DECIDABILITY OF REACHABILITY
IN VECTOR ADDITION SYSTEMS

Perliminary Version

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Abstract

A convincing proof of the decidability of reachability in vector addition systems is presented. No drastically new ideas beyond those in Sacerdote and Tenney, and Mayr are made use of. The complicated tree constructions in the earlier proofs are completely eliminated.

I. Introduction

There already exist two proofs for the decidability of reachability in vector addition systems [4,5,6]. The two approaches have many common features. For example, even though the central concept of 'cones' of [5,6] does not appear explicitly in [4], it seems to have played an equally important role in [4] too. (The surprisingly complicated and unconvincing cone construction of [5,6] can be reduced to a trivial construction.) I will discuss these similarities in the final version. In fact it turns out that no significantly new ideas beyond those of [5,6] are needed to solve this problem. The complicated tree constructions of [4,5,6] can be completely disposed of. I view the tree constructions as convenient tools to test some simple properties of the systems. However, in [4,5,6] certain complicated trees are first constructed and the proof of decidability is built on top of these.

The rest of this section is devoted to pointing out some elementary properties. The main results are presented in the next two sections. Let Z and N be the set of integers and the set of nonnegative integers, respectively. For every n-tuple x, let $\pi_i(x)$ be the value of the i^{th} component of x. For every $m \geq 0$, let $\bar{m}$ be the vector of all m's. The usual componentwise addition, $+$, and comparison, $\geq$, are assumed for vectors. For any vector x, $-x=\bar{0}-x$.

We assume familiarity with semilinear sets, Presburger formulas and their elementary properties [1]. The following somewhat specialized, but trivial, properties are also useful. A linear set with constant c and periods $p_1,\ldots,p_k$ is denoted by $L(c;p_1,\ldots,p_k)$.

Lemma 1: Let L be a semilinear set, and let A be a subset of $\{1,2,\ldots,n\}$. If L does not satisfy the property that for every $m \geq 1$ there exists an $x \in L$ s.t. for every $j \in A$,

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$\pi_j(x) \geq m$, then there exists a constant c such that when L is expressed as a union of linear sets $\bigcup_{i=1}^{k} L_i$, for every L_i there exists a $j \in A$ s.t. the j^{th} component of every period of L_i has value 0 (the j^{th} component of the sum of the periods of L_i has zero value).

Let $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ and $S = \{i_1, i_2, \ldots, i_k\}$. For any $x \in Z^n$, $x(S) = (\pi_{i_1}(x), \pi_{i_2}(x), \ldots, \pi_{i_k}(x))$. For any $L \subseteq N^n$, let $L(S) = \{x(S) \mid x \in L\}$.

Lemma 2:

(a) For any semilinear set L and any set of coordinates S, L(S) is a semilinear set.
(b) If L is a linear set, then L(S) is also a linear set.
(c) If L is a linear set with one period, then L(S) can be expressed with one period also. If L is L(c;p), then L(S) can be given by L(c(s);p(S)).

Thus the projection of a semilinear set into a subset, $\{i_1, \ldots, i_k\}$, of coordinates results in a semilinear set.

Lemma 3: Let L be a semilinear set, and S a subset of its coordinates. If L has the property that

(a) for every $x \in L$ and every $j \notin S$, $\pi_j(x)$ has a fixed value, and
(b) for every $m \geq 1$ there exists an $x \in L$ s.t. for every $j \in S$, $\pi_j(x) \geq m$,
then L has a linear subset L(c;p) where the j^{th} component of p is nonzero iff $j \in S$.

We also need a few concepts from graph theory. Consider a graph (directed and possibly having parallel arcs) in which all the arcs are labeled distinctly. Let these labels be $t_1, t_2, \ldots, t_k$. Throughout, we allow nodes without any in- and out-arcs. Such nodes are isolated nodes.

The folding of any (directed) path, p, of G is a k-tuple $z \in N^k$ such that $\pi_i(z)$ equals the number of occurrences of t_i in the path. Let this folding be denoted by $\pi(p)$. Given a k-tuple $z \in N^k$, a start node q_1, and an end node q_2, the unfolding of z is any path from q_1 to q_2 whose folding is z. For some combinations of z, q_1 and q_2, an unfolding might not exist. Note also that unfolding is not a functional map.

For every $z \in N^k$ and every node q of G, let the in-degree of q w.r.t. z be $\sum_{t_i \text{ is an in-arc of } q} \pi_i(z)$. This is denoted by in(q,z). The out-degree of q w.r.t. z is similarly defined as $\sum_{t_i \text{ is an out-arc of } q} \pi_i(z)$, and is denoted by out(q,z).

In Lemmas 4 to 7, G is any graph and q_1, q_2 and q are any nodes of G.

Lemma 4: Let G be a strongly connected graph. For any k-tuple $z \geq \bar{1}$, an unfolding of z from q_1 to q_1 exists iff for every q, in(q,z) = out(q,z).

Proof: Obvious extension of the standard proofs for the existence of Euler trails (or walks) in directed graphs [2].

Lemma 5: In G, let z be the folding of any path from q_1 to q_2. Then

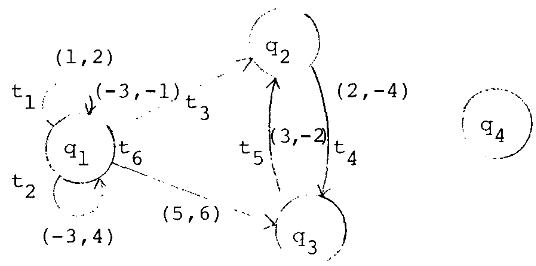
(i) for every $q \notin \{q_1, q_2\}$, in(q,z) = out(q,z), and
(ii) if $q_1 = q_2$ then $\mathrm{in}(q_1,z) = \mathrm{out}(q_1,z)$, and
if $q_1 \neq q_2$ then $(\mathrm{out}(q_1,z) = \mathrm{in}(q_1,z)+1$ and $\mathrm{in}(q_2,z) = \mathrm{out}(q_2,z)+1)$.

Lemma 6: In G, let z_1 and z_2 be the foldings of two paths from q_1 to q_2. If $z_1 - z_2 \geq \bar{1}$, then for every non-isolated q, there exists an unfolding of $z_1 - z_2$ from q to q.

Proof: We leave it to the reader to prove that existence of z_1 and z_2 with $z_1 - z_2 \geq \bar{1}$ implies that all the non-isolated nodes are strongly connected. Now apply Lemma 5 to z_1 and z_2 and deduce that for every node q, $in(q, z_1 - z_2) = out(q, z_1 - z_2)$. Finally apply Lemma 4.

Lemma 7: In G, let z_1 be the folding of a path from q_1 to q_1, and let z_2 be the folding of a path from q_2 to q_2, such that for some i_0, $i_0 z_1 - z_2 \geq \bar{1}$. Then for any non-isolated q, there exists an unfolding of $i_0 z_1 - z_2$ from q to q.

A vector addition system with states (VASS), as in [3], is an fsa or a directed graph in which the label of each arc is an n-tuple of integers. For uniformity of description, we allow the fsa to have states without any in- and out-arcs (isolated states). An example VASS is given below. Note that q_4 is an isolated state. A configuration



of the VASS is given by (q,x) where q is a state and x is a point in Z^n. Given an initial configuration (q_1,x), a path from q_1 in the fsa induces an obvious sequence of configuratons. We informally denote this sequence of configurations or the corresponding sequence of points in Z^n as a path. In the above example for the initial configuration $(q_1,(7,7))$ and the path $q_1 \xrightarrow{t_1} q_1 \xrightarrow{t_2} q_1 \xrightarrow{t_3} q_2 \xrightarrow{t_4} q_3$, the corresponding sequences of configurations and points are $(q_1,(7,7))(q_1,(8,9))(q_1,(5,13))(q_2,(2,12))(q_3,(4,8))$ and $(7,7)(8,9)(5,13)(2,12)(4,8)$, respectively. When the intent is clear, we sacrifice some precision for clarity. For example, we might say that point $(8,9)$ is on the above path.

In a VASS, (q_2,y) is r-reachable from (q_1,x) iff there is a path, p, from (q_1,x) to (q_2,y). We also write this as $(q_2,y) \in r(q_1,x)$ or $(q_2,y) \in r(q_1,x)$ by path p. The corresponding path is an r-path. Observe that an r-path can possibly pass through points in Z^n. The configuration (q_2,y) is R-reachable from the configuration (q_1,x) iff there is a path from (q_1,x) to (q_2,y) s.t. for every configuration (q',v) on the path, $v \in N^n$. We also denote this by $(q_2,y) \in R(q_1,x)$. The corresponding path is an R-path. Note that every R-path must lie completely in the positive orthant. Let A be a subset of the coordinates. The configuration (q_2,y) is semi R-reachable or SR-reachable w.r.t. A from the configuration (q_1,x) iff there is a path from (q_1,x) to (q_2,y) s.t. for every configuration (q',v) on the path and for every $i \in A$, $\pi_i(v) \geq 0$. We also denote this by $(q_2,y) \in SR(q_1,x)$ w.r.t. A. The corresponding path is an SR-path w.r.t. A. Note that when A is the empty set, the SR concept coincides with the r concept, and when A is the set of all n coordinates, the SR concept coincides with the R concept. Now we can state the reachability problem.

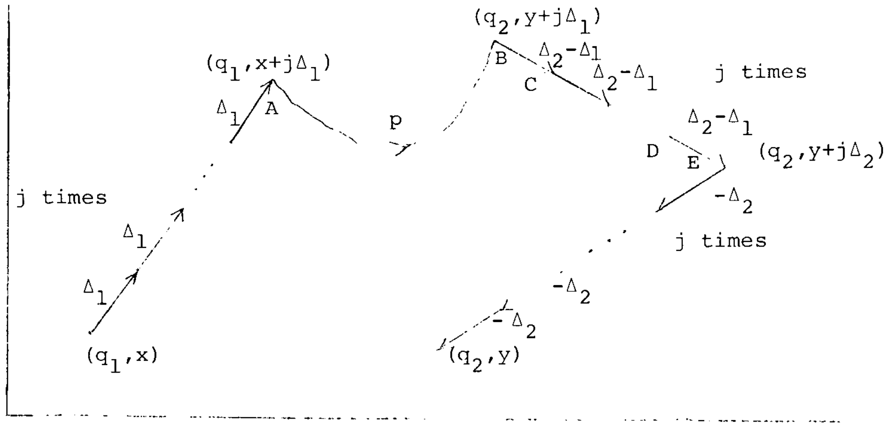
Reachability Problem: Design an algorithm to test for every q_1,x,q_2,y whether $(q_2,y) \in R(q_1,x)$ or not.

(b2) $(q_2, y+\Delta_2) \in R_{rev}(q_2, y)$, and

(c) $(q_2, \Delta_2-\Delta_1) \in r(q_2, \bar{0})$,
then
$(q_2, y) \in R(q_1, x)$.

(Note that (a) states that (q_2,y) is reachable from (q_1,x) by some path in Z^n , (b) states that from (q_1,x) there exists a path in the positive orthant which increases all the coordinates of x , and similarly (q_2,y) can be reached from another point with all coordinates bigger, and (c) specifies certain spanning properties of G .

Proof: Let the r-path from (q_1,x) to (q_2,y) be p (condition (a) assures its existence). Then the R-reachability of (q_2,y) from (q_1,x) can be represented by the following schematic path. Note that the subpaths with Δ_1 and $-\Delta_2$ shifts are R-paths, and the subpaths with $\Delta_2-\Delta_1$ shifts are r-paths. Select the minimum j such that paths



p , BC and DE lie in the positive orthant. (As j increases, every coordinate value of A,B and E increases. Consequently such a j exists). By Lemma 9, the complete path from B to E is in the positive orthant. Thus the complete path is an R-path.

The schematic path shown earlier is very similar to the schematic path in page 11 of [6] and also the proof of Theorem 2.6 in [5]. I want to emphasize this similarity to indicate that the basic idea in [5,6] is sound. In the rest of the paper we generalize this result and solve the reachability problem. The next result is our first variant of Theorem 1.

Theorem 2: In a VASS, G , for every q_1,x,q_2,y if there exist $\Delta_1,\Delta_2 \geq \bar{1}$ s.t.

(a) $(q_2,y) \in r(q_1,x)$,

(b1) $(q_1,x+\Delta_1) \in R(q_1,x)$,

(b2) $(q_2,y+\Delta_2) \in R_{rev}(q_2,y)$, and

(c') $(q_1,\bar{0})$ is r-reachable from $(q_1,\bar{0})$ by a path p s.t. $\pi(p) \geq \bar{1}$,

then $(q_2,y) \in R(q_1,x)$.

Proof: It is sufficient to show that the above conditions imply condition (c) of Theorem 1.

Application of Lemma 8 to condition (b1) leads to $(q_1,\Delta_1) \in r(q_1,\bar{0})$. Similarly

For a VASS, G , let its reverse, denoted G_{rev} , be obtained by reversing the arcs in the fsa and then replacing every label x by -x . The r , R and SR reachabilities of G_{rev} are denoted by r_{rev} , R_{rev} and SR_{rev} , respectively.

Lemma 8: In any VASS, G , if $(q_2,y) \in r(q_1,x)$, then for every Δ , $(q_2,y+\Delta) \in r(q_1,x+\Delta)$. In particular, $(q_2,y-x) \in r(q_1,\bar{0})$.

Proof: Simply shift the starting point, keeping the old path.

If p is a path from $\bar{0}$ to v , then p has the "effect" of shifting any point by v . We denote this effect by shift(p) ; i.e. shift(p) = v .

Lemma 9: In a VASS, G , let $(q_1,\Delta_1) \in r(q_1,\bar{0})$ by a path p . Consider a sequence of configurations (q_1,x) , $(q_1,x+\Delta_1)$, $(q_1,x+2\Delta_1)$, ... , $(q_1,x+k\Delta_1)$ where path p is applied from $(q_1,x+i\Delta_1)$ to $(q_1,x+(i+1)\Delta_1)$ for $i=0,\ldots,k-1$. If the two paths $(q_1,x) \xrightarrow{p} (q_1,x+\Delta_1)$ and $(q_1,(x+(k-1)\Delta_1)) \xrightarrow{p} (q_1,x+k\Delta_1)$ are R-paths, then the complete path is an R-path.

Proof: Observe that for every $1 \le i \le n$ and $1 \le j \le k-1$, $\pi_i(x+j\Delta_1)$ is in between $\pi_i(x)$ and $\pi_i(x+k\Delta_1)$.

Lemma 10: In any VASS , G , and for any initital configuration (q_1,x) , the following hold:

(a) it can be effectively decided whether there exists a $\Delta \ge \bar{1}$ s.t. $(q_1,x+\Delta) \in R(q_1,x)$, and

(b) if there does not exist any $\Delta \ge \bar{1}$ satisfying $(q_1,x+\Delta) \in R(q_1,x)$, then a constant c s.t. every point R-reachable from (q_1,x) has some coordinate value $\le c$ can be effectively computed.

Proof: A trivial tree construction establishes this lemma.

The next lemma is a simple generalization of Lemma 10.

Lemma 11: In any VASS, G , for any set of coordinates A and any initial configuration (q_1,x) , the following hold:

(a) it can be effectively decided whether there exists a $\Delta \in Z^n$, s.t. for every $j \in A$, $\pi_j(\Delta) \ge 1$, and $(q_1,x+\Delta) \in SR(q_1,x)$ w.r.t. A , and

(b) if there does not exist any Δ satisfying (a), then a constant c s.t. every point SR-reachable w.r.t. A from (q_1,x) has i^{th} component value $\le c$ for some i in A can be effectively computed.

Proof: Suppress the coordinates which are not in A from the labels of all the arcs of G and also from x and y , and then apply Lemma 10.

II. A Result on VASS

The following two theorems form the bridge between the approaches of [4,5,6] and the proof given here. They do not form a part of the main approach, and the anxious reader is advised to go directly to Theorem 3. Theorem 4 is the main result that will be applied in the next section. In all these theorems q_1 and q_2 are any states of G , and x and y are in N^n .

Theorem 1: In a VASS , G , for every q_1,x,q_2,y if there exist $\Delta_1,\Delta_2 \ge \bar{1}$ s.t.

(a) $(q_2,y) \in r(q_1,x)$,

(b1) $(q_1,x+\Delta_1) \in R(q_1,x)$

condition (b2) implies $(q_2,\Delta_2) \in r_{rev}(q_2,\bar{0})$, which further implies $(q_2,-\Delta_2) \in r(q_2,\bar{0})$. Let p_1 be an r-path from $(q_1,\bar{0})$ to (q_1,Δ_1) , and let p_2 be an r-path from $(q_2,\bar{0})$ to $(q_2,-\Delta_2)$. Choose the minimum i_0 s.t.

$i_0\ \pi(p) \geq \pi(p_1)+\pi(p_2)+\bar{1}$ (this can be done since $\pi(p) \geq \bar{1}$) . Then application of Lemma 7 twice $(i_0\ \pi(p)-\pi(p_1) \geq \bar{1}$ and $(i_0\pi(p)-\pi(p_1))-\pi(p_2) \geq \bar{1})$, results in the existence of a path from q_2 to q_2 (q_2 is non-isolated due to b2) whose folding is $i_0\ \pi(p) - (\pi(p_1) + \pi(p_2))$. Hence there exists an r-path from $(q_2,\bar{0})$ to $(q_2,\Delta_2-\Delta_1)$. (Note that $\mathrm{shift}(p)=\bar{0}$, $\mathrm{shift}(p_1)=\Delta_1$, and $\mathrm{shift}(p_2)=-\Delta_2$) . Thus condition (c) of Theorem 1 holds.

In this theorem, condition (c') gives a spanning property of the VASS. In the spirit of [5,6], it can be interpreted as that if any vector v is in the "positive span" then $-v$ is also in the positive span.

The next theorem is an important generalization of Theorem 2.

<u>Theorem 3</u>: In a VASS, for every q_1,x,q_2,y and $\Delta x,\Delta y \geq \bar{0}$, if there exist $\Delta_1,\Delta_2 \in Z^n$ and $m_1,m_2 \geq 0$, s.t.

(a) for every $i \geq 1$, $\pi_i(\Delta x)=0 \Longrightarrow \pi_i(\Delta_1) \geq 1$, and

$$\pi_i(\Delta y)=0 \Longrightarrow \pi_i(\Delta_2) \geq 1 ,$$

(b) $(q_2,y) \in r(q_1,x)$,

(c) $(q_1,x+m_1\Delta x+\Delta_1) \in R(q_1,x+m_1\Delta x)$, and

$(q_2,y+m_2\Delta y+\Delta_2) \in R_{rev}(q_2,y+m_2\Delta y)$, and

(d) $(q_1,\Delta y)$ is r-reachable from $(q_1,\Delta x)$ by a path p s.t. $\pi(p) \geq \bar{1}$,

then

$(\exists j_0)(\forall j \geq j_0)(q_2,y+j\Delta y) \in R(q_1,x+j\Delta x))$.

Note that if $\Delta x=\Delta y=\bar{0}$, then this result degenerates into Theorem 2.

<u>Proof</u>: First we prove that the above conditions imply the condition (d') and (e) given below.

(d') $(q_2,\Delta y-\Delta x) \in r(q_2,\bar{0})$, and

(e) there exists an integer $\alpha \geq 1$ s.t.

$$\alpha\Delta x + \Delta_1 \geq \bar{1} ,$$

$$\alpha\Delta y + \Delta_2 \geq \bar{1} , \text{ and}$$

$$(q_2,\alpha(\Delta y-\Delta x)+\Delta_2-\Delta_1) \in r(q_2,\bar{0}) .$$

By Lemma 8, (d) implies that $(q_1,\Delta y-\Delta x) \in r(q_1,\bar{0})$ by path p . Since p makes use of every arc, p goes through every non-isolated state. Remove a prefix of p that corresponds to a subpath from q_1 to q_2 and append it as the suffix of p (q_2 is non-isolated due to (c)) . This establishes (d').

Since $\Delta x,\Delta y \geq \bar{0}$, (a) implies that there exists a $\beta \in N$ s.t. for every $\beta' \geq \beta$:

$$\beta'\Delta x+\Delta_1 \geq \bar{1} , \text{ and}$$

$$\beta'\Delta y+\Delta_2 \geq \bar{1} .$$

Application of Lemma 8 to (c) results in

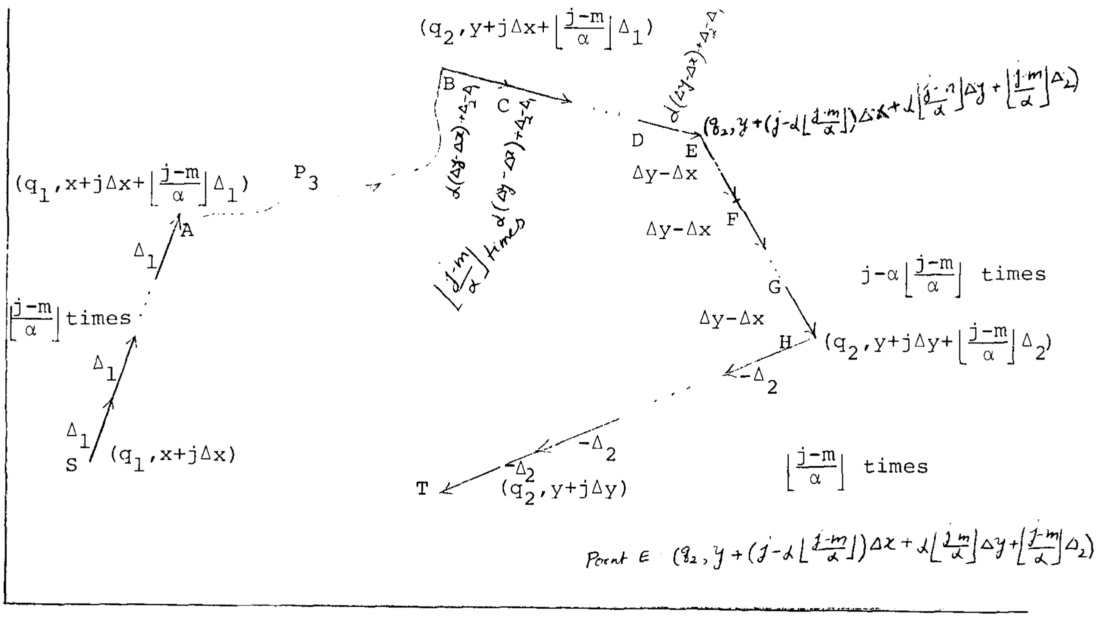
$$(q_1,\Delta_1) \in r(q_1,\bar{0}) , \text{ and}$$

$$(q_2,-\Delta_2) \in r(q_2,\bar{0}).$$

Let p_1 be an r-path from $(q_1,\bar{0})$ to (q_1,Δ_1), and let p_2 be an r-path from $(q_2,\bar{0})$ to $(q_2,-\Delta_2)$. Let γ be the minimum value ≥ 1 s.t. $\gamma\pi(p) \geq \pi(p_1)+\pi(p_2)+\bar{1}$ (this is feasible since $\pi(p) \geq \bar{1}$). For any $\gamma' \geq \gamma$, $\gamma'\pi(p) \geq \pi(p_1)+\pi(p_2)+\bar{1}$. Application of Lemma 7 twice $(\gamma'\pi(p)-\pi(p_1) \geq \bar{1}$, and $(\gamma'\pi(p)-\pi(p_1)-\pi(p_2) \geq \bar{1})$ results in the existence of a path from q_2 to q_2 whose folding is $\gamma'\pi(p) - (\pi(p_1)+\pi(p_2))$. Finally $\gamma'\text{shift}(p)-(\text{shift}(p_1)+\text{shift}(p_2)) = \gamma'(\Delta y-\Delta x)+\Delta_2-\Delta_1$. Then choose $\alpha = \max\{\beta,\gamma\}$, $\beta'=\alpha$, and $\beta'=\alpha$, establishing (e).

Now we shall make use of conditions (b),(c),(d') and (e) to prove the theorem.

Let an r-path from (q_1,x) to (q_2,y) be p_3. The R-reachability of $(q_2,y+j\Delta y)$ from $(q_1,x+j\Delta x)$ can be represented by the following schematic path $(m=\max\{m_1,m_2\})$.



Note that when j increases by a step of α, all the components of A,B,E and H increase (This for E follows easily from the first two conditions of (e). For A,B and H a little more justification is needed. As an example, for A, observe that $x+j\Delta x+\lfloor\frac{j}{\alpha}\rfloor\Delta_1 = x+\lfloor\frac{j}{\alpha}\rfloor(\alpha\Delta_2+\Delta_1)+(j-\alpha\lfloor\frac{j}{\alpha}\rfloor)\Delta x)$. In addition, if $j-m$ is a multiple of α and if j increases by any step in between 1 and α, then no component of A,B,E and H decreases, since $\Delta x,\Delta y \geq \bar{0}$ (caution: $\Delta x+\Delta_1$ and $\Delta y+\Delta_2$ can possibly have negative components). Now keep $j-m$ as a multiple of α and increase it in steps of α until the paths p_3, BC, DE, EF and GH are entirely in the positive orthant. Let the minimum value of j when this happens be j_0. For that choice of j, by Lemma 9, paths BE and EH are in the positive orthant. From the above observations, for every $j \geq j_0$ every component of A,B,E and H is greater than or equal to the corresponding component of j_0. Hence for every $j \geq j_0$, the corresponding path is an R-path.

The next theorem is a minor generalization of Theorem 3, and forms the heart of the decision procedure.

<u>Theorem 4</u>: In a VASS, G, for every q_1,x,q_2,y and $\Delta x,\Delta y \geq \bar{0}$, if there exists $\Delta_1,\Delta_2 \in Z^n$, $A \subseteq \{1,2,\ldots,n\}$ and $m_1,m_2 \geq 0$ s.t.

(a) for every $j \in A$, the j^{th} component of the label of every arc of G has value 0, and
for every $j \notin A$, $\pi_j(\Delta x) = 0 \implies \pi_j(\Delta_1) \geq 1$, and
$\pi_j(\Delta y) = 0 \implies \pi_j(\Delta_2) \geq 1$,

(b) $(q_2,y) \in r(q_1,x)$

(c) $(q_1, x+m_1\Delta x+\Delta_1) \in R(q_1, x+m_1\Delta x)$, and
$(q_2, y+m_2\Delta y+\Delta_2) \in R_{rev}(q_2, y+m_2\Delta y)$, and

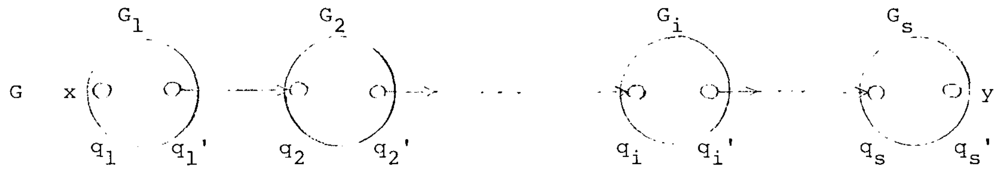
(d) $(q_1,\Delta y)$ is r-reachable from $(q_1,\Delta x)$ by a path p s.t. $\pi(p) \geq \bar{1}$,
then
$(\exists j_0)(\forall j \geq j_0)((q_2, y+j\Delta y) \in R(q_1, x+j\Delta x))$.

Proof: For every $j \in A$, the j^{th} component of the label of every arc has value 0, which implies that for any path in G the j^{th} component will not change. Thus from (b) we can infer that for every $j \in A$, $\pi_j(x) = \pi_j(y)$. Condition (d) implies that for every $j \in A$, $\pi_j(\Delta x) = \pi_j(\Delta y)$. Hence for every $i \geq 0$ and every $j \in A$, $\pi_j(x+i\Delta x) = \pi_j(y+i\Delta y)$. Now from the labels of all the arcs of G and also from all the vectors involved suppress the coordinates in A and apply Theorem 3. Any R-path so obtained from $(q_1,x+i\Delta x)$ to $(q_2,y+i\Delta y)$ is also an R-path for the original G (In this statement, reference to $x+i\Delta x$ and $y+i\Delta y$ is technically incorrect, since after the coordinates of A are suppressed the new vectors are $n-|A|$ dimensional. However the intent is clear, and we will not indulge in unnecessary new notation).

In the next section, we will introduce a more general model of VASS's, known as GVASS. We will then give a decision procedure for solving its reachability problem.

3. Generalized VASS (GVASS)

Consider the following chain of n-dimensional VASS's. Every G_i is a VASS and



there is one arc from q_i' to q_{i+1}, for $i=1,\ldots,s-1$. Every r-path from (q_1,x) to (q_s',y) goes through the arc (q_{i-1}',q_i), for $i=2,\ldots,s$, exactly once. When the r-path reaches q_i for the first time, the corresponding n-dimensional point is denoted as the input point of G_i; similarly, when the path reaches q_i' for the last time, the corresponding n-dimensional point is the output point of G_i. For G_1 the input point is x, and for G_s the output point is Y. Note that an r-path from (q_1,x) to (q_s',y) need not have the input and output points of every G_i in the positive orthant. From now on we shall be interested only in r-paths which have all the intermediate input and output points in the positive orthant. With the help of Presburger formulation, it can be shown that whether there exists such an r-path from (q_1,x) to (q_s',y) can be effectively decided.

We further tighten the concept of an r-path as follows. We place the additional constraint that for every G_i certain specified coordinates of its input and output points must have fixed specified values. To make this precise, we introduce a new symbol, ω, as in [4], which stands for "don't care" or simply "≥ 0". For each G_i we impose 2 constraints: an input constraint, V_i, and an output constraint, V_i', where $V_i, V_i' \subseteq$

$(N\cup\{\omega\})^n$. An r-path from (q_1,x) to (q_s',y) is said to satisfy the input constraint of G_i if for every j with $\Pi_j(V_i)\in N$ the j^{th} component of the input point of G_i has the value $\Pi_j(V_i)$, and for every j with $\Pi_j(V_i)=\omega$ the j^{th} component of the input point of G_i has a value ≥ 0 . That is, non-ω's specify exact values, and ω's are don't cares. The path satisfies the output constraint of G_i if for every j with $\Pi_j(V_i')\in N$ the j^{th} component of the output point of G_i has the value $\Pi_j(V_i')$, and for every j with $\Pi_j(V_i')=\omega$ the j^{th} component of the output point of G_i has a value ≥ 0 . Let us denote any r-path which satisfies the input and output constraints for all G_i's as a <u>cr-path</u>. We express this by $(q_s',y)\in cr(q_1,x)$. Note that when a cr-path is inside† a G_i , the corresponding points can have negative components.

A <u>CR-path</u> from (q_1,x) to (q_s',y) is a cr-path from (q_1,x) to (q_s',y) , and, in addition, no intermediate point can have a negative component. We express this by $(q_s',y)\in CR(q_1,x)$. Note that a CR-path is a cr-path and also an R-path.* Then in any cr-path from (q_1,x) to (q_s',y) , the j^{th} component of the input and the output points of G_i must be the same. This holds since any path in G_i does not change the j^{th} component. In addition, when the path is inside G_i the j^{th} component of every point on this subpath is positive (in fact equal to the j^{th} component of the input point which is the same as the j^{th} component of the output point).

The above chain of VASS's with the input and output constraints and the rigid set specified for each G_i is a <u>generalized VASS</u>, denoted by <u>GVASS</u>.

The <u>reachability problem</u> for a GVASS is to test whether $(q_s',y)\in CR(q_1,x)$ for every given q_1,x,q_s' and y .

Observe that when $s=1$, the GVASS becomes a VASS. Consequently decidability of reachability for GVASS's implies decidability of reachability for VASS's. In this section we give an effective procedure for the reachability problem for GVASS's. Throughout, G refers to the GVASS shown above.

Let the total number of arcs in $G_1,\ldots,G_s$ be k , where each G_i has k_i arcs $(k=\sum k_i)$. Any cr-path from (q_1,x) to (q_s',y) can be mapped into a $(2sn+k)$ - tuple, with the following interpretation.

1st n block	2nd n block	...	$2s^{th}$ n block	k block
input point of G_1 = x	output point of G_1		output point of G_s = y	arcs

For $i=1,\ldots,s$, the $(2i-1)^{th}$ n block specifies the input point of G_i ; for $i=1,\ldots,s$, the $2i^{th}$ n block specifies the output point of G_i ; the last k block specifies the number of times each arc in $G_1,\ldots,G_s$ is used in the path. Let us denote this "extended folding" by Π^e . Let

$$L_G=\{\Pi^e(p)\ |p\text{ is a cr-path from }(q_1,x)\text{ to }(q_s',y)\}\ .$$

It can be easily shown, with the help of Presburger formulation, that L_G is an effectively computable semilinear set. If we project L_G into the $2n+k_i$ dimensional subspace corresponding to the input (n components), and output (n components) points and the arcs (k_i components) of G_i , then the resulting set is semilinear (Lemma 2a), and this set gives the extended foldings of subpaths in G_i . Let this projection be denoted by $L_G[i]$.

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† "path inside a G_i" refers to the subpath every one of whose states is in G_i .
* Some definitions were accidentally omitted; they are given on the bottom of p. 281.

In V_i , replace every ω component by 0 , and let the new vector be denoted by v_i ; similarly v_i' is obtained from V_i' by replacing every ω by 0 . Now we define a very crucial property of GVASS's.

The GVASS, G , with initial configuration (q_1,x) and final configuration (q_s',y) satisfies <u>property θ</u> iff the following conditions are satisfied:

$\theta 1$: for every $m \geq 1$, there exists a cr-path from (q_1,x) to (q_s',y) s.t.

(a) every arc in every G_i is used at least m times, and

(b) for every i and j , if $j \notin S_i$ then the j^{th} component of the input point of G_i has a value $\geq m$, and if $j \notin S_i'$ then the j^{th} component of the output point of G_i has a value $\geq m$, and

$\theta 2$: for every i , there exist $\Delta_i, \Delta_i' \subseteq Z^n$ s.t. for every $j \in S_i - R_i$, $\pi_j(\Delta_i) \geq 1$ and for every $j \in S_i' - R_i$, $\pi_j(\Delta_i') \geq 1$, and

(a) $(q_i, v_i + \Delta_i) \in SR(q_i, v_i)$ w.r.t. $S_i - R_i$ in G_i , and

(b) $(q_i', v_i' + \Delta_i') \in SR_{rev}(q_i', v_i')$ w.r.t. $S_i' - R_i$ in G_i .

(Informally condition $\theta 1(b)$ states that every unconstrained input or output coordinate of every G_i must have a value $\geq m$. Condition $\theta 2$ states that for every G_i in the subspace of its constrained, but nonrigid, input coordinates, all the components of its input constraint vector can be simultaneously increased by an R-path; and in the subspace of its constrained, but nonrigid, output coordinates, all the components of its output constraint vector can be simultaneously increased by an R-path in $G_{i_{rev}}$).

<u>Theorem 5</u>: In the GVASS, G , there is a CR-path from (q_1,x) to (q_s',y) if G satisfies property θ .

<u>Proof</u>: Recall that $L_G = \{\Pi^e(p) \mid p$ is a cr-path from (q_1,x) to $(q_s',y)\}$ is a semilinear set. Since G satisfies property θ , for every m , L_G contains an element whose components corresponding to every unconstrained coordinate and every arc have values $\geq m$ (from $\theta 1$). Thus by Lemma 3, L_G contains a linear subset of the form $L(c;p)$, denoted $\hat{L}_G$, where p has nonzero components corresponding to every unconstrained coordinate and every arc, and zero components corresponding to every constrained coordinate. Project $\hat{L}_G$ into G_i ; i.e. consider $\hat{L}_G[i]$. Every element of $\hat{L}_G[i]$ is in N^{2n+k_i} and is the extended folding of a subpath of a cr-path from (q_1,x) to (q_s',y). By Lemma (2c), $\hat{L}_G[i]$ is a linear set and can be written as $L((x^i,y^i,z^i);\ (\Delta x^i,\Delta y^i,\Delta z^i))$. Where $x^i,y^i,\Delta x^i$ and Δy^i are n-tuples and z^i and Δz^i are k_i-tuples. The x's and y's correspond to the input and the output points, respectively, of G_i . The z's correspond to the foldings of subpaths in G_i . By Lemma (2c) (period of projection=p(s)) , we can infer that for every j ,

$$\pi_j(\Delta x^i) \geq 1 \text{ iff } j \notin S_i \text{ , and } \pi_j(\Delta y^i) \geq 1 \text{ iff } j \notin S_i' \text{ , and}$$

$$\Delta z^i \geq \bar{1} \quad \ldots\ldots (*)$$

In the following we establish that G_i satisfies conditions (a) to (d) of Theorem 4 (with $q_1 = q_i, x = x^i$, $q_2 = q_i'$, $y = y^i$, $\Delta x = \Delta x^i$, $\Delta y = \Delta y^i$, $\Delta_1 = \Delta_i$, $\Delta_2 = \Delta_i'$ and $A = R_i$) .

For every $j \in R_i$ the j^{th} component of the label of every arc in G_i has value 0 . For every $j \notin R_i$, if $\pi_j(\Delta x^i) = 0$, then $j \in S_i - R_i$ (from *) , which, by condition $\theta 2$, implies that $\pi_j(\Delta_i) \geq 1$. For every $j \notin R_i$, if $\pi_j(\Delta y^i) = 0$, then $j \in S_i' - R_i$ (from *), which, by condition $\theta 2$, implies that $\pi_j(\Delta_i') > 0$. Hence for G_i condition (a) of Theorem 4 holds.

From the definition of $\hat{L}_{G}[i]$, it is easily seen that, for every $j \geq 0$, there is an r-path in G_i from $(q_i, x^i + j\Delta x^i)$ to $(q_i', y^i + j\Delta y^i)$, the folding of the path being $z^i + j\Delta z^i$. For $j = 0$, the above implies that $(q_i', y^i) \in r(q_i, x^i)$, which establishes condition (b) of Theorem 4.

Since there is a cr-path from (q_1, x) to (q_s', y) with the input and output points of G_i being x^i and y^i , respectively (from the fact that $(x^i, y^i, z^i) \in \hat{L}_G[i])$, we can infer (from the definition of a cr-path) that for every $j \in S_i$, $\pi_j(V_i) = \pi_j(x^i)$, and for every $j \in S_i'$, $\pi_j(V_i') = \pi_j(y_i)$.

Due to θ2(a), there exists an SR-path, say p_1 , w.r.t. $S_i - R_i$ in G_i from (q_i, v_i) to $(q_i, v_i, +\Delta_i)$. Since path p_1 is entirely in G_i , $v_i \geq \bar{0}$, and for every $j \in R_i$ the j^{th} component of every arc in G_i has value 0 , we can infer that path p_1 is an SR-path w.r.t. S_i from (q_i, v_i) to $(q_i, v_i + \Delta_i)$. From the definition of v_i , it is easily seen that for every $j \in S_i$, $\pi_j(v_i) = \pi_j(V_i)$. Hence for every $j \in S_i$, $\pi_j(v_i) = \pi_j(V_i) = \pi_j(x^i)$. Consequently, path p_1 is an SR-path w.r.t. S_i in G_i from (q_i, x^i) to $(q_i, x^i + \Delta_i)$ (shift the starting point of p_1 from v_i to x^i) . Since $\Delta x^i \geq \bar{0}$, and for every $j \notin S_i$, $\pi_j(\Delta x^i) \geq 1$ (from *), there exists an m_1, s.t. the path p_1 from $(q_i, x^i + m_1\Delta x^i)$ to $(q_i, x^i + m_1\Delta x^i + \Delta_i)$ is an R-path in G_i (shift the starting point from x^i to $x^i + m_1\Delta x^i$) . Making use of θ2(b) and following a similar argument we can prove that there exists an m_2 s.t. there is an R-path in $G_{i_{rev}}$ from $(q_i', y^i + m_2\Delta y^i)$ to $(q_i', y^i + m_2\Delta y^i + \Delta_i')$. Hence condition (c) of Theorem 4 holds.

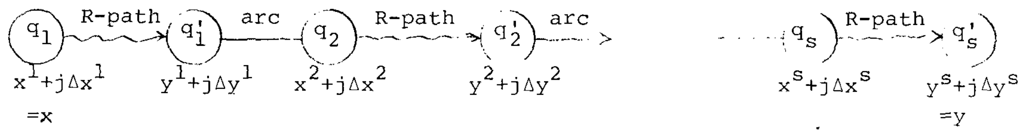
Recall that for every $j \geq o$, there is an r-path in G_i from $(q_i, x^i + j\Delta x^i)$ to $(q_i', y^i + j\Delta y^i)$ the folding of the path being $z^i + j\Delta z^i$. Consider two such paths, one for $j = 0$ and the other for $j = 1$, and apply Lemma 6. This yields that there is an unfolding of Δz^i from $(q_i, \Delta x^i)$ to $(q_i, \Delta y^i)$. Since $\Delta z^i \geq \bar{1}$ (condition *), $(q_i, \Delta y^i)$ is r-reachable from $(q_i, \Delta x^i)$ by a path whose folding is $\geq \bar{1}$. This proves condition (d) of Theorem 4. Hence for G_i , by Theorem 4,

$$(\exists j_0)(\forall j \geq j_0)((q_i', y^i + j\Delta y^i) \in R(q_i, x^i + j\Delta x^i)) .$$

For every i there is a constant j_0 as given by the above statement. Now choose the maximum among such constants, as i varies from 1 to s . Let this be J . Then

$$(\forall i)(\forall j \geq J)(q_i', y^i + j\Delta y^i) \in R(q_i, x^i + j\Delta x^i)) .$$

For any such j , the above statement specifies a CR-path from (q_1, x) to (q_s', y) , which is schematically shown below. Note that $x^1 = x$, $\Delta x^1 = \bar{0}$, $y^s = y$ and $\Delta y^s = \bar{0}$)



Recall that for every i, the linear set $L((x^i, y^i, z^i); (\Delta x^i, \Delta y^i, \Delta z^i))$ was obtained by projecting $\hat{L}_G$ into G_i . Hence for every $j \geq 0$,
$(x^1, y^1, x^2, y^2, \ldots, x^s, y^s, z^1, z^2, \ldots, z^s) + j(\Delta x^1, \Delta y^1, \Delta x^2, \Delta y^2, \ldots, \Delta x^s, \Delta y^s, \Delta z^1, \Delta z^2, \ldots, \Delta z^s) \in \hat{L}_G$.
Consequently, for every $i = 1, \ldots, s-1$ and $j \geq 0$, there is an arc from $y^i + j\Delta y^i$ to $x^{i+1} + j\Delta x^{i+1}$ (in fact this is the arc (q_i', q_{i+1})). Thus the complete path shown above is an R-path.

We exhibited an infinite number of CR-paths from (q_1, x) to (q_s', y) , even though it is sufficient to exhibit just one path.

Now we want to show that the conditions of Theorem 5 can be effectively tested.

Theorem 6: It is effectively decidable whether a GVASS, G, satisfies property θ.

Proof: To test for condition (1), compute the semilinear set L_G as a finite union of linear sets. At least one of these linear sets must have the property that the sum of its periods has a nonzero entry corresponding to every unconstrained coordinate and also every arc. (It trivially follows from the definition of L_G that for every linear subset of L_G, every one of its periods, and hence the sum of its periods, has a 0 value corresponding to every constrained coordinate). This can be easily tested. To test for property 2, apply Lemma 11 to each G_i twice (once for G_i and another time for $G_{i_{rev}}$).

Now we shall establish that if G does not satisfy property θ, then the "size" of the GVASS can be reduced. For every G_i, define its size by a triple (n_{i1}, n_{i2}, n_{i3}) where:

n_{i1} = n- number of rigid coordinates of $G_i (=n-|R_i|)$,

n_{i2} = number of arcs of $G_i (=k_i)$, and

n_{i3} = number of unconstrained input and output coordinates of $G_i (=2n-|S_i|-|S_i'|)$.

The size of G, denoted $SS(G)$, is given by the multiset of sizes of its G_i's. Let '<' refer to the dictionary order among triples; i.e. $(a_1,a_2,a_3) < (b_1,b_2,b_3)$ iff $((a_1 < b_1)$ or $(a_1 = b_1$ and $a_2 < b_2)$ or $(a_1 = b_1, a_2 = b_2$ and $a_3 < b_3))$. In Theorem 7, we shall establish that if G does not satisfy property θ, then we can replace G by a finite number of GVASS's, $G^1, G^2, \ldots$, such that for every i, $SS(G^i)$ can be obtained from $SS(G)$ by replacing a triple by a finite number of smaller triples, and in addition, the CR-reachability of G has a 'yes' answer iff the corresponding problem for some G^i has a 'yes' answer. This establishes that only a finite number of modifications are possible. If the procedure terminates without satisfying property θ, the size of every G_i will be $(0,0,0)$; i.e. every coordinate of G_i is rigid and G_i is a single node or many isolated nodes without any arcs. If property θ does not hold, at this stage, then it can be reported that there is no CR-path from (q_1,x) to (q_s',y).

Theorem 7: If the GVASS, G, does not satisfy property θ and if $SS(G)$ contains an element different from $(0,0,0)$, then G can be replaced by a finite number of GVASS's, $G^1, G^2, \ldots$, such that

(1) for every i, $SS(G^i)$ can be obtained from $SS(G)$ by replacing a triple by a finite number of triples, each of which is less than the triple being replaced, and

(2) the CR-reachability of G has a 'yes' answer iff the CR-reachability of some G^i has a 'yes' answer.

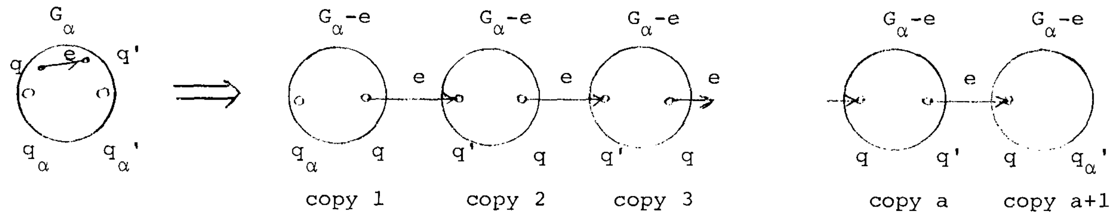
Proof: L_G is a semilinear set and can be expressed as the union of a finite number of linear sets:

$$L_G = L_1 \cup L_2 \cup \ldots \cup L_\beta \text{, where } L_i = L(c_i, p_{i1}, p_{i2}, \ldots, p_{i\gamma_i}).$$

If L_G = empty set, then remove all arcs of G and set all unconstrained coordinates to 0's.

Condition $\theta 1(a)$

If condition $\theta 1(a)$ fails, then we shall replace G by β GVASS's, $G^1, \ldots, G^\beta$, the G^i being based on L_i. When $\theta 1(a)$ fails, by Lemma 1, for every i there exists an $i0$ s.t. the $i0^{th}$ component of $p_{i1}+p_{i2}+\ldots+p_{i\gamma_i}$ is 0, and this $i0^{th}$ component corresponds to some arc. Let this arc be $e = (q, \hat{q}')$, and let it be in some G_α. Note also that $\pi_{i0}(c_i)$ specifies the number of times arc e in G_α gets used on each cr-path whose extended folding is in L_i.



G_α-e is obtained from G_α by simply removing the arc e . The chain shown above has a+1 copies of G_α-e , and for each copy let the set of rigid components be R_α . The input constraint of copy 1 is V_α and the output constraint of copy a+1 is V_α' . The remaining input and output constraints are identical (say = W) as given by:

$$\pi_j(W) = \begin{cases} \omega & \text{if } j \notin R_\alpha \\ \pi_{i0}(V_\alpha) & \text{if } j' \in R_\alpha \end{cases} .$$

The first two components of the size of each copy of G_α-e are $n_{\alpha 1}$ and $n_{\alpha 2}-1$. Thus the size of each G_α-e is less than the size of G_α .

After the above transformation, let the new GVASS be G^i . It can be easily shown that:

> in G there exists a CR-path, from (q_1,x)
> to (q_s',y) , whose extended folding is in L_i
> iff in G^i there exists a CR-path from (q_1,x)
> to (q_s',y) .

Thus the theorem holds in this case.

Condition θ1(b)

If condition θ1(b) fails, then we shall replace G by β GVASS's, $G^1,\ldots,G^\beta$, the G^i being based on L_i . By Lemma 1, for every i there exists an i0 s.t. the $i0^{th}$ component of $p_{i1}+\ldots+p_{i\gamma_i}$ is 0 , and the $i0^{th}$ component corresponds to some unconstrained input or output coordinate in some G_α . Note that $\pi_{i0}(c_i)$ specifies the required fixed value of that coordinate which had previously the value ω . Replace that ω in the corresponding constraint vector (input or output) of G_α by $\pi_{i0}(c_i)$, and let the resulting GVASS be G^i . Note that the size of G_α changes from $(c_{\alpha 1},c_{\alpha 2},c_{\alpha 3})$ to $(c_{\alpha 1},c_{\alpha 2},c_{\alpha 3}-1)$. Now we can complete the argument as in θ1(a).

Condition θ2

Conditions θ2(a) and θ2(b) can be handled similarly, and we show the transformation when θ2(a) fails. Let θ2(a) fail for some G_α . Then there does not exist a Δ_α s.t. for every $i \in S_\alpha - R_\alpha$ $\pi_i(\Delta_\alpha) \geq 1$ and $(q_\alpha,v_\alpha+\Delta_\alpha) \in SR(q_\alpha,v_\alpha)$ w.r.t. $S_\alpha - R_\alpha$ in G_α . Thus by Lemma 11, we can effectively compute a constant c s.t. every point SR-reachable w.r.t. $S_\alpha - R_\alpha$ from (q_α,v_α) has i^{th} component value $\leq c$ for some i in $S_\alpha - R_\alpha$. Since every CR-path from (q_1,x) to (q_s',y) must satisfy the input constraint of G_α , when such a CR-path is inside G_α , any point on it must have a component value $\leq c$ for some coordinate in $S_\alpha - R_\alpha$. We shall treat the reduction as $|S_\alpha-R_\alpha|$ cases, one corresponding to each element in $S_\alpha - R_\alpha$. In each case we shall modify G_α into at most c+1 new G_α's . In total we generate at most $|S_\alpha - R_\alpha|(c+1)$ GVASS's.

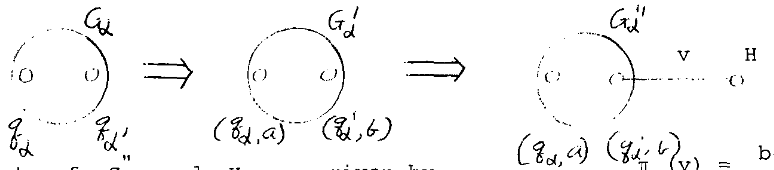
Select any i in $S_\alpha - R_\alpha$.

(i) If $\pi_i(V'_\alpha) = \omega$, then replace that ω by $0,1,2,\ldots,c$ each giving a new G_α , which results in $c+1$ new GVASS's. Note that this additional restriction will not remove any old CR-path, from (q_1,x) to (q'_s,y) , which satisfies the property that when the path is inside G_α (including its outputs point) its i^{th} coordinate is $\leq c$.

(ii) Let $\pi_i(V'_\alpha) \in N$.

Let $\pi_i(V_\alpha) = a$ and $\pi_i(V'_\alpha) = b$. Note that for any CR-path from (q_1,x) to (q'_s,y) , the input and the output points of G_α have i^{th} component values of a and b , respectively. In addition when the path is inside G_α , the i^{th} component of the corresponding points must lie in between 0 and c. By making use of the i^{th} components of the labels of the arcs of G_α we can construct an fsa which captures all paths from a to b with intermediate values in between 0 and c. If we take the cross-product of G_α and this fsa, the new VASS, G'_α , has the additional constraint that the i^{th} component of each of its intermediate point $\mathfrak{I}$ is in between 0 and c . At this stage, if the i^{th} components of the labels of all the arcs of G'_α are replaced by 0's the paths inside G'_α will not be affected but at the output point the i^{th} component will have a value a instead of b . This can be corrected by an arc, whose label is a vector of all 0's , except that its i^{th} component is $b-a$. Even though this is a sufficient description, we precisely describe the procedure below and show that the size decreases in the transformation.

Let G_α have g states. Then create $g(c+1)$ states, the label of each state being a pair (q,α) where q is a state of G_α and $0 \leq \alpha \leq c$. There exists an arc labeled u from (q,α) to (q',α') iff in G_α there is an arc labeled u from q to q' and $\alpha' = \alpha+\pi_i(u)$. Let the input and the output states of G'_α be (q_α,a) and $(q'_\alpha,b))$, respectively. Now in G'_α replace the i^{th} component of the label of every arc by 0 and add an arc, v , to a new state from (q'_α,b) as shown below,



$$\pi_j(v) = \begin{cases} b-a & \text{if } j=i \\ 0 & \text{otherwise} \end{cases}$$

The constraints of G''_α and H are given by.

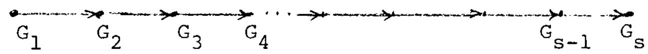
Input constraint of $G''_\alpha = V_\alpha$, output constraint of $G''_\alpha = V$, set of rigid coordinates of $G''_\alpha = R_\alpha \cup \{i\}$, input constraint of $H = V'_\alpha$, output constraint of $H = V'_\alpha$, and set of rigid coordinates of $H = R_\alpha \cup \{i\}$, where V is given by $\pi_j(V) = \begin{cases} a & \text{if } j=i \\ \pi_j(V'_i) & \text{otherwise.} \end{cases}$

Note that the first component of the sizes of G''_α and H is $n_{i1}-1$, which is less than n_{i1} . Hence the size of each of G''_α and H is less than the size of G_α . After the replacement of G_α by G''_α and H let the GVASS be G^i . It is easily seen that there exists a CR-path in G s.t. when the path is inside G_α the i^{th} coordinate of every point is $\leq c$ iff there exists a CR-path in G^i . Thus the theorem holds.

The next theorem gives the termination condition.

<u>Theorem 8</u>: For the GVASS, G, if every member of its size set is (0,0,0) and if G does not satisfy property θ , then there is no CR-path from (q_1,x) to (q'_s,y) .

<u>Proof</u>: Note that there are no arcs within any G_i . If for some $G_i, q_i \neq q'_i$ then there is no directed path from q_1 to q'_s , hence there is no CR-path from (q_1,x) to (q'_s,y). Otherwise, excluding the isolated nodes, G is as shown below.



For such a GVASS, property θ degenerates into existence of a cr-path from (q_1, x) to (q_s', y). Also note that in such a GVASS every cr-path is a CR-path.
Now we can outline the decision procedure for the reachability problem of GVASS's.

```
L: Test whether the GVASS satisfies property θ
   if property θ holds then report 'yes' and halt
   else
      if the size set has a member ≠(0,0,0) then reduce
              the problem size, and goto L
      else report 'no' and halt.
```

IV. Conclusions:

We are able to establish the decidability of reachability by making use of known simple observations. It is particularly gratifying that the simple idea embodied in Theorem 1 can be used to solve the general problem. At about the time [5] was announced, I was able to establish the decidability of reachability in 4 dimensions. The result became outdated even before I could work out the details, due to [3] and [5]. However what intrigued me most was that my approach was based on a special form of Theorem 1. By such a theorem, I was able to reduce the dimensionality by 1, but I was forced to fall back on a technique analogous to that of [7]. What was lacking was the inductive step, which [5] claimed to have developed. Even though [5] does not explicitly consider the extension of VAS's to VASS's as in here and also in [3,4], it does make use of chains of VAS's in a manner not too different from the technique here. In fact, after going through our proof one should not have any difficulty in reformulating this technique for chians of VAS's.

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For G_i, let $S_i = \{j \mid \pi_j(V_i) \neq \omega\}$, and $S_i' = \{j \mid \pi_j(V_i') \neq \omega\}$. The coordinates in S_i and S_i' are the constrained input and output coordinates respectively of G_i. A subset, R_i, of $S_i \cap S_i'$ is denoted as the set of rigid coordinates, and has the following significance. For every $j \in R_i$, the j'th component of the label of every arc in G_i must have value 0.