Concurrency theory

2023/24

Lecture 3

MAZURKIEWICZ TRACES

elementary net $\longmapsto$ configur. graph $\longmapsto$ configur. tree (branching time)

$\searrow$ language of accepting runs (linear time)

$$L = \{ w \in T^* : M_0 \xrightarrow{t} F \}$$

- independence: $(t,u) \in I \iff {}^\bullet t^\bullet \cap {}^\bullet u^\bullet = \emptyset$
- dependence: $(t,u) \in D \iff (t,u) \notin I$

Fact: $M_0 \xrightarrow{w\,t\,u\,v} M \quad (t,u) \in I \implies M_0 \xrightarrow{w\,u\,t\,v} M$

Dependence alphabet:

$(\Sigma, D) \quad D \subseteq \Sigma^2$ reflexive, symmetric

$I = \Sigma^2 \setminus D$

Trace equivalence: $\equiv_D \subseteq (\Sigma^*)^2$

the smallest s.t. $w\,ab\,v \equiv_D w\,ba\,v$

for every $w, v \in \Sigma^*$, $(a,b) \in I$

Trace = equivalence class of $\equiv_D$ $\quad [w]_{\equiv_D}$, $[w]_D$, $[w]$

Example : $\Sigma = \{a, b, c\}$ $\quad D = a \begin{smallmatrix} \diagup\, b \\ \diagdown\, c \end{smallmatrix}$

$$[abbca]_D = \{abbca,\ abcba,\ acbba\}$$

$M \xrightarrow{[w]} M'$ in elementary net

Fact : concatenation preserves $\equiv_D$ ← congruence in $(\Sigma^*, \cdot)$

$$\Sigma^* / {\equiv_D} = \Sigma^*/D = \{[w]_D : w \in \Sigma^*\}$$

Trace monoid $\left(\Sigma^*/D, \cdot\right)$

Special cases :

- $D = \Sigma^2$ $\quad$ words Σ^*
- $D = Id_\Sigma$ $\quad$ finite multisets $\Sigma^\oplus$
- $D = \Sigma_1^2 \cup \Sigma_2^2$ $(\Sigma_1 \cap \Sigma_2 = \phi)$ $\quad \Sigma_1^* \times \Sigma_2^*$
- D transitive $\quad \Sigma_1^* \times \ldots \times \Sigma_n^*$
- I transitive $\quad \left(\Sigma_1^\oplus \cup \ldots \cup \Sigma_n^\oplus\right)^*$

Example : prefixes of a trace $[abbcab]$

$$\begin{tikzcd}[column sep=small]
 & & {[ac]} \arrow[dr] & & & \\
{[\varepsilon]} \arrow[r] & {[a]} \arrow[ur] \arrow[dr] & & {[abc]} \arrow[dr] & & \\
 & & {[ab]} \arrow[ur] \arrow[dr] & & {[abbc]} \arrow[r] & {[abbca]} \arrow[d] \\
 & & & {[abb]} \arrow[ur] & & {[abbcab]}
\end{tikzcd}$$

$$[abb][cab] = [ac][bbab]$$

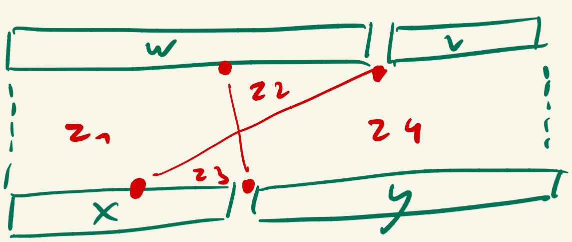
Lemma : $w, v, x, y \in \Sigma^*/D$

$$wv = xy \implies \exists\, z_1\ z_2\ z_3\ z_4 \in \Sigma^*/D$$

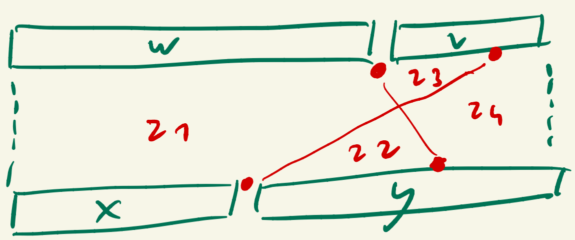
$w = z_1 z_2 \qquad v = z_3 z_4$

$x = z_1 z_3 \qquad y = z_2 z_4$

$z_2 \; I \; z_3$



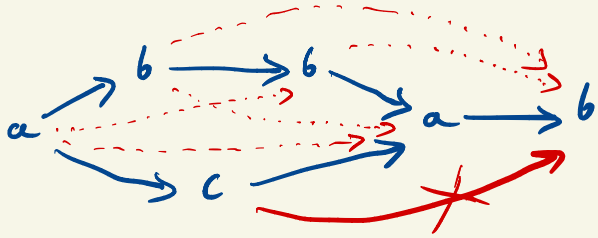
$z_1 = w \sqcap x$

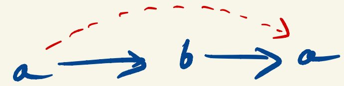


Better representation of traces?

Dependence graphs

Example : $[a\,b\,b\,c\,a\,b]$ $\qquad$ $[a\,b\,a]$





Dependence graph : $(V, E, \ell : V \to \Sigma)$

- finite DAG
- vertices labeled by alphabet letters
- $(v, v') \in E \cup E^{-1} \iff (\ell(v), \ell(v')) \in D$ $\quad v \neq v'$

$$\begin{tikzcd}
\text{traces } \Sigma^*/D \arrow[r, bend left, "?"] & \text{dependence graphs } G_D \arrow[l, bend left, "\text{linearization of labeled partial order}"]
\end{tikzcd}$$

Concatenation of graphs :

$V := V_1 \uplus V_2$

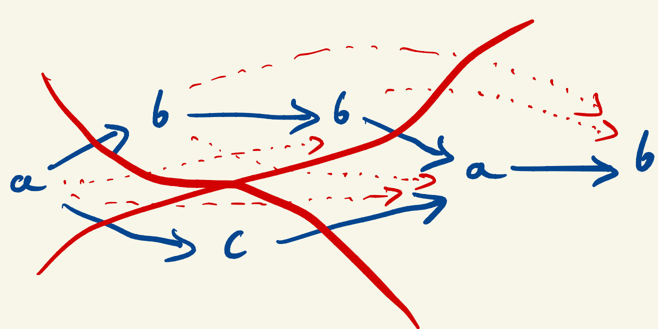
$\ell := \ell_1 \uplus \ell_2$

$E := E_1 \cup E_2 \cup \{(v_1, v_2) \in V_1 \times V_2 : (\ell(v_1), \ell(v_2)) \in D\}$

Lemma : $(\Sigma^*/D, \cdot) \cong (G_D, \cdot)$

Example :

$[a\,b\,b]\,[c\,a\,b] = [a\,c]\,[b\,b\,a\,b]$



$S \subseteq \Sigma^*/_D \qquad L \subseteq \Sigma^*$

trace languages $\approx$ word languages closed under $\equiv_D$

$$\begin{tikzcd} \text{word languages closed under } \equiv_D \\ \text{word languages} \arrow[u, "cl(-)"'] \end{tikzcd}$$

$$cl(L) = \{ w \in \Sigma^* : \exists v \in L.\ v \equiv_D w \}$$

Regular trace languages :

- $cl(L)$, $L \subseteq \Sigma^*$ regular
- $L = cl(L)$, $L \subseteq \Sigma^*$ regular

Example : $\Sigma = \{a, b\}$, $D = Id$, $L = (ab)^*$

$$cl(L) = \{ w : \#a(w) = \#b(w) \}$$

$S = h^{-1}(A)$, $A \subseteq M$

$L = h^{-1}(A)$, $A \subseteq M$

$$\begin{tikzcd} \Sigma^*/_D \arrow[dr, "h"] & \\ \Sigma^* \arrow[u] \arrow[r, "h"] & M \end{tikzcd}$$

finite

Question: do elementary nets recognize all regular trace languages?

$$L(S) = \{ w \in T^* : M_0 \xrightarrow{w} F \}$$

accepting configurations

Example: $L = \{ \varepsilon, a, aa \}$

$L = \{ \varepsilon, a, ba \}$

Model of automaton that recognizes exactly regular trace languages?

Distributed alphabet $\Sigma_1, \dots, \Sigma_n$

$\Sigma = \Sigma_1 \cup \dots \cup \Sigma_n$

$D = \Sigma_1^2 \cup \dots \cup \Sigma_n^2 \subseteq \Sigma^2$

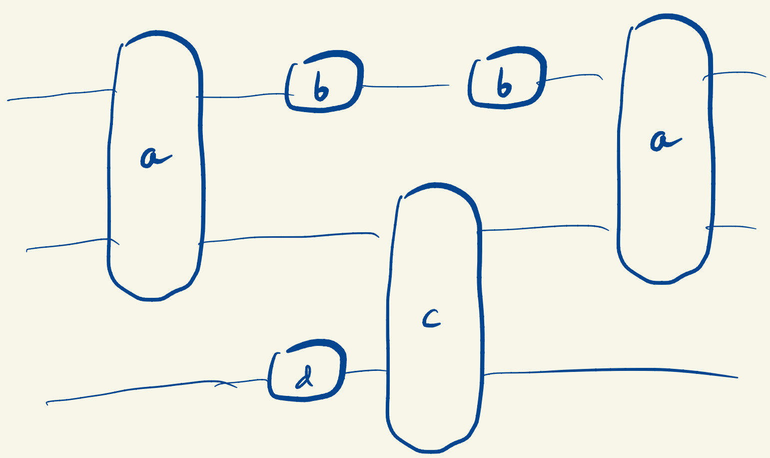
Monoid $\Sigma_1^* \times \dots \times \Sigma_n^*$

pointwise concatenation

Example: $\{a,b,c,d\} = \{a,b\} \cup \{a,c\} \cup \{c,d\}$

(abba, aca, dc)

historia



~~(cabba, ca, dc)~~

nie

Histories:

$\pi_i : \Sigma^* \longrightarrow \Sigma_i^*$ projections

$\pi : \Sigma^* \longrightarrow \Sigma_1^* \times \dots \times \Sigma_n^* \qquad w \longmapsto (\pi_1(w), \dots, \pi_n$

$H_D = \pi(\Sigma^*) \subseteq \Sigma_1^* \times \dots \times \Sigma_n^*$ histories

submonoid generated by

atomic histories $\{\pi(a) : a \in \Sigma\}$

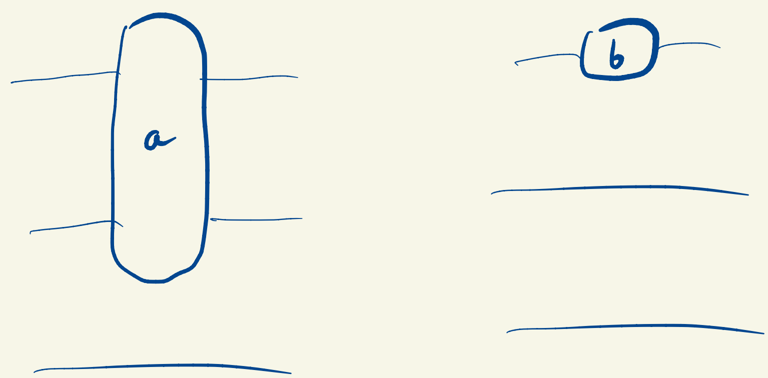
Example: $\pi(abdcba) = (abba, aca, dc)$

$\pi(a) = (a, a, \varepsilon)$

$\pi(b) = (b, \varepsilon, \varepsilon)$

$\pi(c) = (\varepsilon, c, c)$

$\pi(d) = (\varepsilon, \varepsilon, d)$



Lemma: $\Sigma = \Sigma_1 \cup \dots \cup \Sigma_n, \quad D = \Sigma_1^2 \cup \dots \cup \Sigma_n^2$

$$(\Sigma^*/D, \cdot) \cong (H_D, \cdot)$$