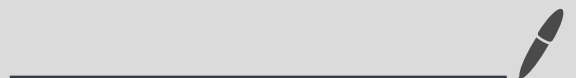


Teoria współczesności

2022/23

Wykład 12



\sim or
 \approx as the greatest fixed point of F : 17

$$\text{Rel} = \mathcal{P}(\text{Proc} \times \text{Proc})$$

• (binary) relations on processes, \subseteq is a complete lattice

• $F: \text{Rel} \rightarrow \text{Rel}$ is monotonic

\Downarrow Knaster-Tarski thm

F has the greatest fixed point equal to

$$\sim = \bigcup \{ S : S \subseteq F(S) \}$$

least upper bound

pre fixed-points

and also to

$$\sim = \bigcap_i F^i(T)$$

full relation $\text{Proc} \times \text{Proc}$

greatest lower bound

approximants indexed by ordinals:

$$F^{i+1}(T) = F(F^i(T))$$

$$F^\lambda(T) = \bigcap_{i < \lambda} F^i(T)$$

Examples:

- \sim
- \approx
- simulation pre-order

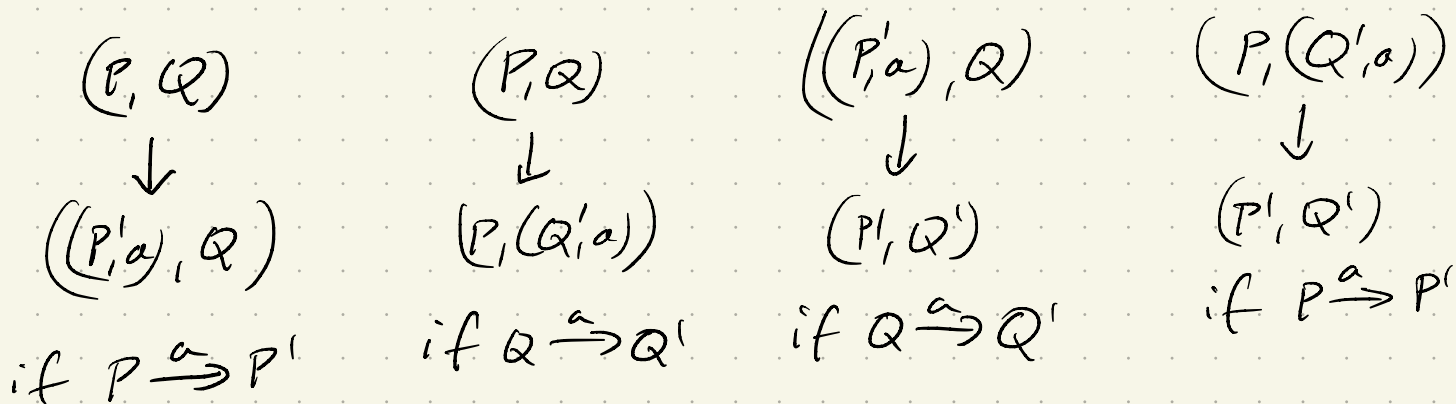
Bisimulation game

Two players: Spoiler, Duplicator

Arena: Spoiler's positions $V_S = Proc \times Proc$
Duplicator's positions $V_D = (Proc \times \Sigma) \times Proc \cup Proc \times (Proc \times \Sigma)$

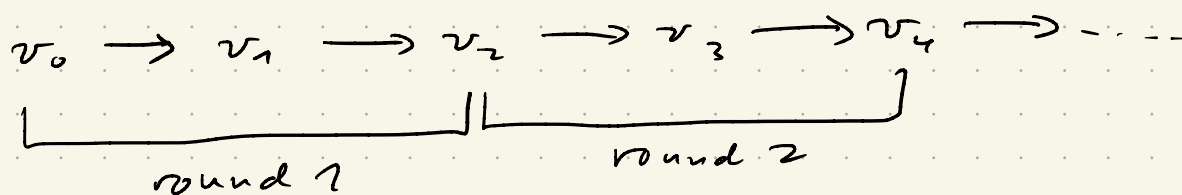
Spoiler moves:

Duplicator's moves:



Initial position: $v_0 = (P_0, Q_0) \in V_S$

Play: maximal sequence



Winner: finite play \rightarrow stuck player loses
infinite play \rightarrow Duplicator wins

Thm: $P_0 \sim Q_0$ iff Duplicator has a winning strategy.

Duplicator's strategy $s: (V_S V_D)^* \xrightarrow{\text{partial}} V_S$
s.t. whenever $s(v_0 v_1 \dots v_n) = v$ then $v_n \rightarrow v$.
 s is winning if it guarantees Duplicator's win.

Spoiler's strategy $s: (V_S V_D)^* V_S \xrightarrow{\text{partial}} V_D$

Questions: game for

- \approx
- $=$
- \sim_n
- \sim_ω
- $\sim_{\omega+1}$
- \approx_n
- simulation pre-order
- simulation equivalence

Coinduction

Lists A^*

constructors:

$nil : List\ A$

$cons : A \times List\ A \rightarrow List\ A$

(CO) INDUCTIVE DEFINITIONS

inductive definition of H :
define value of H on all
constructors applied to x

Streams A^ω

destructors:

$head : Stream\ A \rightarrow A$

$tail : Stream\ A \rightarrow Stream\ A$

coinductive definition of H :
define values of destructors
on $H(x)$

$f : A \rightarrow B$

$map\ f : List\ A \rightarrow List\ B$

$map(f, nil) = nil$

$map(f, cons(a, x)) =$
 $cons(f(a), map(f, x))$

$map\ f : Stream\ A \rightarrow Stream\ B$

$head(map(f, x)) =$
 $f(head(x))$

$tail(map(f, x)) =$
 $map(f, tail(x))$

$H(c(x)) := c^+(H(x))$

$d(H(x)) := H(d^+(x))$

optional

$zip : (Stream\ A)^2 \rightarrow Stream\ A$

$head(zip(x, y)) = head(x)$

$tail(zip(x, y)) =$

$zip(y, tail(x))$

$even : Stream\ A \rightarrow Stream\ A$

$head(even(x)) = head(x)$

$tail(even(x)) =$

$even(tail(tail(x)))$

$odd(x) = even(tail(x))$

$$E(x) ::= z(e(x), o(x)) = x$$

$$\bullet z(e(\text{nil}), o(\text{nil})) = \text{nil}$$

$$\bullet z(e(x), o(x)) = x \Rightarrow$$

$$z(e(\text{cons}(a, x), o(\text{cons}(a, x)))) = \text{cons}(a, x)$$

~~$$z(e(x), o(x)) = x$$~~



~~$$\text{head}(z(e(x), o(x))) = \text{head}(x)$$~~

~~$$\text{tail}(z(e(x), o(x))) = \text{tail}(x)$$~~

$$E(x) \Rightarrow E(c(x))$$

$$E(x) \Rightarrow d(E(x))$$

Def:

$S \subseteq \text{Stream } A \times \text{Stream } A$ is a bisimulation if $\forall x, y$

$$x S y \Rightarrow \text{head}(x) = \text{head}(y) \wedge \text{tail}(x) S \text{tail}(y)$$

$$x S y \Rightarrow x d(S) y$$

$$S := \{ (z(e(x), o(x)), x) : x \in \text{Stream } A \}$$

is a bisimulation:

$$\bullet \text{head}(z(e(x), o(x))) = \text{head}(e(x)) = \text{head}(x)$$

$$\begin{aligned} \bullet \text{tail}(z(e(x), o(x))) &= z(o(x), \text{tail}(e(x))) = \\ &= z(e(\text{tail}(x), o(\text{tail}(x)))) = \\ &= z(e(\text{tail}(x), o(\text{tail}(x)))) \end{aligned}$$

$$S \text{tail}(x)$$

Processes by destructors

Proc

$$\text{Succ} : \text{Proc} \rightarrow \mathcal{P}(\text{Proc})^\Sigma$$

$$\text{Proc} \times \Sigma \rightarrow \mathcal{P}(\text{Proc})$$

$$S \subseteq \text{Proc} \times \text{Proc}$$

$$\text{Succ}(S) = \{ (x, y) : \forall a \in \Sigma.$$

$$\forall p \in \text{Succ}(x, a) \exists q \in \text{Succ}(y, a). p S q$$

$$\forall q \in \text{Succ}(y, a) \exists p \in \text{Succ}(x, a). p S q \}$$

$$\text{Bisimulation} : S \subseteq \text{Succ}(S)$$