Universal algebra

Basics of universal algebra:

- signatures and algebras
- homomorphisms, subalgebras, congruences
- equations and varieties
- equational calculus
- equational specifications and initial algebras
- variations: partial algebras, first-order structures

Plus some hints on applications in

> *foundations of software semantics, verification, specification, development. . .*

TINY data type

Its *signature* Σ (syntax):

sorts $Int, Bool;$
opns $0, 1 \colon Int;$
$plus, times, minus \colon Int \times Int \to Int;$
$false, true \colon Bool;$
$lteq \colon Int \times Int \to Bool;$
$not \colon Bool \to Bool;$
$and \colon Bool \times Bool \to Bool;$

and Σ-*algebra* $\mathcal{A}$ (semantics):

carriers $\mathcal{A}_{Int} = \mathbf{Int}, \mathcal{A}_{Bool} = \mathbf{Bool}$
operations
$0_{\mathcal{A}} = 0, 1_{\mathcal{A}} = 1$
$plus_{\mathcal{A}}(n,m) = n + m, times_{\mathcal{A}}(n,m) = n * m$
$minus_{\mathcal{A}}(n,m) = n - m$
$false_{\mathcal{A}} = \mathbf{ff}, true_{\mathcal{A}} = \mathbf{tt}$
$lteq_{\mathcal{A}}(n,m) = \mathbf{tt}$ **if** $n \leq m$ **else** $\mathbf{ff}$
$not_{\mathcal{A}}(b) = \mathbf{tt}$ **if** $b = \mathbf{ff}$ **else** $\mathbf{ff}$
$and_{\mathcal{A}}(b,b') = \mathbf{tt}$ **if** $b = b' = \mathbf{tt}$ **else** $\mathbf{ff}$

$\mathcal{A}'_{Int} = \{min, \ldots, -1, 0, 1, \ldots, max\} \ldots$
redefines (the semantics of) TINY

Signatures

Algebraic signature:

$$\boxed{\Sigma = (S, \Omega)}$$

- *sort names*: S
- *operation names, classified by arities and result sorts*: $\Omega = \langle \Omega_{w,s} \rangle_{w \in S^*, s \in S}$

Alternatively:

$$\boxed{\Sigma = (S, \Omega, arity, sort)}$$

with *sort names* S, *operation names* Ω, and *arity and result sort functions*
$arity : \Omega \to S^*$ and $sort : \Omega \to S$.

- $f : s_1 \times \ldots \times s_n \to s$ stands for $s_1, \ldots, s_n, s \in S$ and $f \in \Omega_{s_1 \ldots s_n, s}$

Compare the two notions

Fix a signature $\Sigma = (S, \Omega)$ for a while.

Algebras

- Σ-*algebra*:

$$A = (|A|, \langle f_A \rangle_{f \in \Omega})$$

  - *carrier sets:* $|A| = \langle |A|_s \rangle_{s \in S}$
  - *operations*: $f_A \colon |A|_{s_1} \times \ldots \times |A|_{s_n} \to |A|_s$, for $f \colon s_1 \times \ldots \times s_n \to s$

- the class of all Σ-algebras:

$$\mathbf{Alg}(\Sigma)$$

Can $\mathbf{Alg}(\Sigma)$ be empty? Finite?
Can $A \in \mathbf{Alg}(\Sigma)$ have empty carriers?

Subalgebras

- for $A \in \mathbf{Alg}(\Sigma)$, a Σ-*subalgebra* $A_{sub} \subseteq A$ is given by subset $|A_{sub}| \subseteq |A|$ closed under the operations:
  – for $f : s_1 \times \ldots \times s_n \to s$ and $a_1 \in |A_{sub}|_{s_1}, \ldots, a_n \in |A_{sub}|_{s_n}$,
  $$f_{A_{sub}}(a_1, \ldots, a_n) = f_A(a_1, \ldots, a_n)$$
- for $A \in \mathbf{Alg}(\Sigma)$ and $X \subseteq |A|$, the *subalgebra of A generated by X*, $\langle A \rangle_X$, is the least subalgebra of A that contains X.
- $A \in \mathbf{Alg}(\Sigma)$ is *reachable* if $\langle A \rangle_\emptyset$ coincides with A.

Fact: *For any $A \in \mathbf{Alg}(\Sigma)$ and $X \subseteq |A|$, $\langle A \rangle_X$ exists.*

Proof (idea):

- generate the generated subalgebra from X by closing it under operations in A; or
- the intersection of any family of subalgebras of A is a subalgebra of A.

Homomorphisms

- for $A, B \in \mathbf{Alg}(\Sigma)$, a Σ-*homomorphism* $h\colon A \to B$ is a function $h\colon |A| \to |B|$ that preserves the operations:
  - for $f\colon s_1 \times \ldots \times s_n \to s$ and $a_1 \in |A|_{s_1}, \ldots, a_n \in |A|_{s_n}$,
    $$h_s(f_A(a_1, \ldots, a_n)) = f_B(h_{s_1}(a_1), \ldots, h_{s_n}(a_n))$$

Fact: *Given a homomorphism $h\colon A \to B$ and subalgebras A_{sub} of A and B_{sub} of B, the image of A_{sub} under h, $h(A_{sub})$, is a subalgebra of B, and the coimage of B_{sub} under h, $h^{-1}(B_{sub})$, is a subalgebra of A.*

Fact: *Given a homomorphism $h\colon A \to B$ and $X \subseteq |A|$, $h(\langle A \rangle_X) = \langle B \rangle_{h(X)}$.*

Fact: *Identity function on the carrier of $A \in \mathbf{Alg}(\Sigma)$ is a homomorphism $id_A\colon A \to A$. Composition of homomorphisms $h\colon A \to B$ and $g\colon B \to C$ is a homomorphism $h;g\colon A \to C$.*

Isomorphisms

- for $A, B \in \mathbf{Alg}(\Sigma)$, a Σ-*isomorphism* is any Σ-homomorphism $i\colon A \to B$ that has an *inverse*, i.e., a Σ-homomorphism $i^{-1}\colon B \to A$ such that $i;i^{-1} = id_A$ and $i^{-1};i = id_B$.
- Σ-algebras are *isomorphic* if there exists an isomorphism between them.

Fact: *A Σ-homomorphism is a Σ-isomorphism iff it is bijective ("1-1" and "onto").*

Fact: *Identities are isomorphisms, and any composition of isomorphisms is an isomorphism.*

Congruences

- for $A \in \mathbf{Alg}(\Sigma)$, a Σ-*congruence on* A is an equivalence $\equiv\ \subseteq |A| \times |A|$ that is closed under the operations:
  - for $f\colon s_1 \times \ldots \times s_n \to s$ and $a_1, a_1' \in |A|_{s_1}, \ldots, a_n, a_n' \in |A|_{s_n}$,
    if $a_1 \equiv_{s_1} a_1', \ldots, a_n \equiv_{s_n} a_n'$ then $f_A(a_1, \ldots, a_n) \equiv_s f_A(a_1', \ldots, a_n')$.

Fact: *For any relation* $R \subseteq |A| \times |A|$ *on the carrier of a* Σ*-algebra* A*, there exists the least congruence on* A *that contains* R*.*

Fact: *For any* Σ*-homomorphism* $h\colon A \to B$*, the kernel of* h*,* $K(h) \subseteq |A| \times |A|$*, where* $a\ K(h)\ a'$ *iff* $h(a) = h(a')$*, is a* Σ*-congruence on* A*.*

Quotients

- for $A \in \mathbf{Alg}(\Sigma)$ and Σ-congruence $\equiv \subseteq |A| \times |A|$ on A, the *quotient algebra* $A/{\equiv}$ is built in the natural way on the equivalence classes of $\equiv$:
  - for $s \in S$, $|A/{\equiv}|_s = \{[a]_\equiv \mid a \in |A|_s\}$, with $[a]_\equiv = \{a' \in |A|_s \mid a \equiv a'\}$
  - for $f : s_1 \times \ldots \times s_n \to s$ and $a_1 \in |A|_{s_1}, \ldots, a_n \in |A|_{s_n}$,
    $$f_{A/\equiv}([a_1]_\equiv, \ldots, [a_n]_\equiv) = [f_A(a_1, \ldots, a_n)]_\equiv$$

Fact: *The above is well-defined; moreover, the natural map that assigns to every element its equivalence class is a Σ-homomorphisms $[\_]_\equiv : A \to A/{\equiv}$.*

Fact: *Given two Σ-congruences $\equiv$ and $\equiv'$ on A, $\equiv \subseteq \equiv'$ iff there exists a Σ-homomorphism $h : A/{\equiv} \to A/{\equiv'}$ such that $[\_]_\equiv;h = [\_]_{\equiv'}$.*

Fact: *For any Σ-homomorphism $h : A \to B$, $A/K(h)$ is isomorphic with $h(A)$.*

Products

- for $A_i \in \mathbf{Alg}(\Sigma)$, $i \in \mathcal{I}$, the *product of* $\langle A_i \rangle_{i \in \mathcal{I}}$, $\prod_{i \in \mathcal{I}} A_i$ is built in the natural way on the Cartesian product of the carriers of A_i, $i \in \mathcal{I}$:
  - for $s \in S$, $|\prod_{i \in \mathcal{I}} A_i|_s = \prod_{i \in \mathcal{I}} |A_i|_s$
  - for $f : s_1 \times \ldots \times s_n \to s$ and $a_1 \in |\prod_{i \in \mathcal{I}} A_i|_{s_1}, \ldots, a_n \in |\prod_{i \in \mathcal{I}} A_i|_{s_n}$, for $i \in \mathcal{I}$, $f_{\prod_{i \in \mathcal{I}} A_i}(a_1, \ldots, a_n)(i) = f_{A_i}(a_1(i), \ldots, a_n(i))$

Fact: *For any family $\langle A_i \rangle_{i \in \mathcal{I}}$ of Σ-algebras, projections $\pi_i(a) = a(i)$, where $i \in \mathcal{I}$ and $a \in \prod_{i \in \mathcal{I}} |A_i|$, are Σ-homomorphisms $\pi_i : \prod_{i \in \mathcal{I}} A_i \to A_i$.*

Define the product of the empty family of Σ-algebras.
When the projection π_i is an isomorphism?

Terms

Consider an S-sorted set X of variables.

- *terms* $t \in |T_\Sigma(X)|$ are built using variables X, constants and operations from Ω in the usual way: $|T_\Sigma(X)|$ is the least set such that
  - $X \subseteq |T_\Sigma(X)|$
  - for $f\colon s_1 \times \ldots \times s_n \to s$ and $t_1 \in |T_\Sigma(X)|_{s_1}, \ldots, t_n \in |T_\Sigma(X)|_{s_n}$, $f(t_1, \ldots, t_n) \in |T_\Sigma(X)|_s$
- for any Σ-algebra A and valuation $v\colon X \to |A|$, *the value* $t_A[v]$ *of a term* $t \in |T_\Sigma(X)|$ *in* A *under* v is determined inductively:
  - $x_A[v] = v_s(x)$, for $x \in X_s$, $s \in S$
  - $(f(t_1, \ldots, t_n))_A[v] = f_A((t_1)_A[v], \ldots, (t_n)_A[v])$, for $f\colon s_1 \times \ldots \times s_n \to s$ and $t_1 \in |T_\Sigma(X)|_{s_1}, \ldots, t_n \in |T_\Sigma(X)|_{s_n}$

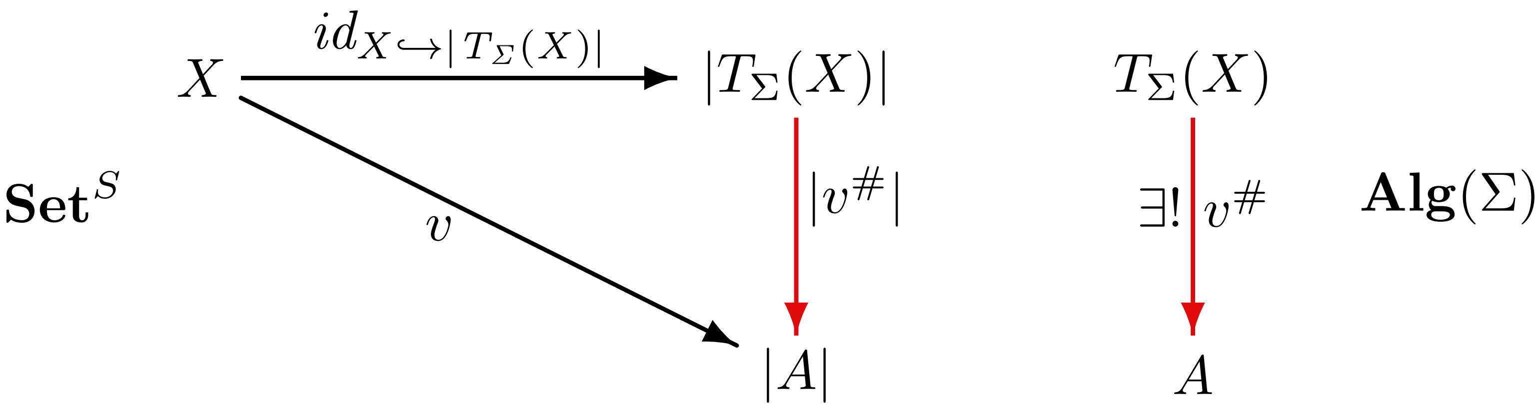
Above and in the following: assuming unambiguous "parsing" of terms!

Term algebras

Consider an S-sorted set X of variables.

- The *term algebra* $T_\Sigma(X)$ has the set of terms as the carrier and operations defined "syntactically":
  - for $f\colon s_1 \times \ldots \times s_n \to s$ and $t_1 \in |T_\Sigma(X)|_{s_1}, \ldots, t_n \in |T_\Sigma(X)|_{s_n}$, $f_{T_\Sigma(X)}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$.

Fact: *For any S-sorted set X of variables, Σ-algebra A and valuation $v\colon X \to |A|$, there is a unique Σ-homomorphism $v^{\#}\colon T_\Sigma(X) \to A$ that extends v. Moreover, for $t \in |T_\Sigma(X)|$, $v^{\#}(t) = t_A[v]$.*



Equations

- *Equation*:

$$\forall X . t = t'$$

where:

  - X is a set of variables, and
  - $t, t' \in |T_\Sigma(X)|_s$ are terms of a common sort.

- *Satisfaction relation*: Σ-algebra A *satisfies* $\forall X . t = t'$

$$A \models \forall X . t = t'$$

when for all $v \colon X \to |A|$, $t_A[v] = t'_A[v]$.

Semantic entailment

$$\Phi \models_\Sigma \varphi$$

Σ-equation φ is a semantic consequence of a set of Σ-equations Φ if φ holds in every Σ-algebra that satisfies Φ.

BTW:

- *Models* of a set of equations: $Mod(\Phi) = \{A \in \mathbf{Alg}(\Sigma) \mid A \models \Phi\}$
- *Theory* of a class of algebras: $Th(\mathcal{C}) = \{\varphi \mid \mathcal{C} \models \varphi\}$
- $\Phi \models \varphi \iff \varphi \in Th(Mod(\Phi))$
- Mod and Th form a *Galois connection*

Equational calculus

$$\frac{}{\forall X.t = t} \qquad \frac{\forall X.t = t'}{\forall X.t' = t} \qquad \frac{\forall X.t = t' \quad \forall X.t' = t''}{\forall X.t = t''}$$

$$\frac{\forall X.t_1 = t'_1 \quad \ldots \quad \forall X.t_n = t'_n}{\forall X.f(t_1 \ldots t_n) = f(t'_1 \ldots t'_n)} \qquad \frac{\forall X.t = t'}{\forall Y.t[\theta] = t'[\theta]} \text{ for } \theta\colon X \to |T_\Sigma(Y)|$$

Mind the variables!

$a = b$ does *not* follow from $a = f(x)$ and $f(x) = b$, unless...

Proof-theoretic entailment

$$\Phi \vdash_{\Sigma} \varphi$$

Σ-*equation* φ *is a proof-theoretic consequence of a set of* Σ-*equations* Φ *if* φ *can be derived from* Φ *by the rules.*

How to justify this?

Semantics!

Soundness & completeness

Fact: *The equational calculus is sound and complete:*

$$\Phi \models \varphi \iff \Phi \vdash \varphi$$

- soundness: "all that can be proved, is true" ($\Phi \models \varphi \Longleftarrow \Phi \vdash \varphi$)
- completeness: "all that is true, can be proved" ($\Phi \models \varphi \Longrightarrow \Phi \vdash \varphi$)

Proof (idea):

- soundness: easy!
- completeness: not so easy!

One motivation

Software systems (data types, modules, programs, databases. . .):

sets of data with operations on them

- Disregarding: code, efficiency, robustness, reliability, . . .
- Focusing on: CORRECTNESS

Universal algebra from rough analogy:

module interface $\leadsto$ signature

module $\leadsto$ algebra

module specification $\leadsto$ class of algebras

Equational specifications

$$\langle \Sigma, \Phi \rangle$$

- signature Σ, to determine the static module interface
- axioms (Σ-equations), to determine required module properties

BUT:

Fact: *A class of Σ-algebras is equationally definable iff it is closed under subalgebras, products and homomorphic images.*

"if" is delicate

Equational specifications typically admit a lot of undesirable "modules"

Example

$$\textbf{spec}\ \textsc{NaiveNat} = \textbf{sort}\ Nat$$
$$\textbf{opns}\ 0\colon Nat;$$
$$succ\colon Nat \to Nat;$$
$$\_+\_\colon Nat \times Nat \to Nat$$
$$\textbf{axioms}\ \forall n{:}Nat.n + 0 = n;$$
$$\forall n, m{:}Nat.n + succ(m) = succ(n + m)$$

Now:

$$\textsc{NaiveNat} \not\models \forall n, m{:}Nat.n + m = m + n$$

Perhaps worse:

There are models $M \in Mod(\textsc{NaiveNat})$ such that $M \models 0 = succ(0)$, or even:

$$M \models \forall n, m{:}Nat.n = m$$

How to fix this

- *Constraints*:

  initiality: *"no junk" & "no confusion"*

  Also: *reachability* ("no junk"), and their more general versions (freeness, generation).

  BTW: Constraints can be thought of as special (higher-order) formulae.

- Other (stronger) *logical systems*: conditional equations, first-order logic, higher-order logics, other bells-and-whistles

  – more about this elsewhere...

  Institutions!

There has been a population explosion among logical systems...

Initial models

Fact: *Every equational specification* $\langle\Sigma,\Phi\rangle$ *has an initial model: there exists a* Σ*-algebra* $I \in Mod(\Phi)$ *such that for every* Σ*-algebra* $M \in Mod(\Phi)$ *there exists a unique* Σ*-homomorphism from* I *to* M*.*

Proof (idea):

- I is the quotient of the algebra of ground Σ-terms by the congruence that glues together all ground terms t, t' such that $\Phi \models \forall\emptyset.t = t'$.
- I is the reachable subalgebra of the product of "all" (up to isomorphism) reachable algebras in $Mod(\Phi)$.

BTW: This can be generalised to the existence of a free model of $\langle\Sigma,\Phi\rangle$ over any (many-sorted) set of data.

BTW: Existence of initial (and free) models carries over to specifications with conditional equations, **but not much further!**

Example

$$
\begin{aligned}
\textbf{spec } \mathrm{N{\small AT}} = \textbf{initial } \{\ & \textbf{sort } Nat \\
& \textbf{opns } 0\colon Nat; \\
& \qquad succ\colon Nat \to Nat; \\
& \qquad \_+\_\colon Nat \times Nat \to Nat \\
& \textbf{axioms } \forall n{:}Nat.n + 0 = n; \\
& \qquad \forall n, m{:}Nat.n + succ(m) = succ(n + m) \\
\}\ &
\end{aligned}
$$

Now:

$$\mathrm{N{\small AT}} \models \forall n, m{:}Nat.n + m = m + n$$

Try another example

$$
\begin{array}{ll}
\textbf{spec } \mathrm{NATPRED} = & \textbf{sort } Nat \\
& \textbf{opns } 0\colon Nat; error\colon Nat; \\
& \quad succ\colon Nat \to Nat; \\
& \quad \_+\_\colon Nat \times Nat \to Nat; \\
& \quad pred\colon Nat \to Nat \\
& \textbf{axioms } \forall n{:}Nat. n + 0 = n; \\
& \quad \forall n, m{:}Nat. n + succ(m) = succ(n+m); \\
& \quad \forall n{:}Nat. pred(succ(n)) = n; \\
& \quad pred(0) = error; \\
& \quad pred(error) = error; succ(error) = error; \\
& \quad \forall n{:}Nat. error + n = error; \forall n{:}Nat. n + error = error
\end{array}
$$

Looks okay. But try to add multiplication:

$$0 * n = 0; succ(m) * n = n + (m * n);$$
$$error * n = error; n * error = error$$

and now everything collapses!

Partial algebras

- *Algebraic signature* Σ: as before
- *Partial Σ-algebra*:

$$\boxed{A = (|A|, \langle f_A \rangle_{f \in \Omega})}$$

as before, but operations $f_A : |A|_{s_1} \times \ldots \times |A|_{s_n} \rightharpoonup |A|_s$, for $f : s_1 \times \ldots \times s_n \to s$, may now be *partial functions*.

BTW: Constants may be undefined as well.

- $\mathbf{PAlg}(\Sigma)$ stands for the class of all partial Σ-algebras.

Few further notions

- *subalgebra* $A_{sub} \subseteq A$: given by subset $|A_{sub}| \subseteq |A|$ closed under the operations; (BTW: at least two other natural notions are possible)
- *homomorphism* $h \colon A \to B$: map $h \colon |A| \to |B|$ that preserves definedness and results of operations; it is *strong* if in addition it reflects definedness of operations; (strong) homomorphisms are closed under composition; (BTW: very interesting alternative: *partial* map $h \colon |A| \rightharpoonup |B|$ that preserves results of operations)
- *congruence* $\equiv$ on A: equivalence $\equiv \subseteq |A| \times |A|$ closed under the operations whenever they are defined; it is *strong* if in addition it reflects definedness of operations; (strong) congruences are kernels of (strong) homomorphisms;
- *quotient algebra* $A/{\equiv}$: built in the natural way on the equivalence classes of $\equiv$; the natural homomorphism from A to $A/{\equiv}$ is strong if the congruence is strong.

Formulae

(Strong) equation:	*Definedness formula*:
$\forall X.t \stackrel{s}{=} t'$ as before	$\forall X.def\ t$ where X is a set of variables, and $t \in \lvert T_\Sigma(X)\rvert_s$ is a term

Satisfaction relation

partial Σ-algebra A *satisfies* $\forall X.t \stackrel{s}{=} t'$	partial Σ-algebra A *satisfies* $\forall X.def\ t$
$A \models \forall X.t \stackrel{s}{=} t'$ when for all $v: X \to \lvert A\rvert$, $t_A[v]$ is defined iff $t'_A[v]$ is defined, and then $t_A[v] = t'_A[v]$	$A \models \forall X.def\ t$ when for all $v: X \to \lvert A\rvert$, $t_A[v]$ is defined

An alternative

- *(Existence) equation*:

$$\forall X. t \stackrel{e}{=} t'$$

where:

  – X is a set of variables, and
  – $t, t' \in |T_\Sigma(X)|_s$ are terms of a common sort.

- *Satisfaction relation*: Σ-algebra A *satisfies* $\forall X. t \stackrel{e}{=} t'$

$$A \models \forall X. t \stackrel{e}{=} t'$$

when for all $v \colon X \to |A|$, $t_A[v] = t'_A[v]$ — both sides are defined and equal.

BTW:

- $\forall X. t \stackrel{e}{=} t'$ iff $\forall X. (t \stackrel{s}{=} t' \wedge def\ t)$
- $\forall X. t \stackrel{s}{=} t'$ iff $\forall X. (def\ t \iff def\ t') \wedge (def\ t \implies t \stackrel{e}{=} t')$

Example

$$
\begin{array}{l}
\textbf{spec } \textsc{NatPred} = \textbf{initial } \{ \ \textbf{sort } Nat \\
\qquad \textbf{opns } 0 \colon Nat; \\
\qquad\qquad succ \colon Nat \to Nat; \\
\qquad\qquad \_+\_ \colon Nat \times Nat \to Nat; \\
\qquad\qquad pred \colon Nat \to ? \ Nat \\
\qquad \textbf{axioms } \forall n{:}Nat . n + 0 = n; \\
\qquad\qquad \forall n, m{:}Nat . n + succ(m) = succ(n+m); \\
\qquad\qquad \forall n{:}Nat . pred(succ(n)) \stackrel{s}{=} n \\
\}
\end{array}
$$

First-order structures

- *First-order signature* $\Sigma = (S, \Omega, \Pi)$: algebraic signature (S, Ω) plus *predicate names*, classified by arities: $\Pi = \langle \Pi_w \rangle_{w \in S^*}$
- *First-order* Σ*-structure*:

$$A = (|A|, \langle f_A \rangle_{f \in \Omega}, \langle p_A \rangle_{p \in \Pi})$$

  consists of:
  - (S, Ω)-algebra $(|A|, \langle f_A \rangle_{f \in \Omega})$
  - *predicates* (relations): $p_A \subseteq |A|_{s_1} \times \ldots \times |A|_{s_n}$, for $p: s_1 \times \ldots \times s_n$ (i.e., $p \in \Pi_{s_1 \ldots s_n}$)
- $\mathbf{Str}(\Sigma)$ stands for the class of all first-order Σ-structures.

Few further notions

- *substructure* $A_{sub} \subseteq A$: given by subset $|A_{sub}| \subseteq |A|$ closed under the operations and such that the inclusion preserves truth of predicates; the substructure is *closed* if the inclusion also preserves falsity of predicates;
- *homomorphism* $h \colon A \to B$: map $h \colon |A| \to |B|$ that preserves the results of operations and truth of predicates; it is *closed* if in addition it preserves falsity of predicates; (closed) homomorphisms are closed under composition;
- *congruence* $\equiv$ on A: equivalence $\equiv \subseteq |A| \times |A|$ closed under the operations; it is *closed* if in addition it preserves truth (and falsity) of predicates; (closed) congruences are kernels of (closed) homomorphisms;
- *quotient structures* $A/{\equiv}$: built in the natural way on the equivalence classes of $\equiv$ so that the natural map from A to $A/{\equiv}$ is a homomorphism; it is closed if the congruence is closed.

Formulae

- *atomic* Σ*-formulae* over set X of variables:
  - $t = t'$, where $t, t' \in |T_{(S,\Omega)}(X)|_s$, $s \in S$
  - $p(t_1, \ldots t_n)$, where $p\colon s_1 \times \ldots \times s_n$, $t_1 \in |T_{(S,\Omega)}(X)|_{s_1}, \ldots t_n \in |T_{(S,\Omega)}(X)|_{s_n}$
- Σ*-formulae* contain atomic formulae and are closed under logical connectives and quantification; Σ*-sentences* are Σ-formulae with no free variables
- *Satisfaction relation* defined as usual between Σ-structures A and Σ-sentences φ

$$A \models \varphi$$

As before, this yields the usual notions of the *class of models* for a set of sentences, the *semantic consequences* of a set of sentences, the *theory* of a class of models, etc.

Initial (and free) models exist for first-order specifications with universally quantified conditional atomic formulae, *but in general may fail to exist!*