Fundamenta Informaticae 31 (1997) 13–25
IOS Press

Petri Nets, Commutative Context-Free Grammars, and Basic Parallel Processes *

Javier Esparza
Institut für Informatik
Technische Universität München
München, Germany
esparza@informatik.tu-muenchen.de

Abstract. The paper provides a structural characterisation of the reachable markings of Petri nets in which every transition has exactly one input place. As a corollary, the reachability problem for this class is proved to be NP-complete. Further consequences are: the uniform word problem for commutative context-free grammars is NP-complete; weak-bisimilarity is semidecidable for Basic Parallel Processes.

Keywords: Petri nets, Commutative Context-free Grammars, Basic Parallel Processes, Weak bisimilarity.

1. Introduction

The reachability problem plays a central rôle in Petri net theory, and has been studied in numerous papers (see [5] for a comprehensive list of references). In the first part of this paper we study it for the nets in which every transition needs exactly one token to occur. Following [8], we call them communication-free nets, because no cooperation between places is needed in order to fire a transition; every transition is activated by one single token, and the tokens may flow freely through the net independently of each other. We obtain a structural characterisation of the set of reachable markings of communication-free nets, and use it to prove that the reachability problem for this class is NP-complete. Another consequence of the characterisation is that the set of reachable markings of communication-free nets is effectively semilinear (this same result has been proved for many other net classes, see again [5] for a survey).

In the second part of the paper we apply the results of the first part to two different problems in the areas of formal languages and process algebras.

The first problem concerns commutative grammars. Huynh proved in [13] the NP-completeness of the uniform word problem for commutative context-free grammars. The proof is rather involved (8 journal pages). It is easy to see that this problem coincides with the reachability problem for communication-free nets. Therefore, our results lead immediately to a new and considerably shorter proof of Huynh's result. In passing, we also derive a

*A former version of this paper appeared in the Proceedings of Fundamentals of Computer Theory '95, LNCS 965, 221–232. This work has been partially supported by the Teilprojekt A3 of the Sonderforschungsbereich 342.

new proof of Parikh's theorem showing that the Parikh mapping of a context-free language is a semilinear set [6].

The second problem concerns the decidability of process equivalences for infinite-state systems (see [9, 5] for a survey of results in this area). Strong bisimulation equivalence [15] has been shown to be decidable for the processes of Basic Process Algebra (BPA) [2, 10], and the Basic Parallel Processes (BPPs) [3, 11], a natural subset of Milner's CCS. Since weak bisimulation is more useful than strong bisimulation for verification problems, it is natural to ask about the decidability of weak bisimulation for these classes. Using our results, we prove that weak bisimulation is semidecidable for BPPs, which hopefully will be a first step towards a decidability proof.

2. Petri nets and labelled Petri nets

For the purposes of this paper it is convenient to describe Petri nets using some notations on monoids.

Given a finite alphabet $V = \{v_1, \dots, v_n\}$, the symbols $V^\star$, $V^\oplus$ denote the free monoid and free commutative monoid generated by V, respectively. Given a word w of $V^\star$ or $V^\oplus$, and an element v of V, $w(v)$ denotes the number of times that v appears in w.

A word w of $V^\oplus$ will be represented in two different ways:

- as a multiset of elements of V (for instance, $\{v_1, v_1, v_2\}$ is the word containing two copies of v_1 and one of v_2);
- as the vector $(w(v_1), \dots, w(v_n))$.

The context will indicate which representation is being used at each moment.

The *Parikh mapping* $\mathcal{P}: V^\star \to V^\oplus$ is defined by $\mathcal{P}(w) = (w(v_1), \dots, w(v_n))$.

Given $u, v \in V^\oplus$, $u + v$ denotes the concatenation of u and v, which corresponds to addition of multisets or sum of vectors.

A *net* is a triple $N = (S, T, W)$, where S is a finite set of *places*, T is a finite set of *transitions*, and $W: (S \times T) \cup (T \times S) \to I\!N$ is a *weight function*. Graphically, places are represented by circles, and transitions by boxes. If $W(x, y) > 0$, then there is an arc from x to y labeled by $W(x, y)$ (when $W(x, y) = 1$, the arc is not labeled for clarity). We denote by ${}^\bullet x$ the set $\{y \mid W(y, x) > 0\}$ and by $x^\bullet$ the set $\{y \mid W(x, y) > 0\}$. For a set X, ${}^\bullet X$ ($X^\bullet$) denotes the union of ${}^\bullet x$ ($x^\bullet$) for every element x of X. A *path* of N is a sequence $x_1 x_2 \dots x_n$ of places and transitions such that $W(x_i, x_{i+1}) > 0$ for every $1 \leq i \leq n-1$. A *circuit* is a path $x_1 x_2 \dots x_n$ such that $W(x_n, x_1) > 0$ and $x_i \neq x_j$ for every $1 \leq i < j \leq n$.

An element of $S^\oplus$ is called a *marking* of N. A marking M is graphically represented by putting $M(s)$ *tokens* (black dots) in each place s. A *Petri net* is a pair (N, M_0), where N is a net and M_0 is a marking of N called the *initial marking.*

A marking M of a net $N = (S, T, W)$ *enables* a transition t if $M(s) \geq W(s, t)$ for every place $s \in {}^\bullet t$. If t is enabled at M, then it can *occur*, and its occurrence leads to the marking M', given by $M'(s) = M(s) + W(t, s) - W(s, t)$ for every place s. This is denoted by $M \xrightarrow{t} M'$. Given a sequence $\sigma = t_1 t_2 \dots t_n$ and markings M and M', $M \xrightarrow{\sigma} M'$ denotes that there exist markings $M_1, M_2, \dots, M_{n-1}$ such that

$$M \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \dots M_{n-1} \xrightarrow{t_n} M' \;;$$

We say that σ is an *occurrence sequence*, and that M' is *reachable from* M or *reachable in* (N, M). $M \xrightarrow{\sigma}$ (read "σ is enabled at M") denotes that $M \xrightarrow{\sigma} M'$ for some marking M'.

The *reachability problem* for a class of Petri nets consists of deciding, given a Petri net (N, M_0) of the class and a marking M of N, if M is reachable form M_0.

The following result is well known:

Proposition 2.1. Let $N = (S, T, W)$ be a net, and let M, M' be markings of N. $M \xrightarrow{\sigma} M'$ implies

$$M'(s) = M(s) + \sum_{t \in T}(W(t,s) - W(s,t)) \cdot \mathcal{P}(\sigma)(t)$$

for every place s. □

In Section 5 we shall consider labelled Petri nets. A *labelled net* is a fourtuple (S, T, W, l), where (S, T, W) is a net and $l: T \to Act$ is a labelling function on a set Act of actions. Markings, enabledness and ocurrence of transitions are defined as for unlabelled nets. If a is an action, then $M \xrightarrow{a} M'$ denotes that $M \xrightarrow{t} M'$ for some transition t such that $l(t) = a$.

3. Communication-free Petri nets

Definition 3.1. A net $N = (S, T, W)$ is *communication-free* if $|{}^\bullet t| = 1$ for every $t \in T$, and $W(s,t) \leq 1$ for every $s \in S$ and every $t \in T$. A Petri net (N, M_0) is communication-free if N is communication-free .

We assume for convenience that communication-free nets contain no *isolated places*, i.e., that each place has some input transition or some output transition.

We provide a structural characterisation of the set $\{\mathcal{P}(\sigma) \mid M_0 \xrightarrow{\sigma}\}$ for a communication-free Petri net (N, M_0). We need the notions of *subnet generated by a set of transitions* and *markable place.*

Definition 3.2. Let $N = (S, T, W)$ be a net and let $U \subseteq T$. The *subnet of N generated by U* is $({}^\bullet U \cup U^\bullet, U, W_U)$, where W_U is the restriction of W to the pairs (x, y) such that x or y is a transition of U. Given $X \in T^\oplus$, the net N_X is defined as the subnet of N generated by the set of transitions that appear in X.

Definition 3.3. Let $N = (S, T, W)$ be a net and let $M \in S^\oplus$ be a marking of N.

A place $s \in S$ is *markable from M* if $M'(s) > 0$ for some marking $M' \in S^\oplus$ reachable in (N, M).

Let $N' = (S', T', W')$ be a subnet of N generated by transitions.

A place $s \in S'$ is *markable from M in N'* if $M'(s) > 0$ for some marking $M' \in S'^\oplus$ reachable in $(N', M_{S'})$, where $M_{S'}$ is the projection of M onto S'.

The following lemma states a basic property of communication-free nets.

Lemma 3.1. Let N be a communication-free net and let M be a marking of N. A place s is markable from M iff N contains a path leading from some place s' satisfying $M(s') > 0$ to s.

Proof:
In order to mark s we just let the transitions of the path occur. □

We are now ready to state the characterisation of the set $\{\mathcal{P}(\sigma) \mid M_0 \xrightarrow{\sigma}\}$.

Theorem 3.1. Let (N, M_0) be a communication-free Petri net with a set T of transitions, and let $X \in T^\oplus$. There exists a sequence $\sigma \in T^\star$ such that $M_0 \xrightarrow{\sigma}$ and $\mathcal{P}(\sigma) = X$ iff

(a) $M_0(s) + \sum_{t \in T}(W(t,s) - W(s,t)) \cdot X(t) \geq 0$ for every place s of N, and

(b) every place of N_X is markable from M_0 in N_X.

Before proving the result we illustrate it with an example. Consider the communication-free Petri net of Figure 1. There is no occurrence sequence with Parikh mapping $X = (1,0,0,1)$, because for the place s_5 we have

$$M_0(s_5) + \sum_{t \in T} (W(t,s_5) - W(s_5,t)) \cdot X(t) = -X(t_4) = -1$$

and therefore (a) does not hold. There is no occurrence sequence with Parikh mapping $X = (0,1,1,0)$ either. In this case, (a) holds, but not (b): the net N_X contains the transitions t_2, t_3, and the places s_2, s_3, s_4, s_5, none of which is markable from M_0 in N_X. Finally, the reader may check that $(1,1,0,1)$ satisfies both (a) and (b), and in fact $t_1\, t_2\, t_4$ is an occurrence sequence having this Parikh mapping.

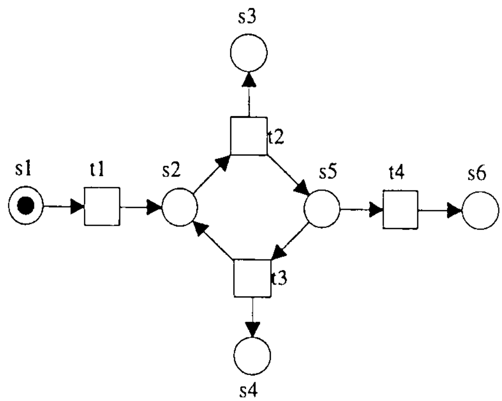

Figure 1. Illustration of Theorem 3.1.

Proof:
($\Rightarrow$) (a) follows from Proposition 2.1. To prove (b), let s be an arbitrary place of N_X. Since N_X is generated by a set of transitions, s has an input or output transition t such that $X(t) > 0$. So t occurs in σ, and therefore s is marked by at least one of the markings reached during the occurrence of σ. So s is markable from M_0 in N_X.

($\Leftarrow$) By induction on $n = \sum_{t \in T} X(t)$.

If $n = 0$ then $X = (0, \ldots, 0)$, and σ can be taken as the empty sequence.

Assume $n > 0$. Let s be a place of N_X such that $M_0(s) > 0$ and $s^\bullet \neq \emptyset$ – the existence of s follows from (b) and the fact that N contains no isolated places. We consider two cases, according to whether s is contained in some circuit of N_X or not.

(1) Some circuit γ of N_X contains s.

Let t_1 be the unique output transition of s contained in γ. Let M_1 be the marking reached by the occurrence of t_1, i.e., $M_0 \xrightarrow{t_1} M_1$, and let $X_1 \in T^{\oplus}$ be given by $X_1(t_1) = X(t_1) - 1$ and $X_1(t) = X(t)$ for every $t \neq t_1$. We claim that (N, M_1) and X_1 satisfy (a) and (b).

(a) is easy to prove. Let r be an arbitrary place of S. We have:

$$\begin{aligned} & M_1(r) + \sum_{t \in T} (W(t,r) - W(r,t)) \cdot X_1(t) \\ = & \{M_1(r) = M_0(r) + W(t_1, r) - W(r, t_1)\} \end{aligned}$$

$$M_0(r) + W(t_1, r) - W(r, t_1) + \sum_{t \in T} (W(t, r) - W(r, t)) \cdot X_1(t)$$
$$= \{\text{Definition of } X_1\}$$
$$M_0(r) + \sum_{t \in T} (W(t, r) - W(r, t)) \cdot X(t)$$
$$\geq \{\text{Condition (a) holds for } M_0 \text{ and } X\}$$
$$0$$

To prove (b), let r be an arbitrary place of N_{X_1}. We consider two cases:

(1.1) Some path π of N_X leads from t_1 to r.
Let s_1 be an output place of t_1 contained in π. Every node of π with the possible exception of t_1 belongs to N_{X_1}. Therefore, the path leading from s_1 to r obtained by removing t_1 from π is a path of N_{X_1}. Since $M_1(s_1) > 0$, r is markable in N_{X_1} under M_1.

(1.2) No path of N_X leads from t_1 to r.
Since r is markable from M_0 in N_X, some path π of N_X leads from a place s' satisfying $M_0(s') > 0$ to r. This path does not contain any node of γ, since otherwise N_X would contain a path leading from t_1 to r. In particular, π contains neither s nor t_1. So $M_1(s') = M_0(s') > 0$, and moreover π is contained in N_{X_1}, which implies that r is markable from M_1 in N_{X_1}.

Since (a) and (b) hold for (N, M_1) and X_1, the induction hypothesis guarantees the existence of a sequence σ_1 such that $M_1 \xrightarrow{\sigma_1}$ and $\mathcal{P}(\sigma_1) = X_1$. Since $M_0 \xrightarrow{t_1} M_1$, the sequence $\sigma = t_1\, \sigma_1$ is enabled at M_0 and satisfies $\mathcal{P}(\sigma) = X$.

(2) No circuit of N_X contains s.

The proof is similar to that of case (1), but obviously we can no longer choose t_1 as a transition of $s^\bullet$ contained in some circuit. Fortunately, we can just let t_1 be an arbitrary transition of $s^\bullet$, and define M_1 and X_1 as above. We show that (N, M_1) and X_1 satisfy (a) and (b).

Condition (a) is proved as in case (1).

To prove (b) let r be an arbitrary place of N_{X_1}. By assumption r is markable from M_0 in N_X, and so N_X contains a path π leading from a place s' satisfying $M_0(s') > 0$ to r. Without loss of generality we assume that s' is the unique place of π marked under M_0. We consider three cases:

(2.1) $s' \neq s$.
Then π is also a path of N_{X_1}, and we have $M_1(s') = M_0(s') > 0$. So r is markable from M_1 in N_{X_1}.

(2.2) $s' = s$ and t_1 is the successor of s in π.
Let s_1 be the successor of t_1 in π. We have $M_1(s_1) > 0$. Moreover, the path obtained by removing s and t_1 from π, which leads from s_1 to r, is contained in N_{X_1}. So r is markable in N_{X_1} under M_1.

(2.3) $s' = s$ and t_1 is not the successor of s in π.
This case needs to be divided into two:

(2.3.1) s has some input transition t_2 in N_X.
This transition has an input place s_2, which also belongs to N_X. By assumption, s_2 is markable from M_0 in N_X, and so N_X contains a path π_2 leading from some place s_3 satisfying $M_0(s_3) > 0$ to s_2. Since no circuit of N_X contains s, we have

$s_2 \neq s \neq s_3$, and so $M_1(s_3) = M_0(s_3) > 0$. Moreover, $\pi_2 t_2 \pi$ is a path of N_{X_1} which leads from s_3 to r. So r is markable from M_1 in N_{X_1}.

(2.3.2) s has no input transitions in N_X.
Then s has no input transitions in N_{X_1} too. Since the successor of s in π is different from t_1, it belongs to N_{X_1}, and so we have

$$\sum_{t \in T} (W(t,s) - W(s,t)) \cdot X_1(t) < 0$$

Since (a) holds for (N, M_1) and X_1, we also have

$$M_1(s) + \sum_{t \in T} (W(t,s) - W(s,t)) \cdot X_1(t) \geq 0$$

From these two inequalities we derive $M_1(s) > 0$. Since π is a path of N_{X_1}, r is markable from M_1 in N_{X_1}.

Since (a) and (b) hold for (N, M_1) and X_1, we can apply the induction hypothesis as in case (1). □

We easily derive two results:

Theorem 3.2. The reachability problem for communication-free Petri nets is NP-complete.

Proof:
NP-hardness follows from a straightforward reduction from the satisfiability problem for boolean formulae in conjunctive normal form. Given such a formula F, we construct a communication-free Petri net (N, M_0) and a marking M such that F is satisfiable iff M is reachable from M_0. Figure 2 shows the communication-free Petri net corresponding to the formula $C_1 \wedge C_2$, where

$$C_1 = x_1 \vee \overline{x_2} \vee x_3 \quad C_2 = \overline{x_1} \vee \overline{x_2} \vee \overline{x_3}$$

The marking M puts one token on the places C_1 and C_2, and no tokens everywhere else.

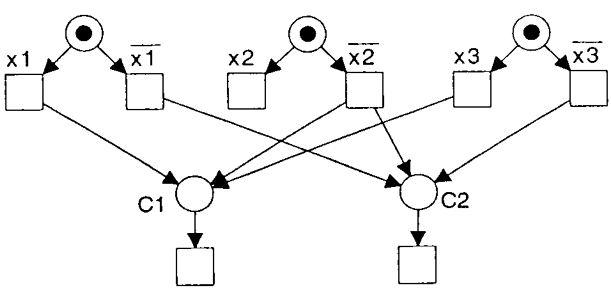

Figure 2. Communication-free Petri net for the formula of the text. The formula is satisfiable iff the marking $\{C_1, C_2\}$ is reachable

To prove membership in NP, use the following nondeterministic algorithm, whose correctness follows immediately from Theorem 3.1. Given a communication-free Petri net (N, M_0) and a marking M, guess a set U of transitions of N. Check in polynomial time that every place of N_U is markable from M_0 in N_U. Construct the system of linear equations containing for each place s the equation

$$M(s) = M_0(s) + \sum_{t \in T} (W(t,s) - W(s,t)) \cdot X(t)$$

Add the equation $X(t) = 0$ for every $t \notin U$ and $X(t) > 0$ for every $t \in U$. Check in nondeterministic polynomial time that the enlarged system has an integer solution. This is possible because Integer Linear Programming belongs to NP ([12], p. 339). □

For the second result we need to introduce semilinear sets. A subset $\mathcal{X}$ of a free commutative monoid $V^{\oplus}$ is *linear* if

$$\mathcal{X} = \{v + a_1 v_1 + \ldots + a_n v_n \mid a_1, \ldots, a_n \geq 0\}$$

for some $v, v_1, \ldots, v_n \in V^{\oplus}$. $\mathcal{X}$ is *semilinear* if it is a finite union of linear sets. $\mathcal{X}$ is *effectively semilinear* if the generators $v, v_1, \ldots, v_n \in V^{\oplus}$ of these linear sets can be effectively computed. Notice that we can also speak of semilinear subsets of cartesian products of free commutative monoids, because $V_1^{\oplus} \times V_2^{\oplus}$ can be identified with $V^{\oplus}$, where V is the disjoint union of V_1 and V_2.

Semilinear sets are closed under boolean operations and under projection onto $V'^{\oplus}$, where $V' \subseteq V$. Even further, they are exactly the sets expressible in Presburger arithmetic, the first order theory of addition [7].

We can now formulate our second result:

Theorem 3.3. Let $N = (S, T, W)$ be a communication-free Petri net, and let *Reach* be the set of triples $(M, X, M') \in S^{\oplus} \times T^{\oplus} \times S^{\oplus}$ such that $M \xrightarrow{\sigma} M'$ for some sequence σ satisfying $\mathcal{P}[\sigma] = X$. The set *Reach* is effectively semilinear.

Proof:
Let $\mathcal{M}(S')$ be the set of markings $M \in S^{\oplus}$ such that $M(s) > 0$ iff $s \in S'$, and let $\mathcal{X}(T')$ be the set of elements X of $T^{\oplus}$ such that $X(t) > 0$ iff $t \in T'$. Define $Reach(S', T')$ as the set of triples (M, X, M') of *Reach* such that $M \in \mathcal{M}(S')$ and $X \in \mathcal{X}(T')$. Clearly, *Reach* is the union over all $S' \subseteq S$ and $T' \subseteq T$ of the sets $Reach(S', T')$. Since the number of these sets is finite, it suffices to prove that $Reach(S', T')$ is effectively semilinear for every S', T'.

We apply Theorem 3.1. Since all markings of $\mathcal{M}(S')$ mark exactly the same places, a place is markable from a marking of $\mathcal{M}(S')$ iff it is markable from *every* marking of $\mathcal{M}(S')$, for instance from the marking $M_{S'}$ which puts one token on each place of S' and no token anywhere else. Let $N_{T'}$ be the subnet of N generated by T', and consider two cases:

- Every place of $N_{T'}$ is markable from $M_{S'}$ in $N_{T'}$.

  Then, by Theorem 3.1., the set $Reach(S', T')$ contains exactly the triples (M, X, M') satisfying the equation

  $$M(s) = M_0(s) + \sum_{t \in T} (W(t, s) - W(s, t)) \cdot X(t)$$

  for every place s. Since the set of integer solutions of a system of linear equations is an effectively semilinear set, $Reach(S', T')$ is effectively semilinear.
- Some place of $N_{T'}$ is not markable from $M_{S'}$ in $N_{T'}$.

  Then, by Theorem 3.1. we have $Reach(S', T') = \emptyset$, which is an effectively linear set. □

By projection of the set *Reach* onto an appropriate submonoid we obtain as corollary that the set of reachable markings of a communication-free Petri net is semilinear.

4. Context-free and commutative context-free grammars

A *context-free grammar* is a 4-tuple $G = (Non, Ter, A, P)$, where *Non* and *Ter* are disjoint sets, called the sets of nonterminals and terminals, respectively, A is an element of *Non* called the *axiom*, and P is a finite subset of $Non \times (Non \cup Ter)^*$, called the set of *productions*. The *language* $L(G)$ of a context-free grammar G is defined as usual.

A *commutative context-free grammar* (ccf-grammar) is a 4-tuple $G^c = (Non, Ter, A, P^c)$, where Non, Ter, and A are as above, and $P^c \subseteq Non \times (Non \cup Ter)^{\oplus}$. That is, free monoids are replaced by free commutative monoids.

Given two commutative words $\alpha, \beta \in (Non \cup Ter)^{\oplus}$, α *directly generates* β, written $\alpha \longrightarrow \beta$, if $\alpha = \alpha_1 + \gamma$, $\beta = \alpha_1 + \delta$, and $(\gamma, \delta) \in P^c$. α *generates* β if $\alpha \stackrel{*}{\longrightarrow} \beta$, where $\stackrel{*}{\longrightarrow}$ denotes the reflexive and transitive closure of $\longrightarrow$.

The following ccf-grammar with $Non = \{A, B\}$ and $Ter = \{a, b, c\}$ generates the language $\{\{a, b^{2n}, c^n\} \mid n \geq 0\}$.

$$A \rightarrow \{a\}$$
$$A \rightarrow \{b, b, A, B\}$$
$$B \rightarrow \{c\}$$

Given a ccf-grammar $G^c = (Non, Ter, A, P^c)$, we assign it a Petri net (S, T, W, M_0). The Petri net of Figure 3 is the one assigned to the grammar above.

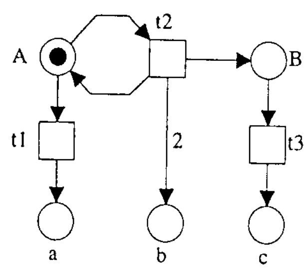

Figure 3. A Petri net

The reader can probably guess the definition of the Petri net from this example: S is the set $Non \cup Ter$, i.e., there is a place for each terminal and for each nonterminal; T is the set P^c, i.e., there is a transition for each production. Given a place s and a transition $t = (X, w)$, the weight $W(s, t)$ is 1 if $s = X$ and 0 otherwise, whereas the weight $W(t, s)$ is the number of times that s appears in the commutative word w. Finally, M_0 is the marking that puts one token on the axiom A and no tokens on the rest.

It follows directly from this description that commutative context-free grammars are assigned communication-free Petri nets.

Every word w of $(Non \cup Ter)^{\oplus}$ is a marking of the net (S, T, W). The application of a production corresponds to the occurrence of a transition. In particular, the relation $\stackrel{*}{\longrightarrow}$ on words of $(Non \cup Ter)^{\oplus}$ corresponds to the reachability relation on markings.

The uniform word problem for commutative context-free grammars. The *uniform word problem* for commutative grammars is the problem of deciding, given a grammar $G^c = (Non, Ter, A, P^c)$ and a commutative word w of terminals, if $A \stackrel{*}{\longrightarrow} w$. It follows immediately from our description of the net assigned to G^c that $A \stackrel{*}{\longrightarrow} w$ iff w is a reachable marking of the Petri net translation of G^c. So the uniform word problem for ccf-grammars can be reduced in linear time to the reachability problem for communication-free Petri nets, and vice versa. Therefore, we have a new proof for the following result of [13]:

Theorem 4.1. The uniform word problem for commutative context-free grammars is NP-complete. □

The connection between commutative context-free grammars and Petri nets was pointed out in [13], but not used. The proof of membership in NP of [13] takes 8 journal pages, and is rather involved. Our proof is much shorter[1], and it uses only standard techniques of net theory.

Parikh's Theorem As a second consequence of Theorem 3.1., we obtain a new proof of Parikh's Theorem.

Theorem 4.2. Let $G = (Non, Ter, A, P)$ be a context-free grammar. The set $\{\mathcal{P}(w) \in Ter^{\oplus} \mid w \in L(G)\}$ is semilinear.

Proof:
Let $G^c = (Non, Ter, A, P^c)$ be the commutative grammar obtained after replacing the productions of G by their commutative versions. We have:

- if $w \in L(G)$, then $\mathcal{P}(w) \in L(G^c)$;
- if $w \in L(G^c)$, then there exists $w' \in L(G)$ such that $w = \mathcal{P}(w')$.

Let (N, M_0) be the Petri net assigned to G^c. We have $\mathcal{P}(w) \in L(G^c)$ iff $\mathcal{P}(w)$, seen as a marking of N, is reachable from M_0. Therefore, $\mathcal{P}(L(G))$ is the set of reachable markings of (N, M_0) that only put tokens in the places corresponding to terminals. This latter condition can be expressed using linear equations. The result follows now from Theorem 3.3. □

We do not claim this proof to be simpler than Parikh's proof, which is rather straightforward, and takes little more than 3 pages in [6]. However, it shows still another connection between Petri nets and formal language theory.

5. Weak bisimilarity in Basic Parallel Processes

We show that communication-free Petri nets are also strongly related to Basic Parallel Processes (BPPs), a subset of CCS.

Basic Parallel Process expressions are generated by the following grammar:

$$\begin{array}{rll} E ::= & \mathbf{0} & \text{(inaction)} \\ \mid & X & \text{(process variable)} \\ \mid & aE & \text{(action prefix)} \\ \mid & E + E & \text{(choice)} \\ \mid & E \parallel E & \text{(merge)} \end{array}$$

where a is an element of a set Act of actions containing a distinguished *silent action* τ. A BPP is a finite family of recursive equations

$$\mathcal{E} = \{X_i \stackrel{\text{def}}{=} E_i \mid 1 \leq i \leq n\}$$

where the X_i are distinct and the E_i are BPP expressions at most containing the variables $\{X_1, \ldots, X_n\}$. We further assume that every variable occurrence in the E_i is *guarded*, that is, appears within the scope of an action prefix. The variable X_1 is singled out as the *leading variable* and $X_1 \stackrel{\text{def}}{=} E_1$ is the *leading equation.*

[1]The comparison of the lengths is fair, because both proofs rely on the NP-completeness of Integer Linear Programming.

Any BPP determines a labelled transition system, whose transition relations $\xrightarrow{a}$ are the least relations satisfying the following rules:

$$aE \xrightarrow{a} E \qquad \frac{E \xrightarrow{a} E'}{E+F \xrightarrow{a} E'} \qquad \frac{E \xrightarrow{a} E'}{E \parallel F \xrightarrow{a} E' \parallel F'}$$

$$\frac{E \xrightarrow{a} E'}{X \xrightarrow{a} E'} (X \stackrel{\text{def}}{=} E) \qquad \frac{F \xrightarrow{a} F'}{E+F \xrightarrow{a} F'} \qquad \frac{F \xrightarrow{a} F'}{E \parallel F \xrightarrow{a} E \parallel F'}$$

The states of the transition system are the BPP expressions E such that $X_1 \xrightarrow{w} E$ for some string w of actions.

Let $\mathcal{E}$ and $\mathcal{F}$ be two BPPs with disjoint sets of variables. Consider the labelled transition system obtained by putting the transition systems of $\mathcal{E}$ and $\mathcal{F}$ side by side. A binary relation $\mathcal{R}$ between the states of this labelled transition system is a *strong bisimulation* if whenever $E\mathcal{R}F$ then, for each $a \in Act$,

- if $E \xrightarrow{a} E'$ then $F \xrightarrow{a} F'$ for some F' with $E'\mathcal{R}F'$;
- if $F \xrightarrow{a} F'$ then $E \xrightarrow{a} E'$ for some E' with $F'\mathcal{R}E'$;

The relation $\mathcal{R}$ is a *weak bisimulation* if whenever $E\mathcal{R}F$ then, for each $a \in Act \setminus \{\tau\}$,

- if $E \xrightarrow{a} E'$ then $F \xRightarrow{a} F'$ for some F' with $E'\mathcal{R}F'$;
- if $F \xrightarrow{a} F'$ then $E \xRightarrow{a} E'$ for some E' with $F'\mathcal{R}E'$;

and, moreover

- if $E \xrightarrow{\tau} E'$ then $F \xRightarrow{\epsilon} F'$ for some F' with $E'\mathcal{R}F'$;
- if $F \xrightarrow{\tau} F'$ then $E \xRightarrow{\epsilon} E'$ for some E' with $F'\mathcal{R}E'$,

where $\xRightarrow{a} = (\xrightarrow{\tau})^\star \xrightarrow{a} (\xrightarrow{\tau})^\star$, and $\xRightarrow{\epsilon} = (\xrightarrow{\tau})^\star$.

$\mathcal{E}$ and $\mathcal{F}$ are strongly (weakly) bisimilar if their leading variables are related by some strong (weak) bisimulation.

A BPP is in *standard form* if every expression E_i on the right hand side of an equation is of the form $a_1\alpha_1 + \ldots + a_n\alpha_n$, where for each i the expression α_i is a merge of variables. It is shown in [1] that every BPP is strongly bisimilar to a BPP in standard form, which can be effectively (and easily) constructed. Therefore, the problem of deciding strong or weak bisimilarity for BPPs can be reduced to the same problem for BPPs in standard form.

Every BPP in standard form can be translated into a labelled communication-free Petri net. The translation is graphically illustrated by means of an example in Figure 4. The net has a place for each variable X_i. For each subexpression $a_j\alpha_j$ in the defining equation of X_i, a transition is added having the place X_i in its preset, and the variables that appear in α_j in its postset. If a variable appears n times in α_j, then the arc leading to it is given the weight n. The transition is labelled by a_j.

X = a (X || Y) + b (Y || Y)
Y = b (X || Y)

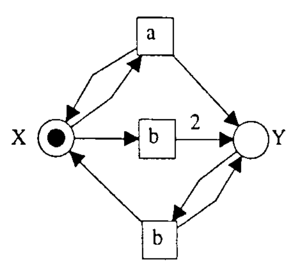

Figure 4. A BPP and its corresponding Petri net

It follows easily from the rules of the operational semantics that every state E of a BPP $\{X_i \stackrel{\text{def}}{=} E_i \mid 1 \leq i \leq n\}$ in standard form can be written (up to commutativity and associativity of parallel composition) as

$$X_1^{i_1} \parallel X_2^{i_2} \parallel \ldots \parallel X_n^{i_n}$$

where $X^i = \underbrace{X \parallel \ldots \parallel X}_{i}$. A state of this form corresponds to the marking $X_1^{i_1} X_2^{i_2} \cdots X_n^{i_n}$ of the labelled Petri net assigned to $\mathcal{E}$, and vice versa. Moreover, if M_E and $M_{E'}$ are the markings corresponding to the states E and E', then $E \xrightarrow{a} E'$ iff $M_E \xrightarrow{a} M_{E'}$. So the transition system of a BPP in standard form and the reachability graph of its labelled Petri net are isomorphic. It follows that two BPPs are strongly bisimilar iff their corresponding labelled communication-free Petri nets are strongly bisimilar, where strong (and weak) bisimilarity for labelled Petri nets is defined as for BPPs, just replacing the states of the labelled transition systems by markings, and the relations $\xrightarrow{a}$ between BPP expressions by the corresponding reachability relations between markings.

5.1. Weak bisimilarity is semidecidable for BPPs

We give a positive test for weak bisimilarity in labelled communication-free Petri nets, which immediately leads to a positive test for BPPs. The test is very similar to the positive test for strong bisimularity presented by Jančar in Section 4.2 of [14]. We do not repeat here Jančars argumentation in detail, just sketch the main points.

The test of [14] works for the class of nets N satisfying the following property[2]: the largest strong bisimulation $\sim$ between markings of N is a semilinear subset of $S^{\oplus} \times S^{\oplus}$, where S is the set of places of N.

The existence of the test is a consequence of the following three facts, which either follow immediately from the definitions, or are given very simple proofs in [14]:

(1) It suffices to give a positive test for strong bisimilarity of pairs of labelled Petri nets having the same underlying net, i.e., Petri nets (N, M_{01}), (N, M_{02}) differing only in their initial markings.
(2) There exists an effective enumeration of the semilinear relations on the markings of a net. Therefore, if we can decide whether a given semilinear relation is a strong bisimulation, then we have a positive test for strong bisimilarity: the semilinear relations containing the pair of markings to be tested are examined in turn, and a "yes" answer is given if one of them is a strong bisimulation.
(3) Let $\mathcal{R}$ be an arbitrary semilinear relation on the markings of a net. The statement "$\mathcal{R}$ is a strong bisimulation" can be encoded into Presburger arithmetic. Since Presburger arithmetic has a decidable truth problem, it is decidable whether a given semilinear relation is a strong bisimulation.

For the test of weak bisimilarity we follow the same line of reasoning. Let $N = (S, T, W)$ be an arbitrary communication-free net. First, we show that communication-free nets satisfy Jančar's condition:

Theorem 5.1. Let N be a labelled communication-free net. The largest weak bisimulation $\approx$ between markings of N is a semilinear subset of $S^{\oplus} \times S^{\oplus}$.

Proof:
It is well-known that $\approx$ is an equivalence relation (for arbitrary transition systems, not only those generated by communication-free nets). For N it is also a *congruence*, i.e., for every

[2]Actually, a sligtly more general class.

$M_1, M_2, M \in S^{\oplus}$ $M_1 \approx M_2$ implies $M_1 + M \approx M_2 + M$. To prove it, it suffices to observe that the relation

$$\mathcal{R} = \{(M_1' + M, M_2' + M) \mid M_1' \approx M_2', M \in S^{\oplus}\}$$

is a weak bisimulation. A result by Eilenberg and Schützenberger, quoted in [14] as Theorem 4.10, shows that congruences over a finitely generated commutative monoid are semilinear. So $\approx$ is a semilinear subset of $S^{\oplus} \times S^{\oplus}$. □

Now, we examine the facts (1) to (3) above. Fact (1) also holds for weak bisimulation (just use the same easy proof of [14]), and Fact (2) is a general statement about semilinear sets. So it remains to extend Fact (3), i.e., to prove that the statement "$\mathcal{R}$ is a *weak* bisimulation" can be encoded into Presburger arithmetic.

An inspection of the definition of weak bisimulation shows that the only problem is the encoding of the statement $M \stackrel{a}{\Longrightarrow} M'$. More precisely, we have to show that there exists a formula of Presburger arithmetic, with vectors of free variables $\mathbf{M}$ and $\mathbf{M}'$, which is true of two markings M and M' iff $M \stackrel{a}{\Longrightarrow} M'$.

At this point we recall Ginsburg and Spanier's result [7]: a set of integers can be encoded into Presburger arithmetic (i.e., there exists formula with a free integer variable which is true exactly of the integers in the set) iff it is semilinear. So it suffices to prove the following result:

Theorem 5.2. Let N be a labelled communication-free net. The set $\{(M, M') \mid M \stackrel{a}{\Longrightarrow} M'\}$, where M and M' are markings of N, is semilinear.

Proof:
The set $\{(M, M') \mid M \stackrel{a}{\Longrightarrow} M'\}$ is the projection onto the first and third components of the intersection of two sets of triples: the set *Reach*, which is semilinear by Theorem 3.3., and the set of triples (M, X, M') satisfying the following two conditions:

- $X(t) = 0$ for every t such that $l(t) \notin \{a, \tau\}$
  (X contains only transitions labelled by a and τ, and
- $\sum_{t \in l^{-1}(a)} X(t) = 1$
  (X contains exactly one transition labelled by a)

It is immediate to prove that this latter set of triples is also semilinear. Use now that the semilinear sets are closed under intersection and projection [7]. □

Jančar also presents in [14] a negative test for strong bisimilarity in arbitrary Petri nets. The test follows again from a simple observation: Petri nets have image-finite labelled transition systems, i.e., a marking has only a finite number of immediate successors. Therefore, the well-known result $\sim = \bigcup_{n \geq 0} \sim_n$, which holds for arbitrary image-finite systems, can be applied ($\sim_n$ is bisimilarity up to n-steps [15]). The relations $\sim_n$ are all decidable, because they only require to examine the transition system of a Petri net up to depth n.

Unfortunately, the test cannot be extended to weak bisimilarity, not even for communication-free Petri nets or BPPs, because the equation $\approx = \bigcup_{n \geq 0} \approx_n$ does not hold. Consider the following two BPPs with leading variables X and Y (this example is due to Hirshfeld):

$$X = a\mathbf{0} + \tau(X \parallel b\mathbf{0}) \qquad \begin{array}{l} Y = aZ + a\mathbf{0} + \tau(Y \parallel b\mathbf{0}) \\ Z = bZ \end{array}$$

We have $X \approx_n Y$ for every $n \geq 0$, but $X \not\approx Y$.

The existence of a negative test for weak bisimilarity of BPPs is still open.

6. Conclusions

We have solved the reachability problem for communication-free Petri nets using well-known techniques of net theory. We have then shown that this solution has several applications to context-free and commutative context-free grammars, and to Basic Parallel Processes. More precisely, and in order of increasing interest, we have obtained a new proof of Parikh's theorem, a simpler proof of the NP-completeness of the uniform word problem for commutative context-free grammars, and a positive test for weak bisimularity of Basic Parallel Processes.

Acknowledgments

Many thanks to Soren Christensen and Yoram Hirshfeld for helpful discussions. Special thanks are also due to Richard Mayr and an anonymous referee for patiently correcting many typos in an earlier version, and for very useful suggestions. Three more referees provided useful comments; in particular, one of them suggested a simplification of Theorem 3.1.

References

[1] S. Christensen. Decidability and Decomposition in Process Algebras. Ph.D. Thesis, University of Edinburgh, CST-105-93, (1993).

[2] S. Christensen, H. Hüttel, and C. Stirling. Bisimulation equivalence is decidable for all context-free processes. Information and Computation 121(2), 143–148 (1995).

[3] S. Christensen, Y. Hirshfeld and F. Moller. Bisimulation equivalence is decidable for basic parallel processes. CONCUR '93, LNCS 715, 143–157 (1993).

[4] S. Eilenberg and M.P. Schützenberger. Rational sets in commutative monoids. Journal of Algebra 13, 173–191 (1969).

[5] J. Esparza and M. Nielsen. Decidability Issues for Petri Nets – a Survey. EATCS Bulletin 52 (1994). Also: Journal of Information Processing and Cybernetics 30(3), 143–160 (1994).

[6] S. Ginsburg. The Mathematical Theory of Context-free Languages. McGraw-Hill (1966).

[7] S. Ginsburg and E.H. Spanier. Semigroups, Presburger formulas and languages. Pacific Journal of Mathematics 16, 285–296 (1966).

[8] Y. Hirshfeld. Petri Nets and the Equivalence Problem. CSL '93, LNCS 832, 165–174 (1994).

[9] Y. Hirshfeld and Faron Moller. Decidability Results in Automata and Process theory. In "Logics for Concurrency", LNCS 1043, 102–148 (1996).

[10] Y. Hirshfeld, M. Jerrum and F. Moller. A polynomial algorithm for deciding bisimularity of normed context-free processes. Theoretical Computer Science 158(1-2), 143–159 (1996).

[11] Y. Hirshfeld, M. Jerrum and F. Moller. A polynomial algorithm for deciding bisimulation of normed basic parallel processes. Journal of Mathematical Structures in Computer Science, 6, 251–259 (1996).

[12] J.E. Hopcroft and J.D. Ullman. Introduction to Automata Theory, Languages and Computation. Addison-Wesley (1979).

[13] D.T. Huynh. Commutative Grammars: The Complexity of Uniform Word Problems. Information and Control 57, 21–39 (1983).

[14] P. Jančar. Undecidability of Bisimilarity for Petri Nets and Some Related Problems. Theoretical Computer Science 148, 281–301 (1995).

[15] R. Milner. Communication and Concurrency. Prentice-Hall (1989).