Theory of concurrency

2023/24

Lecture 7

Approximate analysis of Petri nets using linear algebra

A) invariants
B) continuous reachability

hard problems:

- coverability
- boundedness
- finitness

→ **very hard problems:**

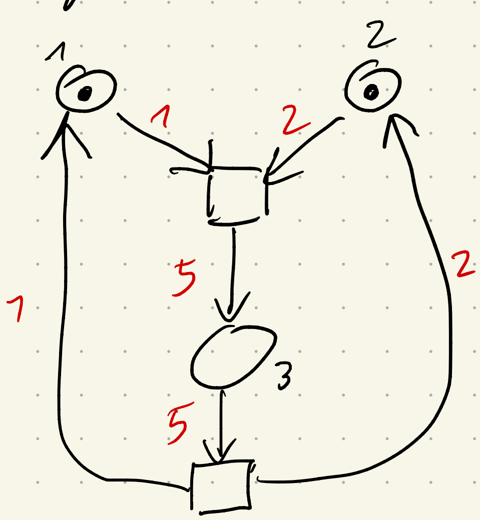
- reachability
- liveness

Model:

- general Petri nets
- counter automata w/o 0-tests
- VASS
- VAS = Petri nets without tight loops

A) invariants

Example:



$\frac{1}{2} p_1 + \frac{1}{2} p_2 + p_3 = \text{const}$

$p_1 + p_3 = \text{const}$

$10 p_1 + 5 p_2 + 4 p_3 = \text{const}$

$18 p_1 + p_2 + 4 p_3 = \text{const}$

$5 p_1 + p_3 = \text{const}$

VAS : $T \subseteq \mathbb{Z}^d$ $\quad n := |T|$

$N \in \mathbb{Z}[d \times n]$

$$[x_1 \ldots x_d] \cdot \begin{bmatrix} | & | & & | \\ t_1 & t_2 & \ldots & t_n \\ | & | & & | \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

Def : P-invariant : $I \in \mathbb{Q}^d$ s.t. $I \cdot N = \vec{0}$

T-invariant : $I \in \mathbb{Q}^n$ s.t. $N \cdot I = \vec{0}$

Petri net :

Fact : $I : P \to \mathbb{Q}$ is a P-invariant $\iff \forall t \in T$.

$$\sum_{p \in {}^\bullet t} I(p) \cdot F(p,t) = \sum_{p \in t^\bullet} I(p) \cdot F(t,p)$$

$I : T \to \mathbb{Q}$ is a T-invariant $\iff \forall p \in P$.

$$\sum_{t \in {}^\bullet p} I(t) \cdot F(t,p) = \sum_{t \in p^\bullet} I(t) \cdot F(p,t)$$

Fact: Let $V \subseteq \mathbb{Q}^d$ subspace generated by T

P-invariants = subspace $\overline{V}$ ortogonal to V

necessary condition:

Fact:

$$\begin{array}{ccc} M \longrightarrow^* M' & \Longrightarrow & M \dashrightarrow^* M' \\ & & M' = M + N \cdot X \\ & & \text{for some } X \in \mathbb{N}^n \\ & & (\text{NP-complete}) \\ \Downarrow & & \Downarrow \\ \forall \text{ P-invariant } I. & & M' = M + N \cdot X \\ I \cdot M = I \cdot M' & \Longleftrightarrow & \text{for some } X \in \mathbb{Q}^n \end{array}$$

$$(M' - M) \text{ ortogonal to } \overline{V} \Longleftrightarrow (M' - M) \in V$$

necessary condition:

all transitions are useful in M

Fact: M live, no isolated places $\Longrightarrow$ $I \cdot M > 0$ for every semi-positive P-invariant I $(I \geq \vec{0},\ I \neq \vec{0})$

Proof: Let $p \in P$ s.t. $I(p) > 0$.

Let $M \xrightarrow{*} M'$ s.t. $M'(p) > 0$

$$\Downarrow$$

$$I \cdot M = I \cdot M' > 0 \qquad \square$$

sufficient condition:

Fact: positive P-invariant $\Longrightarrow$ bounded

Proof: Suppose $M \xrightarrow{*} M'$. Let $p \in P$.

$$I \cdot M = I \cdot M' \geq I(p) \cdot M(p)$$

bounded by $\dfrac{I \cdot M}{I(p)}$

$\square$

necessary and sufficient condition:

$$M \dashrightarrow^{+} M \iff \text{semi-positive T-invariant}$$

B) continuous reachability

$$M \longrightarrow^* M' \Rightarrow M \dashrightarrow^* M'$$

$$M' = M + N \cdot X \text{ for some } X \in \mathbb{N}^n$$

(NP-complete)

$$\Downarrow \qquad \Downarrow$$

(in P) **?** $\Rightarrow$

$$M' = M + N \cdot X \text{ for some } X \in \mathbb{Q}_+^n$$

Def. $M \xrightarrow{t} M' \iff M' = M + t \qquad t \in T$

$M \overset{q\,t}{\dashrightarrow} M' \iff M' = M + q \cdot t \qquad q \in \mathbb{Q}_+$, $M, M' \in \mathbb{Q}_+^d$

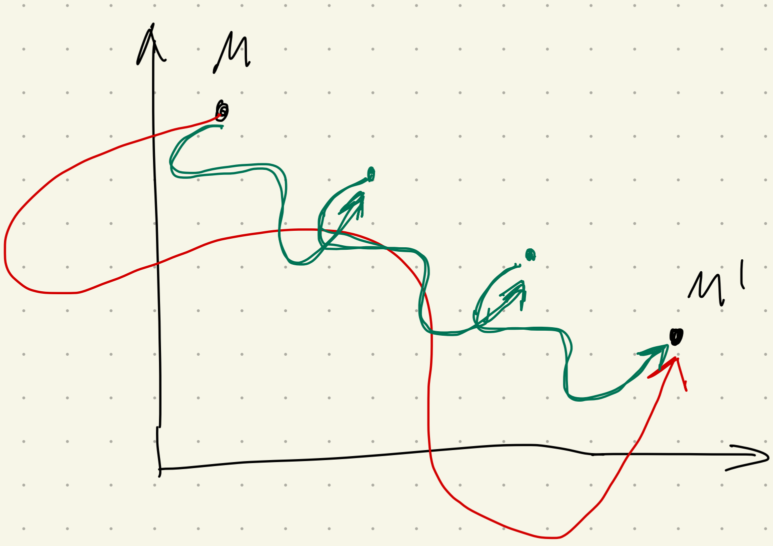
$M \overset{X}{\dashrightarrow} M' \qquad X \in \mathbb{Q}_+^n$

Lemma 1: $M \overset{X}{\dashrightarrow} M' \iff M' = M + N \cdot X$

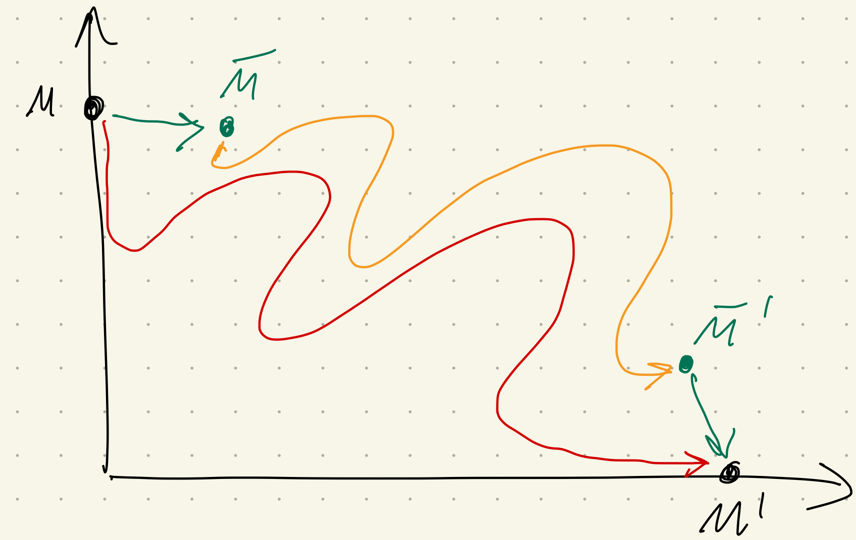
if

$$(*)\begin{cases} \forall t \in T,\ \forall p \in {}^\bullet t, \\ X(t) > 0 \Rightarrow M(p) > 0 \\ \\ \forall t \in T,\ \forall p \in t^\bullet, \\ X(t) > 0 \Rightarrow M'(p) > 0 \end{cases}$$

Proof:



Lemma 2:



continuous run

$\Downarrow$

short continuous run

+

continuous run satisfying (*)

Theorem 1: Continuous reachability is in P.

Proof: encode in linear equations with implications (by Lemma 2 and Thm 2)

short continuous run $\leq d$ steps

+

continuous run satisfying (*)

(by Lemma 1)
system of linear equations with implications with $2d + n$ variables

d variables

$M \xrightarrow{*} M'$

$M \xrightarrow{q_1} M_1 \xrightarrow{q_2} M_2 \dashrightarrow \cdots \xrightarrow{q_d} M_d$

some transition. which?

d variables

$M \xrightarrow{q_1} M_1 \xrightarrow{q_2} M_2 \dashrightarrow \cdots \xrightarrow{q_{d\cdot n}} M_{d\cdot n}$

$t_1 \quad t_2 \quad \cdots \quad t_n$

system of linear equations with $d^2 n + dn$ variables

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Theorem 2: Solvability of linear equations with implications of the form

$$x_i > 0 \Rightarrow x_j > 0$$

over $\mathbb{Q}_+$ is in P.