Teoria współbieżności

2022/23

Wykład 12

$\sim$ or $\approx$ as the greatest fixed point of F:

- ($\underset{\overset{\parallel}{Rel = \mathcal{P}(Proc \times Proc)}}{\text{binary relations on processes}}$, $\subseteq$) is a complete lattice

- $F : Rel \to Rel$ is monotonic

$$\Big\Downarrow \text{ Knaster-Tarski thm}$$

F has the greatest fixed point equal to

$$\sim \;=\; \bigcup \{ S : \underbrace{S \subseteq F(S)}_{\text{pre fixed-points}} \}$$

least upper bound

and also to

full relation $Proc \times Proc$

$$\sim \;=\; \bigcap_i F^i(T)$$

greatest lower bound

approximants indexed by ordinals:

$$F^{i+1}(T) = F(F^i(T))$$

$$F^{\lambda}(T) = \bigcap_{i<\lambda} F^i(T)$$

Examples:

- $\sim$
- $\approx$
- simulation pre-order

Bisimulation game

Two players: Spoiler, Duplicator

Arena: Spoiler's positions $V_S = Proc \times Proc$

Duplicator's positions $V_D = (Proc \times \Sigma) \times Proc \;\cup$
$Proc \times (Proc \times \Sigma)$

Spoiler moves:

$$(P, Q) \to ((P', a), Q) \quad \text{if } P \xrightarrow{a} P'$$

$$(P, Q) \to (P, (Q', a)) \quad \text{if } Q \xrightarrow{a} Q'$$

Duplicator's moves:

$$((P', a), Q) \to (P', Q') \quad \text{if } Q \xrightarrow{a} Q'$$

$$(P, (Q', a)) \to (P', Q') \quad \text{if } P \xrightarrow{a} P'$$

Initial position: $v_0 = (P_0, Q_0) \in V_S$

Play: maximal sequence

$$\underbrace{v_0 \to v_1 \to v_2}_{\text{round 1}} \underbrace{\to v_3 \to v_4}_{\text{round 2}} \to \cdots$$

Winner: finite play $\to$ stuck player looses

infinite play $\to$ Duplicator wins

Thm: $P_0 \sim Q_0$ iff Duplicator has a winning strategy.

Duplicator's strategy $s : (V_S V_D)^* \xrightarrow{\text{partial}} V_S$

s.t. whenever $s(v_0 v_1 \dots v_n) = v$ then $v_n \longrightarrow v$.

s is winning if it guarantees Duplicator's win.

Spoiler's strategy $s : (V_S V_D)^* V_S \xrightarrow{\text{partial}} V_D$

Questions: game for

- $\approx$
- $=$
- $\sim_n$
- $\sim_\omega$
- $\sim_{\omega+1}$
- $\approx_n$
- simulation pre-order
- simulation equivalence

Coinduction

Lists A^*

constructors:

$nil : List\ A$

$cons : A \times List\ A \to List\ A$

Streams A^ω

destructors:

$head : Stream\ A \to A$

$tail : Stream\ A \to Stream\ A$

(CO) INDUCTIVE DEFINITIONS

inductive definition of H: define value of H on all constructors applied to x

coinductive definition of H: define values of destructors on $H(x)$

$$f : A \to B$$

$map\ f : List\ A \to List\ B$

$$map(f, nil) = nil$$

$$map(f, cons(a, x)) = cons(f(a), map(f, x))$$

$map\ f : Stream\ A \to Stream\ B$

$$head(map(f, x)) = f(head(x))$$

$$tail(map(f, x)) = map(f, tail(x))$$

$$H(c(x)) := c^+(H(x))$$

$$d(H(x)) := H(d^+(x))$$

optional

$zip : (Stream\ A)^2 \to Stream\ A$

$$head(zip(x, y)) = head(x)$$

$$tail(zip(x, y)) = zip(y, tail(x))$$

$even : Stream\ A \to Stream\ A$

$$head(even(x)) = head(x)$$

$$tail(even(x)) = even(tail(tail(x)))$$

$$odd(x) = even(tail(x))$$

(CO)INDUCTIVE PROOFS

$$E(x) := \quad z(e(x), o(x)) = x$$

- $z(e(nil), o(nil)) = nil$
- $z(e(x), o(x)) = x \implies$
  $z(e(cons(a,x)), o(cons(a,x))) = cons(a,x)$

$$\cancel{z(e(x), o(x)) = x}$$
$$\cancel{\Downarrow}$$
$$\cancel{head(z(e(x), o(x))) = head(x)}$$
$$\cancel{\wedge}$$
$$\cancel{tail(z(e(x), o(x))) = tail(x)}$$

$$E(x) \implies E(c(x)) \qquad\qquad E(x) \implies d(E(x))$$

Def:
$S \subseteq$ Stream $A \times$ Stream A is a <u>bisimulation</u> if $\forall x, y$

$$x\,S\,y \implies head(x) = head(y) \wedge tail(x)\,S\,tail(y)$$

$$x\,S\,y \implies x\ d(S)\ y$$

$$S := \{ (z(e(x), o(x)), x) : x \in \text{Stream } A \}$$

is a bisimulation:

- $head(z(e(x), o(x))) = head(e(x)) = head(x)$
- $tail(z(e(x), o(x))) = z(o(x), tail(e(x))) =$
  $= z(e(tail(x)), e(tail(tail(x)))) =$
  $= z(e(tail(x)), o(tail(x)))$
  $S\ tail(x)$

Processes by destructors

Proc

$$\mathrm{Succ} : \mathrm{Proc} \longrightarrow \mathcal{P}(\mathrm{Proc})^{\Sigma}$$
$$\mathrm{Proc} \times \Sigma \longrightarrow \mathcal{P}(\mathrm{Proc})$$

$S \subseteq \mathrm{Proc} \times \mathrm{Proc}$

$$\mathrm{Succ}(S) = \{ (x, y) : \forall a \in \Sigma.$$
$$\forall p \in \mathrm{Succ}(x, a)\ \exists q \in \mathrm{Succ}(y, a).\ p\, S\, q$$
$$\forall q \in \mathrm{Succ}(y, a)\ \exists p \in \mathrm{Succ}(x, a).\ p\, S\, q \}$$

Bisimulation: $S \subseteq \mathrm{Succ}(S)$