Teoria współbieżności

2022/23

Wykład 10

Bisimulation equivalence

Example:

$A \stackrel{def}{=} a.A_1$

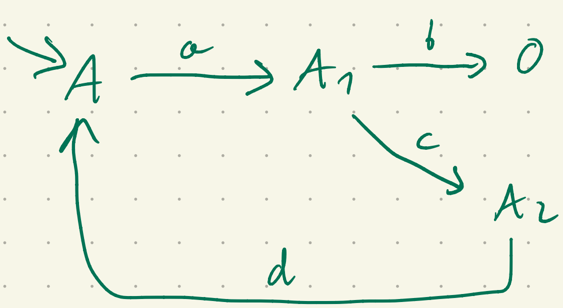
$A_1 \stackrel{def}{=} b.0 + c.A_2$

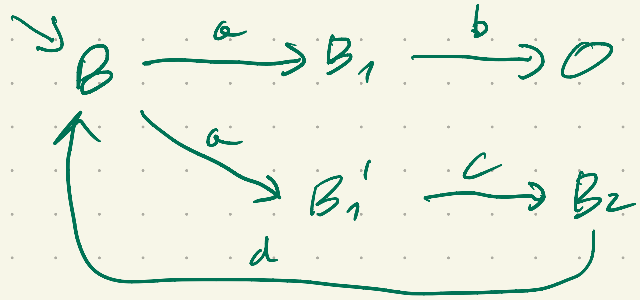
$A_2 \stackrel{def}{=} d.A$

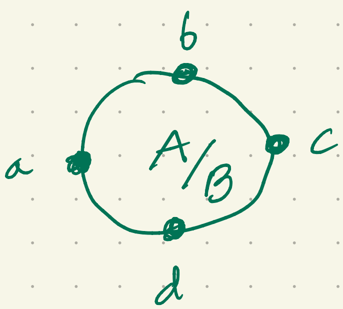
$B \stackrel{def}{=} a.B_1 + a.B_1'$

$B_1 \stackrel{def}{=} b.0 \qquad B_1' \stackrel{def}{=} c.B_2$

$B_2 \stackrel{def}{=} d.B$







$(acd)^* ab + (acd)^\omega$

Idea: P and Q are equivalent

iff

$\forall a \in \Sigma$

every a-successor of P is equivalent to

some a-successor of Q,

and vice versa

(*)

$P \sim Q$ iff $\boxed{\text{for every } a \in \Sigma,}$

- whenever $P \xrightarrow{a} P'$ then, for some Q', $Q \xrightarrow{a} Q'$ and $P' \sim Q'$
- whenever $Q \xrightarrow{a} Q'$ then, for some P', $P \xrightarrow{a} P'$ and $P' \sim Q'$

$P \sim' Q$ $\qquad\qquad$ $P \; F(\sim) \; Q$

Definition: A binary relation S between processes is a bisimulation if $S \subseteq F(S)$.

Fact: $\sim$ satisfying (*) is a bisimulation

Question: Is every bisimulation symmetric?

Example:

$$Sem \overset{def}{=} p.\,v.\,Sem$$

$$Sem_2 \overset{def}{=} p.\,Sem_1$$

$$Sem_1 \overset{def}{=} p.\,Sem_0 + v.\,Sem_2$$

$$Sem_0 \overset{def}{=} v.\,Sem_1$$

$$S = \{ (Sem|Sem,\ Sem_2),\ (v.Sem \mid Sem,\ Sem_1),\ (Sem \mid v.Sem,\ Sem_1),\ (v.Sem \mid v.sem,\ Sem_0) \}$$

Fact. If $S, S', S_i \ (i \in I)$ are bisimulations then

- identity
- S^{-1}
- $S \circ S'$
- $\bigcup_{i \in I} S_i$

are also bisimulations.

sketch of a proof:

$$\begin{tikzcd} P \arrow[r, dash, dotted, "S"] \arrow[d, "a"'] & Q \arrow[r, dash, dotted, "S'"] \arrow[d, "a"'] & R \arrow[d, "a"] \\ P' \arrow[r, dash, dotted, "S"] & Q' \arrow[r, dash, dotted, "S'"] & R' \end{tikzcd}$$

Definition: P and Q are bisimulation equivalent, $P \sim Q$, if $(P, Q) \in S$ for some bisimulation S.

$$\sim \; = \bigcup \{ S : S \text{ is a bisimulation} \}$$

Fact: $\sim$ is the largest bisimulation

$\sim$ is an equivalence

$\sim$ satisfies $(*)$: $\sim \; = F(\sim)$

sketch of a proof:

- $\sim \; \subseteq F(\sim)$
- $F(\sim) \subseteq \; \sim$: $F(\sim)$ is a bisimulation
  $F(\sim) \subseteq F(F(\sim))$

Bisimulation as a proof method — COINDUCTION

Definition: S is a <u>bisimulation up to $\sim$</u> if

whenever $(P,Q) \in S$ then for every $a \in \Sigma$,

- whenever $P \xrightarrow{a} P'$ then, for some Q', $Q \xrightarrow{a} Q'$ and $P' \sim \circ S \circ \sim Q'$,
- whenever $Q \xrightarrow{a} Q'$ then, for some P', $P \xrightarrow{a} P'$ and $P' \sim \circ S \circ \sim Q'$

$$\boxed{S \subseteq F(\sim \circ S \circ \sim)}$$

$$\begin{tikzcd}
P \arrow[rrrr, dash, dotted, "S"] \arrow[d] & & & & Q \arrow[d] \\
P' \arrow[r, phantom, "\sim"] & P'' \arrow[rr, dash, dotted, "S"] & & Q'' \arrow[r, phantom, "\sim"] & Q'
\end{tikzcd}$$

Example: (page 2)

<u>Fact</u>: S is a bisimulation up to $\sim$

$$\Downarrow$$

$\sim \circ S \circ \sim$ is a bisimulation

$$\Downarrow$$

$$S \subseteq \sim$$

L5

$P + Q \sim Q + P$

$P + (Q + R) \sim (P + Q) + R$

$P + 0 \sim P$

$P + P \sim P$

$P | Q \sim Q | P$

$P | (Q | R) \sim (P | Q) | R$

$P | 0 \sim P$

$P \setminus K \setminus L \sim P \setminus (K \cup L)$

$P \setminus L \sim P$ if L never appears in P

$P[f] \setminus L \sim P \setminus f^{-1}(L)\, [f]$

$P[f][f'] \sim P[f' \circ f]$

$P[id] \sim P$

$(P | Q) \setminus L \sim P \setminus L \,|\, Q \setminus L$ if .....

$(P | Q)[f] \sim P[f] \,|\, Q[f]$ if .....

<u>The expansion law</u>:

$$(P_1[f_1] \,|\, \dots \,|\, P_n[f_n]) \setminus L \sim$$

$$\sum \{ f_i(a) . (P_1[f_1] \,|\, \dots \,|\, P_i'[f_i] \,|\, \dots \,|\, P_n[f_n]) \setminus L : P_i \xrightarrow{a} P_i',\ f_i(a) \notin L \cup \overline{L} \}$$

$$+$$

$$\sum \{ \tau . (P_1[f_1] \,|\, \dots \,|\, P_i'[f_i] \,|\, \dots \,|\, P_j'[f_j] \,|\, \dots \,|\, P_n[f_n]) \setminus L : P_i \xrightarrow{a} P_i',\ P_j \xrightarrow{b} P_j',\ f_i(a) = \overline{f_j(b)},\ a, b \neq \tau,\ i < j \}$$

Fact : $\sim$ is a congruence : if $P \sim P'$ then

- $a.P \sim a.P'$
- $P + Q \sim P' + Q$
- $P \mid Q \sim P' \mid Q$
- $P \setminus L \sim P' \setminus L$
- $P[f] \sim P'[f]$

sketch of a proof : $\{(P|Q, P'|Q) : P \sim P'\}$ is a bisimulation

Question : How to make τ unobservable ?

$\sim$ as the greatest fixed point of F :

- $(\underbrace{\text{binary relations on processes}}_{Rel}, \subseteq)$ is a complete lattice
- $F : Rel \to Rel$ is monotonic

$\Downarrow$ Knaster-Tarski thm

F has the greatest fixed point equal to

$$\underbrace{\bigcup}_{\text{least upper bound}} \{ S : \underbrace{S \subseteq F(S)}_{\text{pre fixed-points}} \}$$