \frac{1}{\sqrt{n}}, 0, \cdots)\|.\end{aligned}$$

6.2 1-Subsymmetric does not imply 1-Greedy

Let us recall that a basis (e_n) is *subsymmetric* if it is unconditional and for every increasing sequence of integers $\{n_i\}_{i=1}^{\infty}$, the subbasis $(e_{n_i})_{i=1}^{\infty}$ is equivalent to (e_n). The *subsymmetry constant* of (e_n) is the smallest constant

$C \geq 1$ such that given any scalars $(a_i) \in c_{00}$, we have

$$\|\sum_{i=1}^{\infty} \theta_i a_i e_{n_i}\| \leq C\|\sum_{i=1}^{\infty} a_i e_i\|$$

for all sequences of signs (θ_i) and all increasing sequences of integers $\{n_i\}_{i=1}^{\infty}$. In this case we say that (e_n) is *C-subsymmetric.*

Since a 1-subsymmetric basis (e_n) is 1-unconditional and 1-democratic, by Theorem 1.1 it follows that (e_n) is greedy with greedy constant ≤ 2.

The following example, in combination with Theorem 3.1, shows that a 1-subsymmetric basis need not be 1-greedy. It is interesting to point out here that this was precisely the first counterexample that showed that a subsymmetric basis need not be symmetric (see [2]). Let $(X, \|\cdot\|)$ be the Banach space of all sequences of scalars $x = (x_1, x_2, \cdots)$ for which

$$\|x\| = \sup \sum_{i=1}^{\infty} \frac{|x_{n_i}|}{\sqrt{i}} < \infty,$$

the supremum being taken over all increasing sequences of integers $\{n_i\}_{i=1}^{\infty}$.

The unit vectors (e_i) form a 1-subsymmetric basis of X, but (e_i) fails to be 1-greedy because it does not have Property A. Indeed, take $x = (1, \frac{1}{\sqrt{2}}, 0, 0 \cdots)$ and, for instance, the greedy permutation of x given by $\pi(1) = 3, \pi(2) = 2$. Then, $\|(1, \frac{1}{\sqrt{2}}, 0, 0 \cdots)\| = 1 + \frac{1}{2}$ whereas $\|(0, \frac{1}{\sqrt{2}}, 1, 0 \cdots)\| = \sqrt{2}$.

6.3 Greedy does not imply Subsymmetric

It was proved in [7] that for $1 < p < \infty$, $(H_n^{(p)})_{n=1}^{\infty}$ is a greedy basis in $L_p(0,1)$ with a greedy constant strictly bigger than 1 (unless for $p = 2$ that the greedy constant is $= 1$). Clearly $(H_n^{(p)})_{n=1}^{\infty}$ is not subsymmetric since if we consider $n_k = 2^{k+1} - 1$, $k = 1, 2, \cdots$, then the subbasis $(H_{n_k}^{(p)})_{k=1}^{\infty}$ is isometrically isomorphic to ℓ_p, which is not isomorphic to $L_p(0,1)$.

6.4 1-Greedy does not imply 1-Symmetric

We are going to construct the unit ball of an n-dimensional Banach space as follows. For each $i = 1, 2, \cdots, n$, let $\mathcal{E}_i$ denote the Euclidean unit ball in the hiperplane $H_i = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i = 0\}$, i.e.,

$$\mathcal{E}_i = \Big\{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : x_i = 0 \text{ and } \|x\|_2 = \big(\sum_{j=1}^n |x_j|^2\big)^{1/2} \leq 1\Big\},$$

and let $\mathcal{E}$ be the Euclidean unit ball in $\mathbb{R}^n$. We define the set $\mathcal{A}$ to be

$$\mathcal{A} = \Big\{a = (x_1, \cdots, x_n) \in \mathbb{R}^n : x_1 \geq x_2 \geq \cdots \geq x_n \geq 0 \text{ and } \sum_{j=1}^n |x_j|^2 \leq 1\Big\}.$$

Now, for each different choice of signs $\theta^{(j)} = (\theta_1^{(j)}, \dots, \theta_n^{(j)})$, $j = 1, \dots, 2^n$, put

$$\mathcal{A}_j = \big\{a = (\theta_1^{(j)} x_1, \dots, \theta_n^{(j)} x_n) : (x_i)_{i=1}^n \in \mathcal{A}\big\}.$$

Let us observe that all of the sets $\mathcal{E}_i$'s and $\mathcal{A}_j$'s are convex. Finally, put

$$\mathcal{S} = \big(\bigcup_{i=1}^n \mathcal{E}_i\big) \cup \big(\bigcup_{j=1}^{2^n} \mathcal{A}_j\big).$$

Let

$$\mathcal{B} = \mathrm{co}(\mathcal{S}),$$

the convex hull of $\mathcal{S}$, and let $\|\cdot\|_{\mathcal{B}}$ denote the Minkowski functional of $\mathcal{B}$. $X = (\mathbb{R}^n, \|\cdot\|_{\mathcal{B}})$ is a Banach space. We will prove that the unit vector basis $(e_i)_{i=1}^n$ of X is 1-greedy but it is not 1-symmetric. First we make a few geometric remarks that we will be using in the sequel. Notice that $\mathcal{B}$ consists of all sums $\sum_{j=1}^{2^n} \lambda_j a_j + \sum_{i=1}^n \mu_i b_i$ such that $a_j \in \mathcal{A}_j$ for $j = 1, \dots, 2^n$, $b_i \in \mathcal{E}_i$ for $i = 1, \dots, n$ and $\lambda_1, \dots, \lambda_{2^n}, \mu_1, \dots, \mu_n$ are non-negative real numbers satisfying $\sum_{j=1}^{2^n} \lambda_j + \sum_{i=1}^n \mu_i \leq 1$. It then follows that for $x \in \mathbb{R}^n$

$$\|x\|_{\mathcal{B}} = \inf\Big\{\sum_{j=1}^{2^n} \lambda_j + \sum_{i=1}^n \mu_i : \lambda_j \geq 0, \mu_i \geq 0 \text{ and } x = \sum_{j=1}^{2^n} \lambda_j a_j + \sum_{i=1}^n \mu_i b_i\Big\}. \tag{16}$$

Moreover, it is an easy consequence of the compactness of the sets $\mathcal{A}_j$ and $\mathcal{E}_i$ that this infimum is attained.

Let us see that $(e_k)_{k=1}^n$ is 1-unconditional. Given $\epsilon = (\epsilon_k)_{k=1}^n$, where $\epsilon_k = \pm 1$, let T_ϵ be the map $\sum_{k=1}^n x_k e_k \to \sum_{k=1}^n \epsilon_k x_k e_k$ from X to X. Each such map T_ϵ is linear. We must prove that $\sup_\epsilon \|T_\epsilon\| \leq 1$. Since $\mathcal{B} = \mathrm{co}(\mathcal{S})$, it suffices to show that $T_\epsilon(\mathcal{S}) \subset \mathcal{S}$ for each ϵ. But if $x \in \mathcal{S}$, then either $x \in \mathcal{E}_i$ for some i, in which case $T_\epsilon(x) \in \mathcal{E}_i \subset \mathcal{S}$, or $x \in \mathcal{A}_j$ for some $1 \leq j \leq 2^n$, in which case $T_\epsilon(x) \in \mathcal{A}_{j'} \subset \mathcal{S}$.

To check that $(e_k)_{k=1}^n$ has Property A, let us pick $x = (x_1, \ldots, x_n)$ a vector of the unit ball of X so that at least one of its coordinates is zero (otherwise there is nothing to prove). That is, x belongs to the intersection of $\mathcal{B}$ with at least one hyperplane H_i. Since $\mathcal{S} \subset \mathcal{E}$, it follows that $\mathcal{B}$, the convex hull of $\mathcal{S}$, is also contained in $\mathcal{E}$, hence $\mathcal{B} \cap H_i \subset \mathcal{E} \cap H_i = \mathcal{E}_i$. We conclude that $\mathcal{B} \cap H_i = \mathcal{E}_i$. We will prove that if $x \in \mathcal{E}_i$ then $\|x\|_2 = \|x\|_{\mathcal{B}}$. It may be assumed that $\|x\|_2 = 1$. Taking into account the expression of $\|\cdot\|_{\mathcal{B}}$ given in (16) and the fact that $x \in \mathcal{E}_i$, we deduce that $\|x\|_{\mathcal{B}} \leq 1$. Suppose that $\|x\|_{\mathcal{B}} < 1$. Pick a representation of x,

$$x = \sum_{j=1}^{2^n} \bar{\lambda}_j \bar{a}_j + \sum_{i=1}^n \bar{\mu}_i \bar{b}_i$$

such that $\bar{\lambda}_j \geq 0$, $\bar{\mu}_i \geq 0$, $\bar{a}_j \in A_j$, $\bar{b}_i \in \mathcal{E}_i$ and

$$\|x\|_{\mathcal{B}} = \sum_{j=1}^{2^n} \bar{\lambda}_j + \sum_{i=1}^n \bar{\mu}_i.$$

Hence we would have

$$\begin{aligned} 1 = \|x\|_2 &= \Big\| \sum_{j=1}^{2^n} \bar{\lambda}_j \bar{a}_j + \sum_{i=1}^n \bar{\mu}_i \bar{b}_i \Big\| \\ &\leq \sum_{j=1}^{2^n} \bar{\lambda}_j \|\bar{a}_j\|_2 + \sum_{i=1}^n \bar{\mu}_i \|\bar{b}_i\|_2 \\ &\leq \sum_{j=1}^{2^n} \bar{\lambda}_j + \sum_{i=1}^n \bar{\mu}_i \\ &< 1. \end{aligned}$$

Therefore to evaluate the $\|\cdot\|_{\mathcal{B}}$-norm of an element of $\mathcal{E}_i$ we can use its $\|\cdot\|_2$-norm. Now, if π is a greedy permutation of x, the vector $x_\pi = (x_{\pi(1)}, \ldots, x_{\pi(n)})$ belongs to $\mathcal{E}_k$ for some $1 \le k \le n$ and

$$\|x_\pi\|_{\mathcal{B}} = \|x_\pi\|_2 = \|x\|_2 = \|x\|_{\mathcal{B}}.$$

By Theorem 3.1, $(e_k)_{k=1}^n$ is 1-greedy.

It remains to be proved that $(e_k)_{k=1}^n$ is not 1-symmetric. We will see that there exist vectors $x = (x_1, \ldots, x_n) \in X$ with $\|x\|_{\mathcal{B}} = 1$ such that for some permutation π, the norm of $(x_{\pi(1)}, \ldots, x_{\pi(n)})$ is strictly bigger than 1. Let us take $x = (x_1, \ldots, x_n) \in \mathcal{A}$ such that $x_1 > x_2 > \cdots > x_n > 0$ and $\|x\|_2 = 1$. Since $\mathcal{A} \subset \mathcal{B} \subset \mathcal{E}$, it follows that

$$1 = \|x\|_2 \le \|x\|_{\mathcal{B}} \le 1,$$

hence $\|x\|_{\mathcal{B}} = 1$.
Now consider $x' = (x_n, x_{n-1}, \ldots, x_2, x_1)$. Obviously, $\|x'\|_2 = 1$. We aim to show that $\|x\|_{\mathcal{B}} > 1$. Suppose the contrary. Then, since $\|x'\|_{\mathcal{B}}$ cannot be strictly less than 1, the only option is $\|x'\|_{\mathcal{B}} = 1$.
We choose a representation of x' where its $\|\cdot\|_{\mathcal{B}}$-norm is attained,

$$x' = \sum_{j=1}^{2^n} \bar{\lambda}_j \bar{a}_j + \sum_{i=1}^{n} \bar{\mu}_i \bar{b}_i \quad \text{and} \quad 1 = \sum_{j=1}^{2^n} \bar{\lambda}_j + \sum_{i=1}^{n} \bar{\mu}_i.$$

Clearly, in the above representation it must be $\|\bar{a}_j\|_2 = 1 = \|\bar{b}_i\|_2$ for all $j = 1, \ldots, 2^n$ and all $i = 1, \ldots, n$. This way we have a vector in the Euclidean unit sphere of $\mathbb{R}^n$ written down as a convex combination of vectors in the Euclidean unit sphere of $\mathbb{R}^n$ as well. Using the strict convexity (or rotundity) of $\mathcal{E}$ we infer that $\bar{a}_j = \bar{b}_i = x'$, which is impossible by our choice of x'.

Let us notice that this counterexample is finite-dimensional in nature. Indeed, as n grows larger the basis becomes more and more symmetric. Let $x = (x_1, \ldots, x_n) \in X$ such that $\|x\|_{\mathcal{B}} = 1$. At least one of the coordinates of x, say x_n, is $\le \frac{1}{\sqrt{n}}$. Then, given any permutation $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, we have

$$\begin{aligned}
\|x_1 e_{\pi(1)} + \cdots + x_n e_{\pi(n)}\|_{\mathcal{B}} &\le \|x_1 e_{\pi(1)} + \cdots + x_{n-1} e_{\pi(n-1)}\|_{\mathcal{B}} + \|x_n e_{\pi(n)}\|_{\mathcal{B}} \\
&\le \|x_1 e_1 + \cdots + x_{n-1} e_{n-1}\|_{\mathcal{B}} + \frac{1}{\sqrt{n}} \\
&\le \|x\|_{\mathcal{B}} + \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{n}}.
\end{aligned}$$

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