The Dollar Auction with Spiteful Players
Additional materials

Proof of Lemma 2

Consider a player i with spite coefficient α_i. Out of all states such that $x_i \geq b - s$, $x_j \geq b - s$ and j wins the auction, i achieves the highest possible utility when:

- j pays the highest possible price of b in order to get the stake; and
- i makes the lowest possible bid of $b - s$.

Hence, the highest possible utility of i in the losing states under consideration is:

$$\hat{u}_i = \alpha_i(b - s) + (1 - \alpha_i)(s - b) = (2\alpha_i - 1)(b - s).$$

Out of all states such that $x_i \geq b - s$, $x_j \geq b - s$ and i wins the auction, i achieves the lowest possible utility when:

- i pays the highest possible price of b in order to get the stake; and
- j makes the lowest possible bid of $b - s$.

Hence, the lowest possible utility of i in considered winning states is

$$\hat{U}_i = \alpha_i(b - s) + (1 - \alpha_i)(s - b) = (2\alpha_i - 1)(b - s).$$

Therefore, the highest possible utility of any player i in the losing states under consideration is equal to the lowest possible utility of any player i in the winning states under consideration. From this, we conclude that ending the auction whenever possible in a winning state with a bid of b is always better than giving the opponent an opportunity to achieve greater utility by ending the game in her winning state.

Proof of Lemma 3

First, we focus on the case in which $x_i < b - s \vee x_j < b - s$. Assume by contradictory that $u_i(x_i, x_j) > (2\alpha_i - 1)(b - s)$ and $u_j(x_i, x_j) \geq (2\alpha_j - 1)(b - s)$. In that case by adding both inequalities we have:

$$u_i(x_i, x_j) + u_j(x_i, x_j) > 2(\alpha_i + \alpha_j - 1)(b - s).$$

Without the loss of generality assume that $x_i > x_j$ (the proof for the symmetric case is analogous). We now know that $x_j < b - s$. After expanding the utility formulas we get that:

$$(\alpha_i + \alpha_j - 1)(x_i + x_j - 2b + s) > 0.$$

The same can be rewritten as:

$$(\alpha_i + \alpha_j - 1)((x_i - b) + (x_j - (b - s)) > 0.$$

However, since $x_i \leq b$, $x_j < b - s$ and $\alpha_i + \alpha_j - 1 \geq 0$ (as they are both strongly spiteful players) then the left hand side of the inequality cannot be positive. Therefore, the assumption that $u_i(x_i, x_j) > (2\alpha_i - 1)(b - s)$ and $u_j(x_i, x_j) \geq (2\alpha_j - 1)(b - s)$ has to be false.

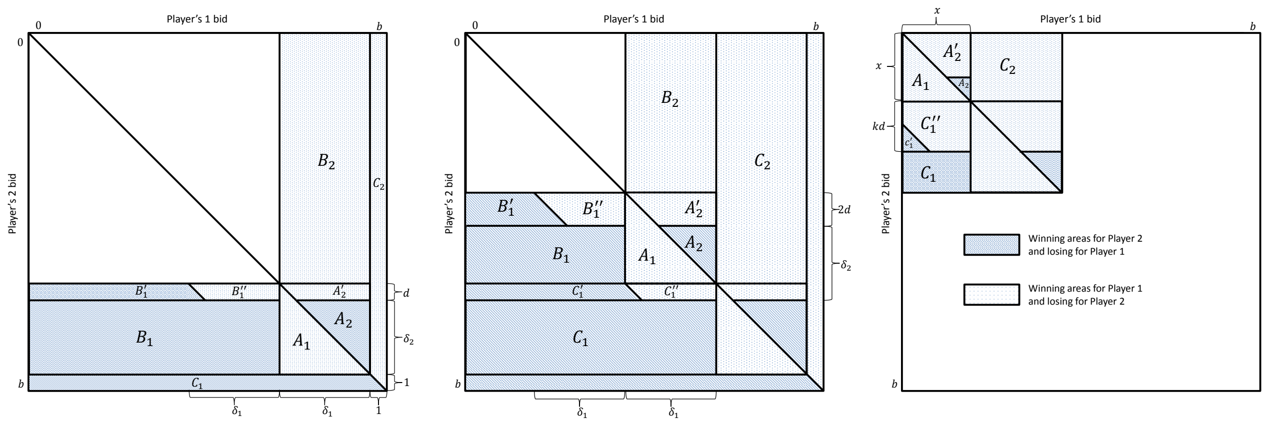
Now assume that $x_i \geq b - s \wedge x_j \geq b - s$. In the proof of Lemma 2, we show that $(2\alpha_i - 1)(b - s)$ is the minimal utility in the winning states and the maximal utility in the losing states for player i. Therefore higher utility can be achieved by i only in the winning states. However, those are the losing states for j, so by symmetric argument the maximal utility in those states for her is $(2\alpha_j - 1)(b - s)$. From Lemma 2, we also know that for $x_i \geq b - s \wedge x_j \geq b - s$ the auction can end only with either $x_i = b$ or $x_j = b$. Therefore, the utility of player i higher than $(2\alpha_i - 1)(b - s)$ can be only achieved for $x_i = b$ and $x_j > b - s$. However, player j would never bid higher than $b - s$ but lower than b.

Proof of Theorem 3

Our proof is the extension of the proof presented by O'Neill for the case of two non-spiteful players. Consider the graph-based representation of the dollar auction, discussed in Section and illustrated in Figure 1.

We call node (x_1, x_2) a *winning node* for player i if, by starting a game from this node, player i is guaranteed to eventually win the stake s without having to raise her bid by more than δ_i. Note that it is irrational for player i to raise her bid by more than δ_i, as, from the definition of the maximal preserving increase, this would not bring her any higher utility and it would be better for her to pass. Finally, recall that if one player bids all her budget, the game ends.

We will now analyse each node of the graph to determine whether it is a winning node for player i or for player j. Figure 1a presents states where at least one bid is close to the whole budget. All figures assume that player 1 is player i, *i.e.*, the player with δ_i no lower than δ_j. The width of areas C_1 and C_2 is one node, while the width of areas B_2 and A_1



(a) The states at the end of the auction (b) Backtracking and expanding (c) The states at the beginning of the auction

Figure 1: Winning and losing areas during different moments of the auction.

is δ_i nodes each. The height of areas B_1 and A_2 is δ_j nodes, while the height of areas B_1', B_1'' and A_2' is d nodes each.

Any node in C_1 is a winning one for player 2, since player 1 cannot bid more than b and her only choice is to pass. An analogous analysis holds for C_2. Nodes in A_1 are winning for player 1, since she can end the auction and get the stake by raising her bid by δ_1 or less. Analogical analysis holds for A_2. For nodes in areas B_2 and A_2' any valid bid increments of 2 are either higher than δ_2 or lead to a state winning for 1. Therefore they are winning for player 1. Analogous analysis holds for areas B_1 and B_1'. that are winning for player 2. Finally, nodes in area B_1'' are winning for 1 because she can make a move from them to a winning area A_2'.

We now analyse states that are gradually closer to $(0, 0)$, as shown in Figure 1b. Previously analysed areas now play the role of areas C_1, C_1', C_1'' and C_2. The height of areas B_1 and A_2 is $\delta_j - d$ nodes each, as player 2 can make a move to C_1 (and avoid area C_1'') in order to get to a winning state. Therefore, B_1 and A_2 are winning areas for player 2. For the same reason the height of areas B_1', B_1'' and A_2' is now $2d$ nodes. Otherwise the previous analysis holds.

We can repeat this process until we reach an area with dimensions that are less or equal than $\delta_i \times \delta_i$, as illustrated in Figure 1c. Width (as well as height) of a left to analyse area is $b \; mod \; \delta_i$ nodes or δ_i nodes if this value is zero. This conditional value can be expressed as $x = (b-1) \; mod \; \delta_i + 1$, as defined in the theorem. In this case, previously analysed area C_2 is winning for player 1, while previously analysed areas C_1' and C_1 are winning for player 2.

Now, let us analyse what are the possible initial bids of player 1. The state corresponding to the lowest possible bid of player 1, *i.e.*, the state $(1, 0)$ is winning for 1 only if area A_2' is of a positive height. Height of area A_2' is $x - (\delta_j - kd)$, as area A_2 contains states where player 2 can get to area C_1 by making bid increment of at most δ_j. In other case, *i.e.*, when area A_2' is non-existing, player 1 has to make a bid of x in order to get to a winning state of hers. That gives us the initial bid of $w = 1$ if $x > \delta_j - kd$, and $w = x$ otherwise.

Let us consider now player 2. In order to get to a winning state of hers, she needs to move to area C_1 (the bid of $x + kd$) or to area C_1' (the bid of δ_i), whichever is smaller. That gives us the initial bid of $W = \min(x + kd, \delta_1)$. Alternatively, if she passes, the player 1 will make the bid of w. To maximize her utility, she should let player 1 do that only when $u_2(0, w) \geq u_2(W, 0)$. After expanding the utilities we get that this is true for:

$$(1 - \alpha_j)W \geq s - \alpha_j w.$$

Therefore, if this is the case, player 2 should let player 1 move first.

It might seem that this is the end of analysis, *i.e.*, the auction should end in either state $(w, 0)$ or $(0, W)$. However, if $w \geq s$ both of those states hold non-positive utility for both players. Therefore, it is optimal for them to never bid and let the auction end without conclusion. If any player would decide to leave this forced stalemate and make a bid lower than s, the other would bid her maximal preserving increase and bring her opponent to negative utility.