Theory of concurrency

2023/24

Lecture 11

Bisimulation equivalence

Example:

$A \stackrel{def}{=} a.A_1$

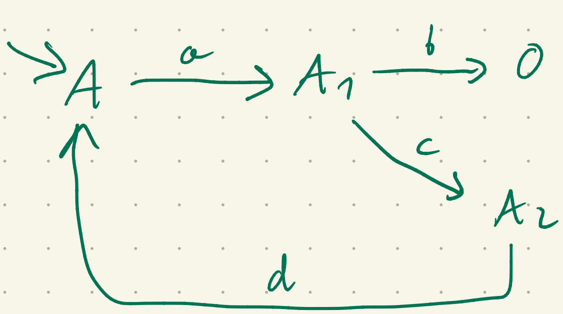
$A_1 \stackrel{def}{=} b.0 + c.A_2$

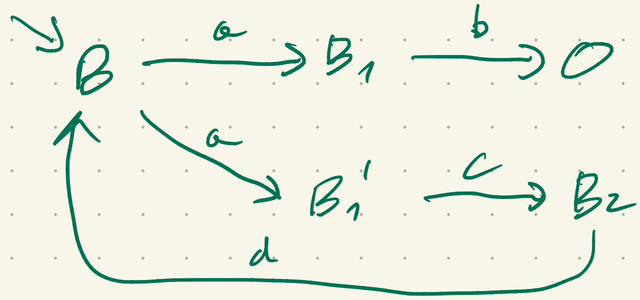
$A_2 \stackrel{def}{=} d.A$

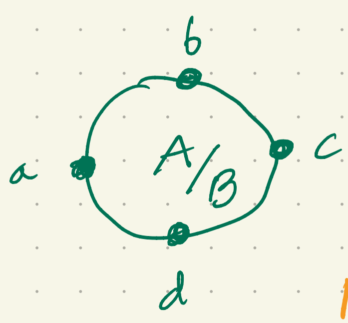
$B \stackrel{def}{=} a.B_1 + a.B_1'$

$B_1 \stackrel{def}{=} b.0 \qquad B_1' \stackrel{def}{=} c.B_2$

$B_2 \stackrel{def}{=} d.B$







$(acd)^* ab + (acd)^\omega$

$P, Q \in Proc$

Idea: P and Q are equivalent

iff

$\forall a \in \Sigma$

every a-successor of P is equivalent to some a-successor of Q,

and vice versa

$\sim = F(\sim)$

$P \sim Q$ iff for every $a \in \Sigma$,

- for every P', if $P \xrightarrow{a} P'$ then, for some Q', $Q \xrightarrow{a} Q'$ and $P' \sim Q'$
- for every Q', if $Q \xrightarrow{a} Q'$ then, for some P', $P \xrightarrow{a} P'$ and $P' \sim Q'$

$P \; F(\sim) \; Q$

Definition: A binary relation S between processes is a **bisimulation** if $S \subseteq F(S)$.

Fact: If $S = F(S)$ then S is a bisimulation.

Question: Is every bisimulation symmetric?

Example:

$Sem \stackrel{def}{=} p.\,v.\,Sem$

$Sem_2 \stackrel{def}{=} p.\,Sem_1$

$Sem_1 \stackrel{def}{=} p.\,Sem_0 + v.\,Sem_2$

$Sem_0 \stackrel{def}{=} v.\,Sem_1$

$S = \{ (Sem \mid Sem, Sem_2), (v.Sem \mid Sem, Sem_1), (Sem \mid v.Sem, Sem_1), (v.Sem \mid v.Sem, Sem_0) \}$

Fact. If S, S', S_i $(i \in I)$ are bisimulations then

- identity
- S^{-1}
- $S' \circ S$
- $\bigcup_{i \in I} S_i$

are also bisimulations.

sketch of a proof:

$$\begin{tikzcd}
P \arrow[r, dash, dotted, "S"] \arrow[d, "a"'] & Q \arrow[r, dash, dotted, "S'"] \arrow[d, "a"'] & R \arrow[d, "a"] \\
P' \arrow[r, dash, dotted, "S"] & Q' \arrow[r, dash, dotted, "S'"] & R'
\end{tikzcd}$$

Definition: P and Q are <u>bisimulation equivalent</u>, $P \sim Q$, if $(P, Q) \in S$ for some bisimulation S.

$$\sim \; = \bigcup \{ S : S \text{ is a bisimulation} \}$$

Fact:

$\sim$ is the largest bisimulation

$\sim$ is an equivalence

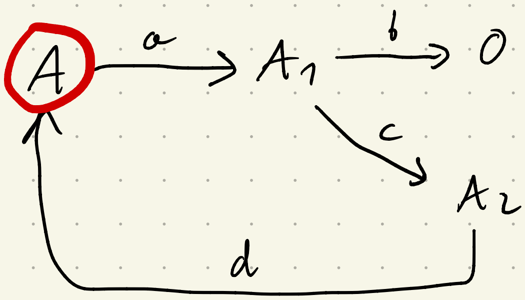
$\sim \; = F(\sim)$

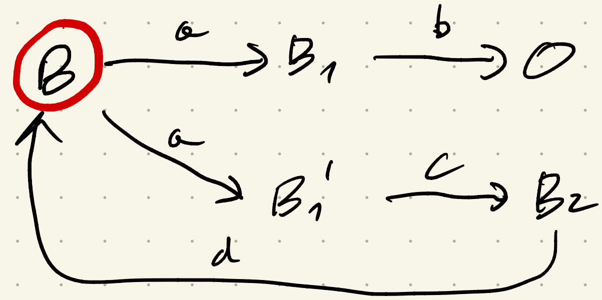
sketch of a proof:

- $\sim \; \subseteq F(\sim)$
- $F(\sim) \subseteq \; \sim$: $F(\sim)$ is a bisimulation
  $F(\sim) \subseteq F(F(\sim))$

Bisimulation game

Two players: Spoiler, Duplicator





Arena:

Spoiler's positions $V_S = Proc \times Proc$

Duplicator's positions $V_D = (Proc \times \Sigma) \times Proc \ \cup$

$Proc \times (Proc \times \Sigma)$

Spoiler moves:

$(P, Q) \rightsquigarrow ((P', a), Q)$ if $P \xrightarrow{a} P'$

$(P, Q) \rightsquigarrow (P, (Q', a))$ if $Q \xrightarrow{a} Q'$

Duplicator's moves:

$((P', a), Q) \rightsquigarrow (P', Q')$ if $Q \xrightarrow{a} Q'$

$(P, (Q', a)) \rightsquigarrow (P', Q')$ if $P \xrightarrow{a} P'$

Initial position: $v_0 = (P_0, Q_0) \in V_S$

Play: maximal sequence

$$\underbrace{v_0 \rightsquigarrow v_1 \rightsquigarrow v_2}_{\text{round 1}} \underbrace{\rightsquigarrow v_3 \rightsquigarrow v_4}_{\text{round 2}} \rightsquigarrow \cdots$$

Winner: finite play $\longrightarrow$ stuck player looses

infinite play $\longrightarrow$ Duplicator wins

Duplicator's strategy $s : (V_S V_D)^+ \xrightarrow{\text{partial}} V_S$

s.t. whenever $s(v_0 v_1 \dots v_n) = v$ then $v_n \rightsquigarrow v$.

s is **winning** if it guarantees Duplicator's win.

Spoiler's strategy $s : (V_S V_D)^* V_S \xrightarrow{\text{partial}} V_D$

Thm: $P_0 \sim Q_0$ iff Duplicator has a winning strategy.

$$P+Q \sim Q+P \qquad P|Q \sim Q|P$$

$$P+(Q+R) \sim (P+Q)+R \qquad P|(Q|R) \sim (P|Q)|R$$

$$P+0 \sim P \qquad P|0 \sim P$$

$$P+P \sim P$$

$$P \setminus K \setminus L \sim P \setminus (K \cup L) \qquad P[f][f'] \sim P[f' \circ f]$$

$P \setminus L \sim P$ if L never appears in P $\qquad P[id] \sim P$

$$P[f] \setminus L \sim P \setminus f^{-1}(L)\,[f]$$

$(P|Q) \setminus L \sim P \setminus L \,|\, Q \setminus L$ if .....

$(P|Q)[f] \sim P[f] \,|\, Q[f]$ if .....

The expansion law:

$$(P_1[f_1] | \ldots | P_n[f_n]) \setminus L \sim$$

$$\sum \{ f_i(a).(P_1[f_1] | \ldots | P_i'[f_i] | \ldots | P_n[f_n]) \setminus L : P_i \xrightarrow{a} P_i',\ f_i(a) \notin L \cup \overline{L} \}$$

$$+$$

$$\sum \{ \tau.(P_1[f_1] | \ldots | P_i'[f_i] | \ldots | P_j'[f_j] | \ldots | P_n[f_n]) \setminus L : P_i \xrightarrow{a} P_i',\ P_j \xrightarrow{b} P_j',\ f_i(a) = \overline{f_j(b)},\ a,b \neq \tau,\ i<j \}$$

Fact: $\sim$ is a congruence: if $P \sim P'$ then

- $a.P \sim a.P'$
- $P + Q \sim P' + Q$
- $P \mid Q \sim P' \mid Q$
- $P \setminus L \sim P' \setminus L$
- $P[f] \sim P'[f]$

sketch of a proof: $\{(P \mid Q, P' \mid Q) : P \sim P'\}$ is a bisimulation

Question: How to make τ unobservable?

$\sim$ as the greatest fixed point of F:

$$Rel = \mathcal{P}(Proc \times Proc)$$

- ($\underset{\text{binary relations on processes}}{\overset{Rel}{\shortparallel}}$, $\subseteq$) is a complete lattice

- $F : Rel \to Rel$ is monotonic

$$\Downarrow \text{ Knaster-Tarski thm}$$

F has the greatest fixed point equal to

$$\bigcup \{ S : \underbrace{S \subseteq F(S)}_{\text{pre fixed-points}} \}$$

least upper bound