Teoria współbieżności

2022/23

Wykład 13

Bisimulation and Logic

Modal logic M:

SYNTAX:

$$\phi ::= true \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \langle a\rangle \phi$$

$a \in \Sigma$

SEMANTICS:

$P \models true$

$P \models \neg\phi$ iff $P \not\models \phi$

$P \models \phi_1 \wedge \phi_2$ iff $P \models \phi_1$ and $P \models \phi_2$

$P \models \langle a\rangle\phi$ iff $P' \models \phi$ for some P' s.t. $P \xrightarrow{a} P'$

SHORTHANDS:

- $[a]\phi := \neg\langle a\rangle\neg\phi$

  $P \models [a]\phi$ iff $P' \models \phi$ for all P' s.t. $P \xrightarrow{a} P'$

- $\phi_1 \vee \phi_2$
- $\langle -\rangle\phi := \bigvee_a \langle a\rangle\phi$
- $false := \neg true$

Example: $\langle a\rangle[b]true \vee [a]\langle b\rangle true$

$\langle -\rangle true \quad [-]false$

Def: $P \equiv_M Q$ iff $\forall \phi \in M. \, (P \models \phi \text{ iff } Q \models \phi)$

Lemma: $P \sim Q$ implies $P \equiv_M Q$

Proof: By structural induction over $\Phi \in M$ show that $P \sim Q$ implies $(P \models \Phi \text{ iff } Q \models \Phi)$

- **Base**: $\Phi = true$
- **Step**: $\Phi = \neg \psi$

  $\phi = \phi_1 \wedge \phi_2$

  $\Phi = \langle a \rangle \psi$

$P \sim Q$

$P \models \langle a \rangle \psi \qquad Q \models \langle a \rangle \psi$

$$\begin{array}{ccc} P & & Q \\ \downarrow a & & \downarrow a \\ P' \models \psi & & Q' \models \psi \end{array}$$

$P' \sim Q'$

Lemma: Under <u>image-finiteness</u> assumption,

$P \equiv_M Q$ implies $P \sim Q$.

Proof: $\equiv_M$ is a bisimulation

$$\begin{array}{ccccc} P & \equiv_M & Q & & \\ a\downarrow & & \downarrow a & \ldots & \searrow a \\ P' & & Q_1 & & Q_n \end{array}$$

$P \models \langle a \rangle true \qquad Q \models \langle a \rangle true$

We need: $P' \equiv_M Q_i$ for some i

Towards contradiction, assume the contrary:

$P' \not\equiv_M Q_i$ for all i

There are $\phi_1, \dots, \phi_n \in M$ s.t. $P' \models \phi_i$

$Q_i \not\models \phi_i$

for every i

Put $\psi := \phi_1 \wedge \dots \wedge \phi_n$

$P \models \langle a \rangle \psi \quad Q \not\models \langle a \rangle \psi$ — a contradiction.

Observation: a modal formula is a witness of bisimulation non equivalence

Remark: one can drop image-finiteness assumption

Question: is negation needed in the last proof?

Variants for:
- $\approx$
- $\sim_n$
- long steps $\langle a_1 \dots a_n \rangle \phi := \langle a_1 \rangle \dots \langle a_n \rangle \phi$

M = bisimulation-invariant fragment of FO

FO-logic:

$$\varphi ::= x = y \mid x \xrightarrow{a} y \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \exists x.\,\varphi_1$$

$\varphi(x)$ – FO formulas with 1 free variable

<u>Def</u>: $\varphi(x)$ is bisimulation-invariant if
$P \sim Q$ implies $(P \models \varphi$ iff $Q \models \varphi)$

<u>Translation</u>: $M \longrightarrow FO$
$\phi \longmapsto \overline{\phi}_x$

$$\overline{\text{true}}_x := x = x$$

$$\overline{\neg\phi}_x := \neg\,\overline{\phi}_x$$

$$\overline{\phi_1 \wedge \phi_2}_x := \overline{\phi_1}_x \wedge \overline{\phi_2}_x$$

$$\overline{\langle a\rangle\phi}_x := \exists y.\; x \xrightarrow{a} y \wedge \overline{\phi}_y$$

(M is a fragment of FO)

<u>Theorem</u> (van Benthem):

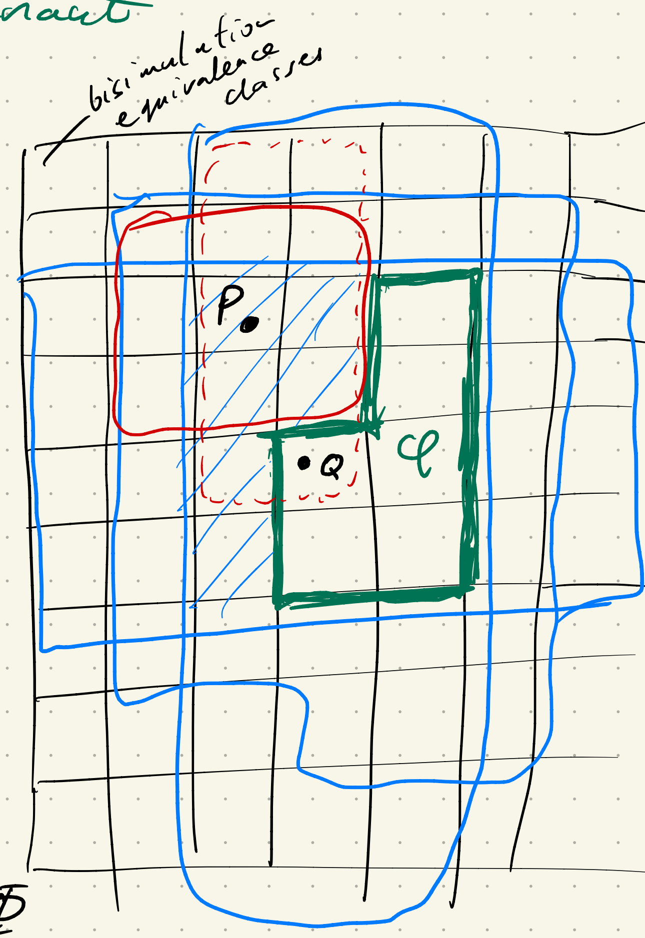
an FO formula $\varphi(x)$ is equivalent to an M formula
iff
it is bisimulation-invariant

Proof: $\Rightarrow$ immediate

$\Leftarrow$: (Under image-finiteness restriction)

Assume $\varphi(x)$ is bisimulation-invariant

Let $\Phi = \{\overline{\phi} : \phi \in M, \varphi \models \overline{\phi}\}$



Claim: $\Phi \models \varphi$

By compactness $\Phi' \models \varphi$ for some $\Phi' \subseteq_{fin} \Phi$

Hence $\varphi \equiv \bigwedge \Phi'$

Proof of claim:

Suppose $P \models \Phi$. We show $P \models \varphi$.

Let $\Psi = \{\overline{\phi} : \phi \in M, P \models \overline{\phi}\} \supseteq \Phi$

$\Psi \cup \{\varphi\}$ is satisfiable – indeed, if $\Psi \models \neg\varphi$ then by compactness $\Psi' \models \neg\varphi$ for some $\Psi' = \{\overline{\phi_1} \dots \overline{\phi_n}\} \subseteq \Psi$,

hence $\{\overline{\phi_1}, \dots, \overline{\phi_n}, \varphi\}$ unsatisfiable

$\{\overline{\phi_1 \wedge \dots \wedge \phi_n}, \varphi)$ unsatisfiable

$\overline{\phi_1 \wedge \dots \wedge \phi_n} \in \Psi$ – a contradiction

$Q \models \Psi$ and $Q \models \varphi$ for some Q

$$P \equiv_M Q \xRightarrow{\text{by image-finiteness}} P \sim Q \xRightarrow{\text{by bisimulation invariance of } \varphi} P \models \varphi$$