Theory of concurrency

2023/24

Lecture 5

A) EXPSPACE lower bound

B) coverability tree

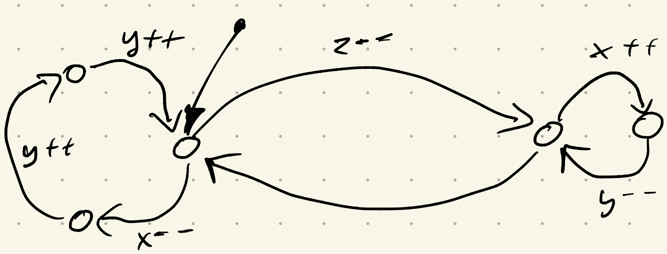
1-bounded nets ⊆ elementary Petri nets ⊆ general Petri nets

↑ all problems are PSPACE-h (1-bounded nets)

↑ (general Petri nets) A) all problems are EXPSPACE-h B) most of them decidable

A) Lipton's construction

Model: counter automata w/o 0-tests = Petri nets



init: $x=1$ $y=0$ $z=100$

convenient notation →

```
loop
|  loop
|  |  x--
|  |  y++
|  |  y++
|  z--
|  loop
|  |  x++
|  |  y--
```

2^{2^n}-bounded readability of a control state in a counter automaton M of size n with 0-tests

(3 counters)

⇝ (↑ quadratic blow-up)

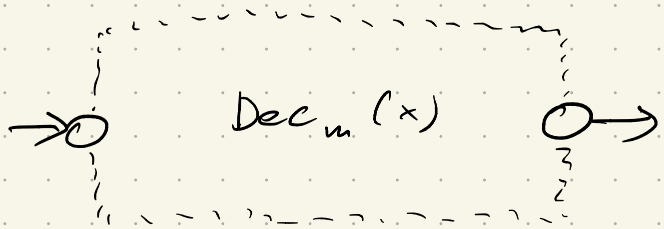
readability of a control state in a counter automaton w/o 0-tests

($O(n)$ counters)

Observation:

$$2^{2^{m+1}} = \left(2^{2^m}\right)^2$$

(1)



Assume "subroutines" $Dec_m(x)$: $(m = 0, 1, \dots, n)$

- $x \geq 2^{2^m} \longrightarrow$ exactly one run succeeds, the effect is: $x \mathrel{-}= 2^{2^m}$ $\quad \hat{x} \mathrel{+}= 2^{2^m}$
- $x < 2^{2^m} \longrightarrow$ all runs fail

(2) simulation of M:

$$\circ \xrightarrow{x++} \circ \qquad\qquad \circ \xrightarrow{x++ \quad \hat{x}--} \circ$$

$$\circ \xrightarrow{x--} \circ \qquad\qquad \circ \xrightarrow{x-- \quad \hat{x}++} \circ$$

$$\circ \xrightarrow{x==0\,?} \circ \qquad\qquad \circ \xrightarrow{Dec_n(\hat{x})} \circ \xrightarrow{Dec_n(x)} \circ$$

Counters: $x, \hat{x}, y, \hat{y}, z, \hat{z}$

init:

$$x = y = z = 0$$

$$\hat{x} = \hat{y} = \hat{z} = 2^{2^n}$$

budget

invariant:

$$x + \hat{x} = 2^{2^n}$$

$$y + \hat{y} = 2^{2^n}$$

$$z + \hat{z} = 2^{2^n}$$

Counters:

$a_0, \hat{a}_0, b_0, \hat{b}_0$

$a_1, \hat{a}_1, b_1, \hat{b}_1$

$\vdots$

$a_{n-1}, \hat{a}_{n-1}, b_{n-1}, \hat{b}_{n-1}$

(3) $Dec_0(x)$:

$x \in \{a_{m+1}, b_{m+1}\}$

$Dec_{m+1}(x)$:

loop
- $a_m{+}{+}$, $\hat{a}_m{-}{-}$
- loop
  - $b_m{+}{+}$, $\hat{b}_m{-}{-}$
  - $\boxed{x{-}{-},\ \hat{x}{+}{+}}$
- $Dec_m(b_m)$

$Dec_m(a_m)$

init:

$a_m = b_m = 0$

$\hat{a}_m = \hat{b}_m = 2^{2^m}$

invariant:

$a_m + \hat{a}_m = 2^{2^m}$

$b_m + \hat{b}_m = 2^{2^m}$

(4) Initiation (bootstrapping)

$Inc_0(\hat{a}_0, \hat{b}_0)$

$Inc_1(\hat{a}_1, \hat{b}_1)$ (using $Dec_0(a_0), Dec_0(b_0)$)

$\vdots$ $\vdots$

$Inc_{n-1}(\hat{a}_{n-1}, \hat{b}_{n-1})$ (using $Dec_{n-2}(a_{n-2}), Dec_{n-2}(b_{n-2})$)

$Inc_n(\hat{x}, \hat{y}, \hat{z})$ (using $Dec_{n-1}(a_{n-1}), Dec_{n-1}(b_{n-1})$)

CHEATING? size of $Dec_n(x)$ is $O(2^n)$!

$Dec_{n+1}(\cancel{x})$:

```
loop
    a_n++ , â_n--
    loop
        b_n++ , b̂_n--
    Dec_n(b_n)
Dec_n(a_n)
```

$\bar{x}$--

x-- $\hat{x}$++

$\bar{x}$++

or

$\bar{y}$--

y-- $\hat{y}$++

$\bar{y}$++

$\bar{a}_n$++ Dec_n $\bar{a}_n$--

$\bar{b}_n$++ Dec_n $\bar{b}_n$--

Question: Does the lower bound apply to:

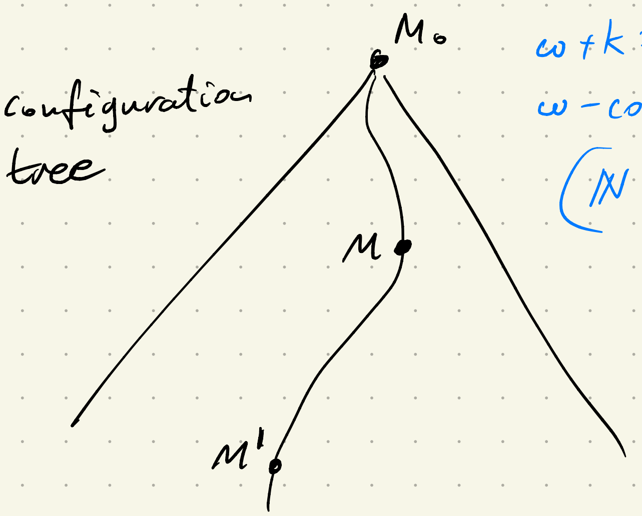
- coverability
- boundedness
- finiteness (termination)

etc

⋮

B) Coverability tree

<u>Model:</u> general Petri nets



$\omega + k = \omega - k = \omega$

ω-conf: $(\mathbb{N} \cup \{\omega\})^d$

<u>Lemma (Dickson):</u>

Every infinite sequence $M_0, M_1, M_2 \ldots$ of ω-configurations admits a domination $M_i \sqsubseteq M_j$, $i < j$

pointwise order in $(\mathbb{N} \cup \{\omega\})^d$

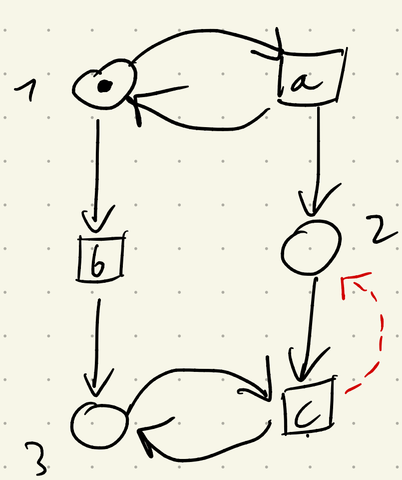
1) $M = M' \quad \rightarrow$ omit M' and all its descendants

2) $M \sqsubset M' \quad \rightarrow$ instead of M' add $\overline{M'}$, where

$$\overline{M'}(i) = \begin{cases} M'(i) & \text{if } M(i) = M'(i) \\ \omega & \text{otherwise.} \end{cases}$$

<u>Fact:</u> Coverability tree is finite

<u>Example:</u>



coverability tree:

$$\begin{tikzcd} 1\,0\,0 \arrow[r, "a"] \arrow[d, "b"] & 1\,\omega\,0 \arrow[r, dashed, "a"] \arrow[d, "b"] & 1\,\omega\,0 \\ 0\,0\,1 & 0\,\omega\,1 \arrow[r, dashed, "c"] & 0\,\omega\,1 \end{tikzcd}$$

configuration graph:

$$
\begin{tikzcd}
100 \arrow[r, "a"] \arrow[d, "b"] & 110 \arrow[r, "a"] \arrow[d, "b"] & 120 \arrow[r, "a"] \arrow[d, "b"] & 130 \arrow[d, "b"] \\
001 & 011 \arrow[l, "c"'] & 021 \arrow[l, "c"'] & 031 \arrow[l, "c"']
\end{tikzcd}
$$

configuration tree:

$$
\begin{tikzcd}
100 \arrow[r, "a"] \arrow[d, "b"] & 110 \arrow[r, "a"] \arrow[d, "b"] & 120 \arrow[r, "a"] \arrow[d, "b"] & 130 \arrow[d, "b"] \\
001 & 011 \arrow[d, "c"] & 021 \arrow[d, "c"] & 031 \arrow[d, "c"] \\
 & 001 & 011 \arrow[d, "c"] & 021 \arrow[d, "c"] \\
 & & 001 & 011 \arrow[d, "c"] \\
 & & & 001
\end{tikzcd}
$$

Fact: (coverability tree "covers" configuration tree)

- $\forall M \in$ configuration tree

  $\exists M' \in$ coverability tree $\quad M \sqsubseteq M'$

- $\forall M' \in$ coverability tree $\quad \forall k \in \mathbb{N}$

  $\exists M \in$ configuration tree $\quad \forall p \in P.$

  $$M(p) = M'(p) \quad \text{if } M'(p) < \omega$$

  $$M(p) \geq k \quad \text{if } M'(p) = \omega$$

Question: How do we decide coverability, boundedness, place-boundedness using coverability trees?