Theory of Concurrency

2023/24

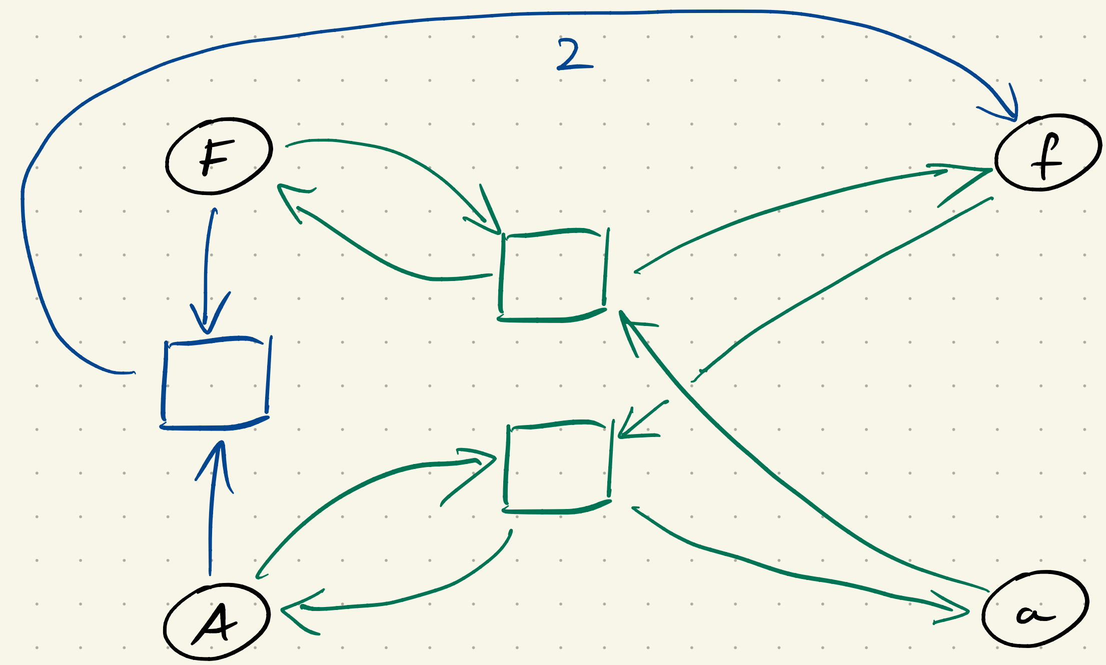
Lecture 9

Example: Debating philosophers

$P = \{ F, A, f, a \}$

rested $\begin{cases} \text{For} \\ \text{Against} \end{cases}$

tired $\begin{cases} \text{for} \\ \text{against} \end{cases}$



$Fa \to Ff$

$Af \to Aa$

$FA \to ff$

Question: starting with n tokens in F and m tokens in A,

does the protocol <u>stabilise</u> to a decision for/against?

Definition: Population protocol

- binary Petri net
- $I \subseteq P$ initial places — $I = \{F, A\}$
- $O : P \to \{0,1\}$ output function — $F, f \mapsto 1$; $A, a \mapsto 0$

Configurations: $\mathbb{N}^P$

Initial configurations: $\mathbb{N}^I \approx \{M \in \mathbb{N}^P : \forall p \notin I.\ M(p) = 0\}$

Runs: $\underset{\text{initial}}{M_0} \longrightarrow M_1 \longrightarrow \dots$ (infinite)

Fair runs:

$\forall M \to M'.\ \{i : M_i = M\}$ infinite $\Rightarrow \{i : M_i = M,\ M_{i+1} = M'\}$ infinite

$\Updownarrow$

$\forall M'.\ \{i : M_i \to M'\}$ infinite $\Rightarrow \{i : M_i = M'\}$ infinite

Output of configuration:

$$O(M) = \begin{cases} b & \text{if } \forall p \in P.\ M(p) > 0 \Rightarrow O(p) = b \\ \bot & \text{otherwise} \end{cases}$$

Run **stabilises** to $b \in \{0, 1, \bot\}$ if $\exists k.\ \forall i > k\ O(M_i) = b$

Protocol **computes** a predicate $\varphi : \mathbb{N}^I \to \{0,1\}$ if every fair run stabilises to $\varphi(M_0)$

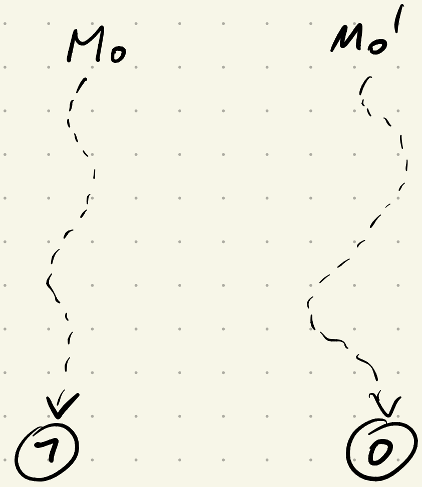
Protocol is **well-specified** if computes some predicate.

Alternative probabilistic semantics:

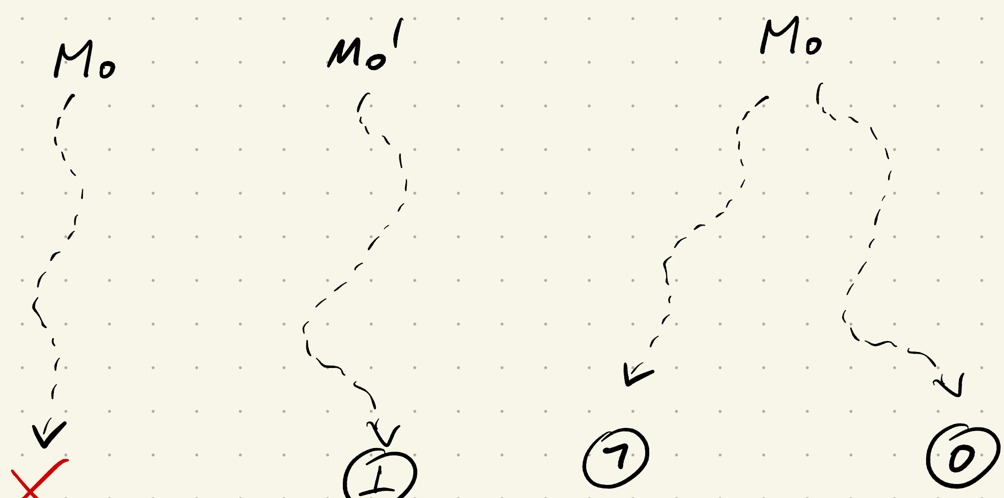
Protocol computes φ if $\forall M_0 \in \mathbb{N}^F$, run stabilises to $\varphi(M_0)$ with probability 1

Every run ends with probability 1 in a bottom SCC.

well-specified:



ill-specified:



Questions:

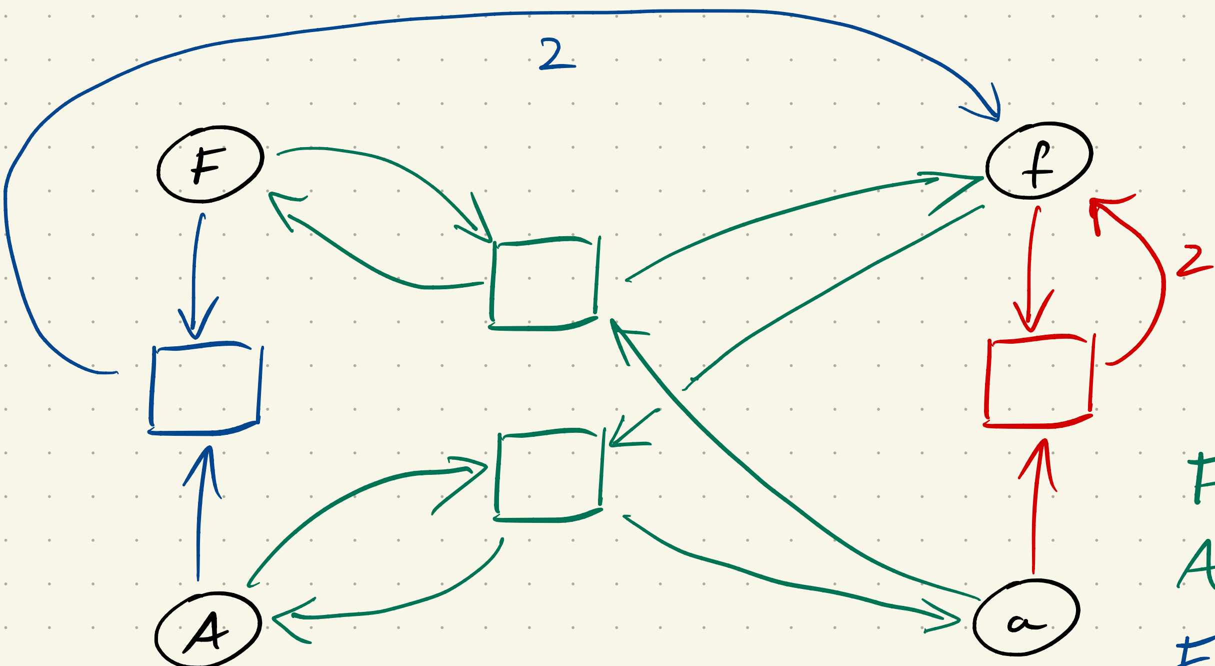
Is Debating philosophers well-specified?

How to correct it?

What predicate it computes?

How to speed it up? $\boxed{\text{TUTORIALS}}$

FFAA ⟶ ffFA ⟶ faFA ⟶ faff



Fa ⟶ Ff

Af ⟶ Aa

FA ⟶ ff

fa ⟶ ff

Thm: (Angluin 2006)

Predicates computable by protocols = semi-linear predicates

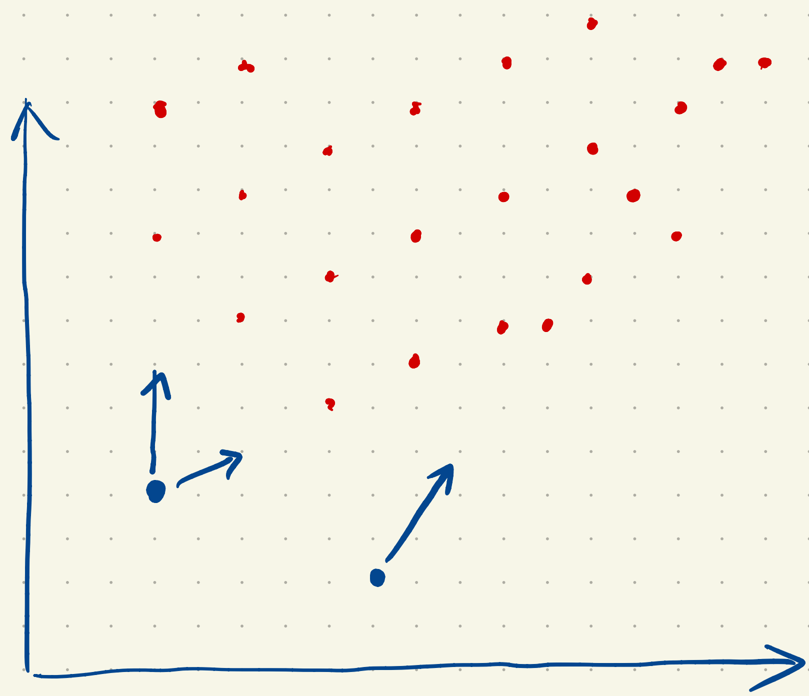
Semi-linear sets: $\bigcup_i b_i + P_i^*$ $\quad b_i \in \mathbb{N}^d$, $P_i \subseteq_{fin} \mathbb{N}^d$

$\|$

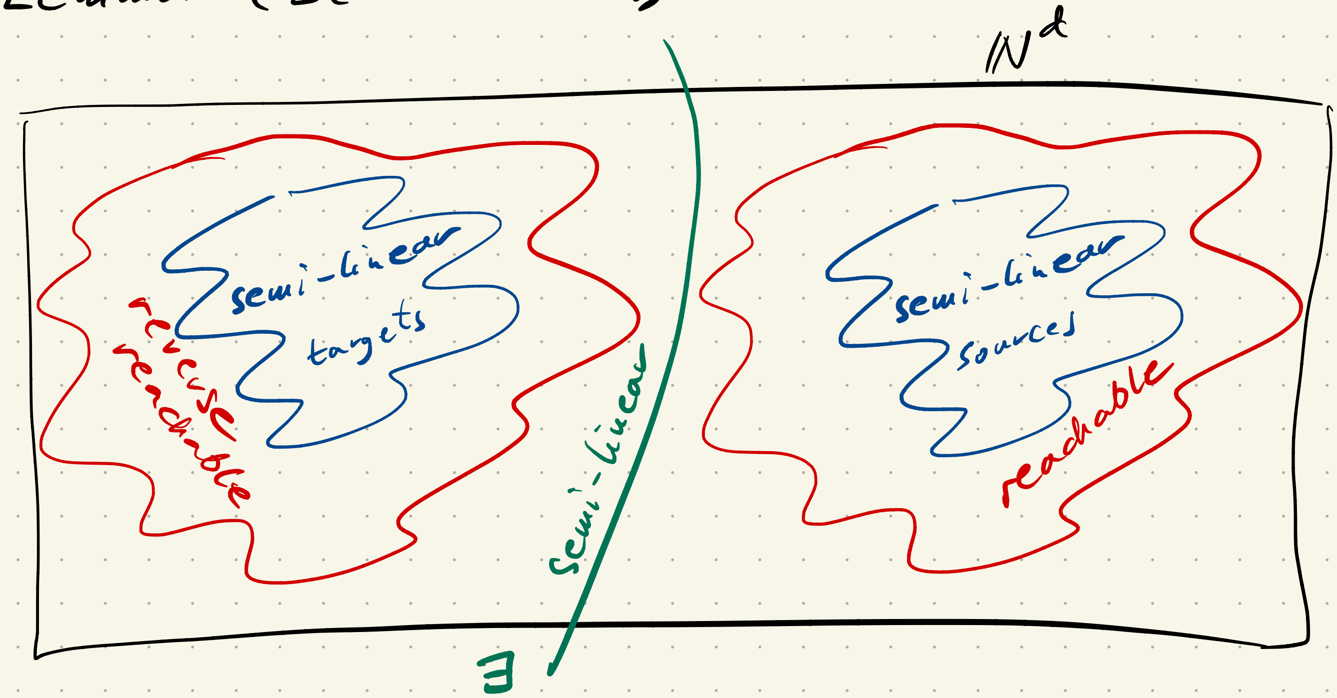
$b + P^* = \{ b + p_1 + \ldots + p_k : k \geq 0, p_1 \ldots p_k \in P \}$

Presburger-definable sets = definable in $FO(\mathbb{N}, +)$

$[\![\varphi(x_1 \ldots x_d)]\!] = \{ (n_1 \ldots n_d) : \varphi(n_1 \ldots n_d) \text{ is true} \}$



Lemma (Leroux 2011)

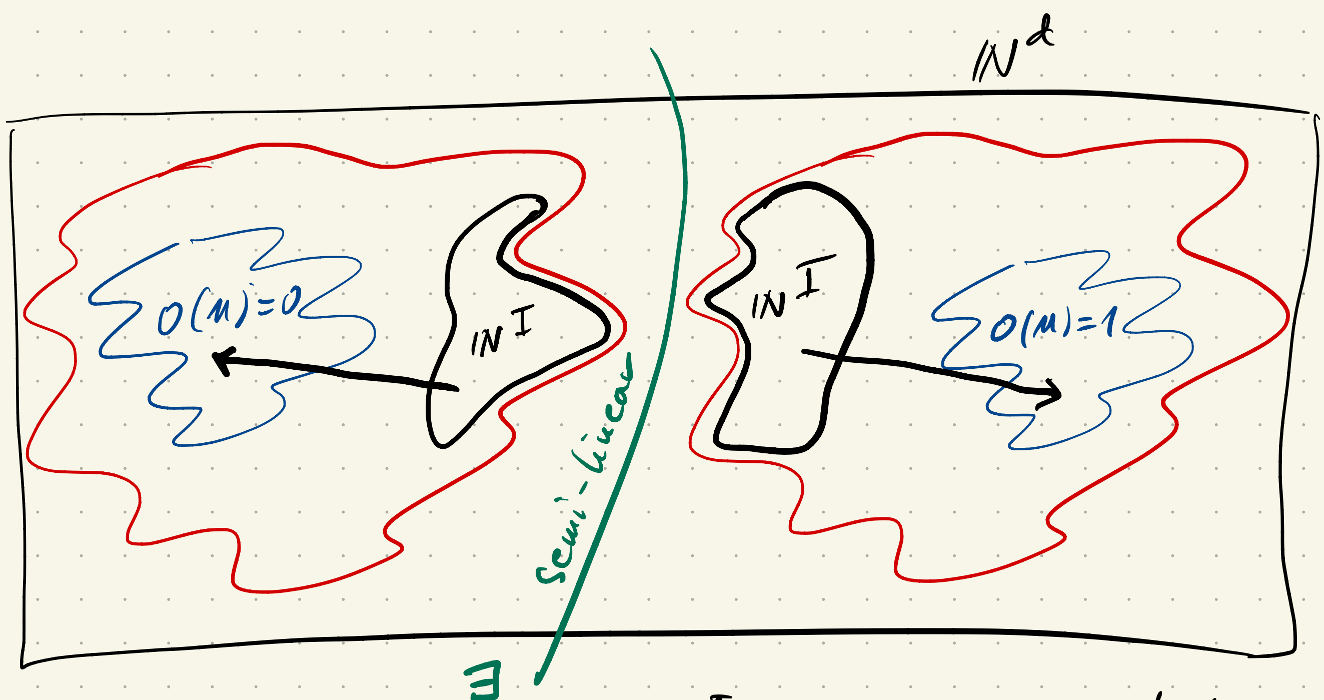


Proof of thm : $\supseteq$ $\boxed{\text{TUTORIALS}}$

$\subseteq$

Consider a well-specified protocol.

Every fair run ends in a bottom SCC with output 0 or 1.



$\mathbb{N}^I = C_0 \cup C_1$ both semi-linear

Computational problems:

reduces to reachability

- Is a given protocol well-specified?
- Does a given protocol compute a given predicate?
- Given a well-specified protocol, compute the predicate computed by the protocol.

Question: How to compute the other problems?