Theory of concurrency

2023/24

Lecture 13

Bisimulation and Logic

Modal logic M:

SYNTAX:

$$\phi ::= true \mid \neg \phi \mid \phi_1 \wedge \phi_2 \mid \langle a \rangle \phi$$

$a \in \Sigma$

SEMANTICS:

$P \models true$

$P \models \neg \phi$ iff $P \not\models \phi$

$P \models \phi_1 \wedge \phi_2$ iff $P \models \phi_1$ and $P \models \phi_2$

$P \models \langle a \rangle \phi$ iff $P' \models \phi$ for some P' s.t. $P \xrightarrow{a} P'$

SHORTHANDS:

- $[a]\phi := \neg \langle a \rangle \neg \phi$

  $P \models [a]\phi$ iff $P' \models \phi$ for all P' s.t. $P \xrightarrow{a} P'$
- $\phi_1 \vee \phi_2$
- $\langle - \rangle \phi := \bigvee_a \langle a \rangle \phi$
- $false := \neg true$

Example: $\langle a \rangle [b] true \vee [a] \langle b \rangle true$

$\langle - \rangle true \quad [-] false$

Def: $P \equiv_M Q$ iff $\forall \phi \in M. \left(P \models \phi \text{ iff } Q \models \phi \right)$

Lemma: $P \sim Q$ implies $P \equiv_M Q$

Proof: By structural induction over $\Phi \in M$ show that $P \sim Q$ implies $(P \models \Phi \text{ iff } Q \models \phi)$

- Base: $\phi = true$
- Step: $\phi = \neg\psi$

  $\phi = \phi_1 \wedge \phi_2$

  $\phi = \langle a \rangle \psi$

$P \sim Q$

$P \models \langle a \rangle \psi \qquad Q \models \langle a \rangle \psi$

$$\begin{array}{ccc} P & & Q \\ \downarrow a & & \downarrow a \\ P' \models \psi & & Q' \models \psi \end{array}$$

$P' \sim Q'$

Lemma: Under image-finiteness assumption,

$P \equiv_M Q$ implies $P \sim Q$.

Proof: $\equiv_M$ is a bisimulation

$$\begin{array}{ccccc} P & \equiv_M & Q & & \\ a \downarrow & & \downarrow a & \dots & \searrow a \\ P' & & Q_1 & & Q_n \end{array}$$

We need: $P' \equiv_M Q_i$ for some i

Towards contradiction, assume the contrary:

$P \models \langle a \rangle true \qquad Q \models \langle a \rangle true$

$P' \not\equiv_M Q_i$ for all i

There are $\phi_1, \dots, \phi_n \in M$ s.t. $P' \models \phi_i$
$Q_i \not\models \phi_i$
for every i

Put $\psi := \phi_1 \wedge \dots \wedge \phi_n$

$P \models \langle a \rangle \psi$ $\quad Q \not\models \langle a \rangle \psi$ — a contradiction.

<u>Observation</u>: a modal formula is a witness of bisimulation non equivalence

<u>Remark</u>: one can drop image-finiteness assumption

<u>Question</u>: is negation needed in the last proof?

Variants for:
- $\approx$
- $\sim_n$
- long steps $\langle a_1 \dots a_n \rangle \phi := \langle a_1 \rangle \dots \langle a_n \rangle \phi$