Coalgebra morphisms subsume open maps

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We relate two abstract notions of bisimulation, induced by open maps and by coalgebras morphisms, respectively. We show that open maps correspond to coalgebra morphisms for a suitable chosen endofunctor in a category of many sorted sets. This demonstrates that the notion of open-maps bisimilarity is of essentially coalgebraic nature. A central role in our development is played by a category of presheaves, which we show to correspond to the subcategory of *consistent* coalgebras with lax cohomomorphisms.

1. Introduction

We investigate two different ways of generalizing the standard notion of bisimulation [13, 14]. First of them is a categorical formulation of bisimulation by means of *open maps* (open morphisms) [7,8], enabling a uniform definition of bisimulation equivalence across a range of different models for parallel computations. This setting turned out appropriate for defining, among others, strong and weak bisimilarity [13], trace equivalence, testing equivalence and strong history preserving bisimulation of event structures (see [3], [8] for an overview). Moreover, in [9] it was shown that the observational equivalence of algebraic abstract data types can be also captured in this framework. Open maps, in general, constitute a subclass of morphisms satisfying certain *computation-lifting* property [8], where a notion of computation is formalized as a morphism from any object in the distinguished subcategory of *paths*. Two objects A and B in a category are bisimilar if they are related by a span of open maps, representing abstractly a bisimulation.

An alternative direction of abstraction, aiming at a generalization of bisimulation, is to turn to a category of coalgebras [5,15,16], induced by an endofunctor on an arbitrary category. Coalgebra morphisms can be understood then as functional bisimulations, while their spans (again) give an abstract notion of a bisimulation. This can be done for different endofunctors, and a relevant special case is the category of transition systems with zig-zag morphisms, equivalent to the category of coalgebras induced by the powerset functor.

A basic motivation for our work was to clarify relationships between both approaches considered distinct (at least technically) and not shown explicitly to be comparable so far. Since the central notion of bisimulation is captured in both cases by spans of appropriate morphisms, the crucial issue is to relate open maps and coalgebra morphisms. To this aim, we define an embedding of any category with a fixed class of open maps to an appropriate category of coalgebras with so called lax cohomomorphisms [4]. A relevant observation is that under this embedding open maps correspond to coalgebra morphisms. We also

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demonstrate that a relation-based generalization of bisimulation related to open maps, called *path bisimulation* in [8], is an instance of coalgebra bisimulation. These results indicate that the coalgebraic treatment of bisimulation is the most basic one, subsuming even as abstract setting as open maps.

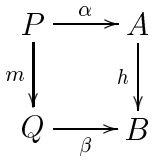
A crucial role in our development is played by categories of presheaves over a path subcategory, studied recently intensively in the context of open maps, e.g. in [2]. On one side, presheaves provide a uniform and simple model for concurrent systems in which open-map bisimilarity arises in a particularly natural way. On the other side, we show that category of presheaves is isomorphic to some subcategory of well-behaved coalgebras (called *consistent* coalgebras here). This motivates a wider research in applying presheaves as a model of computation, since they seem to be a central and fundamental construction especially well suited for studying generalized bisimulations.

In general, category of presheaves has more objects than concrete models, like transition systems or event structures. Therefore an interesting question arises, which presheaves correspond to concrete models in each case. For strong bisimilarity of labelled transition systems this question was answered already in [8], while e.g. a convenient and simple characterization of presheaves corresponding to event structures was proposed only recently in [17]. As a natural consecutive step in such a program, we obtained characterizations of presheaves corresponding to standard and partial algebras – these results are to be reported elsewhere. A preliminary version of this paper appeared as [10].

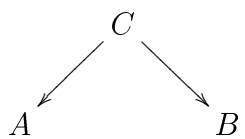
2. Bisimulation by open maps

Let $\mathbb{M}$ be a category of models of computation, in which we choose a (not necessarily full) *path subcategory* $\mathbb{P}$. Any morphism $\alpha : P \to A$ from a path object $P \in |\mathbb{P}|$ is understood as an observable *computation* in A following path P. A morphism $h : A \to B$ between models can be intuitively thought of as a simulation of A in B, since h transforms every computation $\alpha : P \to A$ in A to a computation $\alpha; h$ in B. Moreover, for any morphism $m : P \to Q$ in $\mathbb{P}$, a commuting triangle $\alpha = m; \alpha'$ can be understood as an extension of α to a "longer" computation $\alpha' : Q \to A$ (via m).

Definition 2.1 (Open maps [8]) *A morphism $h : A \to B$ in $\mathbb{M}$ is* $\mathbb{P}$-open *if for any $m : P \to Q$ in $\mathbb{P}$ and two computations $\alpha : P \to A$, $\beta : Q \to B$ in $\mathbb{M}$, whenever the square*



commutes, i.e. $\alpha; h = m; \beta$, there exists a diagonal *morphism $\gamma : Q \to A$ in $\mathbb{M}$ making two induced triangles commutative, i.e. $\alpha = m; \gamma$ and $\beta = \gamma; h$. Two objects A and B are* $\mathbb{P}$-bisimilar*, denoted by $A \sim_{\mathbb{P}} B$, if they are related by a pair of $\mathbb{P}$-open maps in $\mathbb{M}$:*



with a common domain C (a span *of $\mathbb{P}$-open maps).*

We omit the prefix $\mathbb{P}$- when it is obvious from a context. The notion of $\mathbb{P}$-bisimilarity generalizes e.g. strong bisimilarity of labelled transition systems, which was shown in [8] to coincide with bisimilarity induced by the full subcategory $Bran_L$ of transition systems consisting of finite sequences of actions:

$$i \xrightarrow{a_1} s_1 \xrightarrow{a_2} \ldots \xrightarrow{a_n} s_n. \tag{1}$$

This was done in the category TS_L, whose objects are labelled transition systems (over a fixed set of labels L) with one distinguished initial state, and morphisms are those transition-preserving functions from states to states which map an initial state to initial one. The $Bran_L$-open morphisms f of transition systems are characterized by the following *zig-zag* property: whenever $f(s) \xrightarrow{a} s'$, for a reachable state s, there exists a state r such that $s \xrightarrow{a} r$ and $f(r) = s'$.

3. Coalgebraic characterization of open maps

Consider any locally small category $\mathbb{M}$ together with an arbitrary small path subcategory $\mathbb{P}$. We will define an embedding of $\mathbb{M}$ into the category of coalgebras of some endofunctor on the category $Set^{|\mathbb{P}|}$ of $|\mathbb{P}|$-sorted sets ($|\mathbb{P}|$-indexed sets), where $|\mathbb{P}|$ is the set of objects of $\mathbb{P}$. This endofunctor, called $F_{\mathbb{P}}$ in the sequel as it is determined by $\mathbb{P}$, is defined by the following rule:

$$\{X_P\}_{P\in|\mathbb{P}|} \longmapsto \{\prod_{Q\in|\mathbb{P}|} \mathcal{P}(X_Q)^{Hom_{\mathbb{P}}(P,Q)}\}_{P\in|\mathbb{P}|}$$

where $\mathcal{P}(_)$ denotes the powerset, X_P, for $P \in |\mathbb{P}|$, denotes a component of a $|\mathbb{P}|$-sorted set X, and $Hom_{\mathbb{P}}(P,Q)$ stands for the set of all morphisms from P to Q in $\mathbb{P}$. Intuitively, this functor resembles $\mathcal{P}(_)^L$, for a set L of labels , with a difference that the set of states of an $F_{\mathbb{P}}$-coalgebra is indexed by objects of $\mathbb{P}$ and transitions are labelled by appropriate morphisms of $\mathbb{P}$. On morphisms, $F_{\mathbb{P}}$ is defined as usual:

$$F_{\mathbb{P}} : \big(\{f_P\}_{P\in|\mathbb{P}|} : X \to Y\big) \longmapsto \{\textstyle\prod_{Q\in|\mathbb{P}|} h^P_Q\}_{P\in|\mathbb{P}|}, \text{ where}$$
$$h^P_Q : \mathcal{P}(X_Q)^{Hom_{\mathbb{P}}(P,Q)} \to \mathcal{P}(Y_Q)^{Hom_{\mathbb{P}}(P,Q)} : g \longmapsto \big(\lambda m.\{f_Q(x) : x \in g(m)\}\big).$$

Let $\mathcal{C}oalg_{\mathbb{P}}$ denote the category of $F_{\mathbb{P}}$-coalgebras, i.e. pairs $\langle S, t\rangle$ with S an object in $Set^{|\mathbb{P}|}$ and $t : S \to F_{\mathbb{P}}(S)$ a morphism in $Set^{|\mathbb{P}|}$, together with coalgebra morphisms from $\langle S, t\rangle$ to $\langle S', t'\rangle$ being those $\mathbb{P}$-sorted functions $f : S \to S'$ that satisfy the usual "cohomomorphism" condition, i.e. make the following square commutative:

$$\begin{array}{ccc} F_{\mathbb{P}}(S) & \xrightarrow{F_{\mathbb{P}}(f)} & F_{\mathbb{P}}(S') \\ {\scriptstyle t}\uparrow & & \uparrow{\scriptstyle t'} \\ S & \xrightarrow{f} & S' \end{array}$$

Since we are in the category of $|\mathbb{P}|$-sorted sets, a coalgebra structure $t : S \to F_{\mathbb{P}}(S)$ on a set S consists of a family of functions

$$\{t_P : S_P \to \prod_{Q \in |\mathbb{P}|} \mathcal{P}(S_Q)^{Hom_{\mathbb{P}}(P,Q)}\}_{P \in |\mathbb{P}|}.$$

Moreover, any element x of $\prod_{Q \in |\mathbb{P}|} \mathcal{P}(S_Q)^{Hom_{\mathbb{P}}(P,Q)}$ is also a $|\mathbb{P}|$-sorted function,

$$x = \{x_Q : Hom_{\mathbb{P}}(P,Q) \to \mathcal{P}(S_Q)\}_{Q \in |\mathbb{P}|}.$$

For convenience, in the following we feel free to omit the subscript and write $x(m)$ for a morphism $m : P \to Q$ in $\mathbb{P}$ instead of $x_Q(m)$; this is justified by the canonical bijection:

$$\prod_{Q \in |\mathbb{P}|} \mathcal{P}(S_Q)^{Hom_{\mathbb{P}}(P,Q)} \quad \cong \quad \prod_{m \in \biguplus_{Q \in |\mathbb{P}|} Hom_{\mathbb{P}}(P,Q)} \mathcal{P}(S_{\mathrm{codomain}(m)}). \tag{2}$$

For a coalgebra $\langle S, t : S \to F_{\mathbb{P}}(S)\rangle$, a triple $\langle \alpha, m, \beta\rangle$, where $\alpha \in S_P$, $\beta \in S_Q$ and $m : P \to Q$ in $\mathbb{P}$, satisfying $\beta \in t_P(\alpha)(m)$ will be called a transition and denoted by $\alpha \xrightarrow{m} \beta$.

Turning back to the category $\mathbb{M}$, each of its objects determines an $F_{\mathbb{P}}$-coalgebra by taking as states all its computations and as transitions all commuting triangles induced by a pair of computations. Formally, for X in $\mathbb{M}$, the corresponding $F_{\mathbb{P}}$-coalgebra has a carrier $\{Hom_{\mathbb{M}}(P,X)\}_{P \in |\mathbb{P}|}$ and transitions

$$\alpha \xrightarrow{m} \beta \quad \Longleftrightarrow \quad \begin{array}{ccc} P & \xrightarrow{\alpha} & X \\ {\scriptstyle m}\downarrow & \nearrow_{\beta} & \\ Q & & \end{array} \quad \text{commutes in } \mathbb{M}. \tag{3}$$

To extend this mapping to morphisms, we should relax the requirement on coalgebra morphisms. Following [4], a $|\mathbb{P}|$-sorted function $f : S \to S'$ between carriers of coalgebras $C = \langle S, t\rangle$ and $C' = \langle S', t'\rangle$ is called a *lax cohomomorphism* from C to C' if for each $m : P \to Q$ in $\mathbb{P}$ and $s \in S_P$,

$$\{f_Q(r) : r \in t_P(s)(m)\} \subseteq t'_P(f_P(s))(m).$$

Coalgebras together with their lax cohomomorphisms form a category $\mathcal{C}oalg^{lax}_{\mathbb{P}}$, having $\mathcal{C}oalg_{\mathbb{P}}$ as a strict subcategory ($\mathcal{C}oalg_{\mathbb{P}}$ contains those lax cohomomorphisms for which the inclusion above can be replaced by equality). Lax cohomomorphisms are like usual morphisms of transition systems, i.e., transition preserving $|\mathbb{P}|$-sorted functions:

Proposition 3.1 *For any coalgebras $C = \langle S, t\rangle$ and $C' = \langle S', t'\rangle$, a $\mathbb{P}$-sorted function $f : S \to S'$ is a lax cohomomorphism $C \to C'$ iff for each transition $\alpha \xrightarrow{m} \beta$ in C, there is a transition $f_P(\alpha) \xrightarrow{m} f_Q(\beta)$ in C'.*

Moreover, according to the terminology from [4], the lax cohomomorphisms are induced by $F_{\mathbb{P}}$ endowed with a family of pre-orders $\sqsubseteq_{X,Y} \subseteq Hom_{Set^{|\mathbb{P}|}}(X, F_{\mathbb{P}}(Y)) \times Hom_{Set^{|\mathbb{P}|}}(X, F_{\mathbb{P}}(Y))$ defined by

$$f \sqsubseteq g \Longleftrightarrow f_P(x)(m) \subseteq g_P(x)(m), \text{for each } m : P \to Q \text{ in } \mathbb{P} \text{ and } x \in X_P.$$

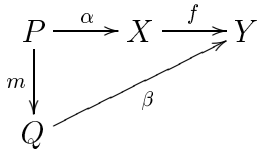
Now we are ready to introduce a functor $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}} : \mathbb{M} \to \mathcal{C}oalg^{lax}_{\mathbb{P}}$, acting on an object as explained in (3) and on a morphism by postcomposing it with computations: for $f : X \to Y$ in $\mathbb{M}$,

$$\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}(f)_P : Hom_{\mathbb{M}}(P, X) \to Hom_{\mathbb{M}}(P, Y) : \alpha \longmapsto (\alpha; f).$$

A lax cohomomorphism yielded by $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}$ is not a coalgebra morphism in general. As the name $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}$ suggests, an object of $\mathbb{M}$ is mapped to a coalgebra representing its whole observable behaviour, expressed in terms of computations and path morphisms. This functor will allow us to relate $\mathbb{P}$-bisimilarity of objects of $\mathbb{M}$ with coalgebraic bisimilarity in $\mathcal{C}oalg_{\mathbb{P}}$. While spans of ($\mathbb{P}$-)open maps represent abstractly bisimulations in the category $\mathbb{M}$, for coalgebras the same role is played by spans of coalgebra morphisms. It turns out that open maps in $\mathbb{M}$ correspond via $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}$ precisely to coalgebra morphisms:

Proposition 3.2 *A morphism $f : X \to Y$ in $\mathbb{M}$ is $\mathbb{P}$-open iff $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}(f)$ is a coalgebra morphism.*

Proof: First, notice that coalgebra morphisms $h : \langle S, t\rangle \to \langle S', t'\rangle$ are those lax cohomomorphisms which satisfy the following zig-zag condition (analogously to the zig-zag morphisms of transition systems): for each $m : P \to Q$ in $\mathbb{P}$ and $\alpha \in S_P$, whenever $h_P(\alpha) \stackrel{m}{\to} \beta$ in $\langle S', t'\rangle$ for some $\beta \in S'_Q$, there is some $\gamma \in S_Q$ such that $\alpha \stackrel{m}{\to} \gamma$ in $\langle S, t\rangle$ and $h_Q(\gamma) = \beta$. Now, recalling the definition of $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}$ on objects (see (3)) and morphisms, the zig-zag condition for $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}(f)$ can be spelled out equivalently: for each $m : P \to Q$ in $\mathbb{P}$ and $\alpha : P \to X$ in $\mathbb{M}$, whenever the following diagram commutes



for some $\beta : Q \to Y$ in $\mathbb{M}$, there exists some $\gamma : Q \to X$ making the following triangles commute:

This is obviously equivalent to $\mathbb{P}$-openness of f. □

A very general and intuitive conclusion from the above fact is that spans of open maps, intended to represent very abstractly and uniformly bisimulations for different models of computation, are essentially spans of coalgebra morphisms. This justifies the opinion that coalgebraic bisimulation captures not only the essence of this notion, but is also widely applicable to many different models. We will further expand on this while studying examples in the following sections.

Not all coalgebras are in the image of the functor. Those which do, are well-behaved in a sense that they are *consistent* with composition of path morphisms:

Definition 3.3 *A $F_{\mathbb{P}}$-coalgebra $\langle S, t\rangle$ is* consistent *if*

1. *$\alpha \stackrel{id_P}{\to} \alpha$, for each $P \in |\mathbb{P}|$ and $\alpha \in S_P$,*
2. *whenever $\alpha \stackrel{m}{\to} \beta$ and $\beta \stackrel{n}{\to} \gamma$, there is a compose transition $\alpha \stackrel{m;n}{\to} \gamma$ in $\langle S, t\rangle$,*
3. *for each $\beta \in S_Q$ and $m : P \to Q$ in $\mathbb{P}$, there is exactly one $\alpha \in S_P$ with $\alpha \stackrel{m}{\to} \beta$.*

A coalgebra satisfying only (3) will be called reachable *in the sequel.*

One easily recognizes among requirements for consistent coalgebras some categorical aspects, like the existence of identities or definedness of composition. It turns out that the consistent coalgebra $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}(X)$, for an object X in $\mathbb{M}$, is another way of looking at the small *category of* $\mathbb{P}$*-computations* of X, denoted below by $\int^{\mathbb{M}}_{\mathbb{P}} X$. Its objects are all pairs (P, α), where P is an object of $\mathbb{P}$ and $\alpha : P \to X$ in $\mathbb{M}$. Its morphisms $(P, \alpha) \to (Q, \beta)$ are those morphisms $m : P \to Q$ from $\mathbb{P}$ for which $\alpha = m; \beta$. (This category can be equivalently defined as the comma category of the inclusion $\mathbb{P} \hookrightarrow \mathbb{M}$ and the constant functor $1 \longmapsto X : \mathbf{1} \to \mathbb{M}$.) States of the coalgebra $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}(X)$ are precisely objects of $\int^{\mathbb{M}}_{\mathbb{P}} X$, while its transitions are labelled by morphisms.

4. Examples

To illustrate definitions from the previous section, let us consider the category TS_L, introduced in Section 2; its objects are transition systems (with initial states) labelled by a fixed set L of labels; its morphisms are functions which preserve transitions and initial state. Recall that a path subcategory appropriate for strong bisimilarity is the full subcategory $Bran_L$ of finite sequences of transitions. Despite that $Bran_L$ is not small, as required in the previous section, it is essentially small, and we implicitly assume working with some its skeleton. This means that a set of states of a F_{Bran_L}-coalgebra is essentially indexed by the set L^* of all finite sequences of labels, and transitions are labelled by all prefix-embeddings. Moreover, the functor $\mathcal{B}eh^{TS_L}_{Bran_L}$ takes a transition system T to a coalgebra representing its unfolding (a synchronization tree), with states corresponding to finite sequences of actions in T starting in the initial state.

What changes when we move to the category TS^0_L of transition systems without distinguished initial states? This category (whose morphisms are all transition-preserving mappings from states to states) is isomorphic to the category of the coalgebras for a powerset functor $\mathcal{P}(_)^L$ together with lax cohomomorphisms (for the canonical order endowment of the powerset endofunctor, cf. [4]). Let us find out how TS^0_L fits into our framework. inducing as open maps precisely The $\mathcal{P}(_)^L$-coalgebra morphisms are all functions $h : S \to S'$ satisfying the following zig-zag property[2]: for any $s \in S$, whenever $h(s) \stackrel{a}{\to} s'$, then $s \stackrel{a}{\to} r$, for some $r \in S$ satisfying $h(r) = s'$. As a path subcategory suitable for them it suffices to take Single_L, the category having as objects a (unique up to isomorphism) one-state transition system with no transitions and all single-transition systems (i.e. those with two states related by one transition); the only morphisms of Single_L, besides identities, are

[2]This is slightly different from the analogous property characterizing $Bran_L$-open morphisms between transition systems with initial states, where a zig-zag condition is only required for reachable states s.

functions f_a, $a \in L$, mapping a one-state system to the starting state of a single transition (labelled by a):

objects : morphisms :

Now, consider category of coalgebras for the endofunctor F_{Single_L} on $Set^{L\cup\{*\}}$ (the indexing set $L \cup \{*\}$ corresponds to objects of Single_L, with $*$ representing the one-state system). The carrier set of a F_{Single_L}-coalgebra $\langle S, t\rangle$ corresponding to a transition system T contains the set of its states (S_*) and for each $a \in L$, the set of a-transitions (S_a) in T. Moreover, $S_* \ni s \xrightarrow{f_a} \langle s, a, s'\rangle \in S_a$ iff $\langle s, a, s'\rangle$ is a transition in T. Reachability amounts in this case to the requirement that each transition has precisely one starting state. We have that $\mathcal{P}(_)^L$-coalgebra morphisms coincide with Single_L-open maps, and thus correspond (via $\mathcal{B}eh^{TS^0_L}_{\mathrm{Single}_L}$) precisely to F_{Single_L}-coalgebra morphisms. But in contrast to $\mathcal{P}(_)^L$-coalgebras, F_{Single_L}-coalgebras have no explicit information about a final state of a transition – this would require introducing into Single_L extra morphisms, mapping the one-state system to the final state of a single transition. But this would change the class of induced open maps and we would loose the abovementioned correspondence.

5. Presheaves as coalgebras

Category of presheaves, i.e. set-valued functors was proposed as a general and uniform model for concurrency [8,2]. For an arbitrary category $\mathbb{P}$, category $\widehat{\mathbb{P}}$ of presheaves consists of all contravariant functors $\mathbb{P}^{op} \to Set$, together with natural transformations as morphisms between them.

It is suitable to consider presheaves over $\mathbb{P}^{op}$, for subcategory $\mathbb{P}$ of paths. For example, category $\widehat{Bran_L}$ corresponds to synchronization forests, i.e. collections of *synchronization trees* (unfoldings of transition systems) (cf. [8]). A presheaf F over $\mathbb{P}^{op}$ is a particularly simple model of behaviour (w.r.t. $\mathbb{P}$), since its value $F(P)$ on P, an object from $\mathbb{P}$, corresponds bijectively, by Yoneda Lemma, to the set of computations $P \to F$ (we treat here $\mathbb{P}$ as a subcategory of $\widehat{\mathbb{P}}$, as it embeds fully and faithfully into $\widehat{\mathbb{P}}$ via Yoneda embedding). Moreover, $F(m)$, for a morphism $m : P \to Q$ in $\mathbb{P}$, describes how "longer" computations from Q restrict uniquely to "shorter" ones (via m). This demonstrates uniformity and simplicity of presheaf models, consisting (up to natural bijection) exclusively of their own computations. Yoneda Lemma gives also a nice characterization of open morphisms in $\widehat{\mathbb{P}}$ [7]: a natural transformation $f : F \Longrightarrow G$ is open iff all naturality squares

$$\begin{array}{ccc} F(D) & \xrightarrow{f_D} & G(D) \\ {\scriptstyle F(\alpha)}\downarrow & & \downarrow{\scriptstyle G(\alpha)} \\ F(C) & \xrightarrow{f_C} & G(C) \end{array}$$

are weak pullbacks in *Set*.

For any category $\mathbb{M}$ and a path subcategory $\mathbb{P}$, we can consider a canonical functor $\mathcal{Y}_{\mathbb{P}}^{\mathbb{M}} : \mathbb{M} \to \widehat{\mathbb{P}}$, mapping X to the contravariant hom-functor $Hom_{\mathbb{M}}(_, X) : \mathbb{P}^{op} \to Set$. For transition systems we have a pleasing situation: the subcategory of synchronization trees is equivalent via such a canonical functor to subcategory of *rooted* presheaves, i.e., those assigning to the initial object in $Bran_L$ (or generally in $\mathbb{P}$) a singleton set.

From now on we consider presheaves over $\mathbb{P}^{op}$, for an arbitrary fixed small category $\mathbb{P}$. The behavior functor $\mathcal{B}eh_{\mathbb{P}}^{\widehat{\mathbb{P}}}$ maps presheaves to consistent coalgebras; moreover, we prove that every consistent coalgebra corresponds to some presheaf. This could be expected, since the consistency condition in Definition 3.3 guarantees in fact a kind of "functorial" properties of transitions, e.g. preservation of composition (2) and identities (1).

Proposition 5.1 *Category $\widehat{\mathbb{P}}$ is equivalent via $\mathcal{B}eh_{\mathbb{P}}^{\widehat{\mathbb{P}}}$ to the full subcategory of consistent $F_{\mathbb{P}}$-coalgebras with lax cohomomorphisms, with the inverse functor $\mathcal{I}nv$ given by:*

$$\begin{array}{rcl} \mathcal{I}nv(\langle S,t\rangle)(P) & = & S_P, \\ \mathcal{I}nv(\langle S,t\rangle)(m : P \to Q)(\beta \in S_Q) & = & \alpha \in S_P, \quad \textit{when } \beta \in t_P(\alpha)(m), \\ \mathcal{I}nv(f : \langle S,t\rangle \to \langle S',t'\rangle) & = & f : \mathcal{I}nv(\langle S,t\rangle) \to \mathcal{I}nv(\langle S',t'\rangle). \end{array}$$

Proof: The action of $\mathcal{I}nv(\langle S,t\rangle)$ on morphisms is determined uniquely due to the condition (3) in Definition 3.3. Moreover, $\mathcal{I}nv$ always yields a functor, i.e. $\mathcal{I}nv(\langle S,t\rangle)$ preserves composition and identities, which is guaranteed by condition (2) and (1), respectively. Both compositions of $\mathcal{B}eh_{\mathbb{P}}^{\widehat{\mathbb{P}}}$ and $\mathcal{I}nv$ result in identities, since they change neither carriers of objects nor morphisms and their only relevant action is to recode an internal structure (a behavior) of an object in one of two ways:

- for a presheaf F, a functional restriction of a "longer" computation $Q \to F$ (which is in $F(Q)$ by Yoneda Lemma) via $m : P \to Q$ to a "shorter" computation $P \to F$, represented by $F(m) : F(Q) \to F(P)$,
- for a coalgebra $\langle S,t\rangle$, the set of all extensions of a shorter computation to a longer one via $m : P \to Q$, represented by a coalgebra structure t.

Concerning morphisms, by Proposition 3.1 it follows that the naturality condition for a $|\mathbb{P}|$-sorted function between two presheaves is equivalent to the lax cohomomorphism condition w.r.t. the corresponding coalgebra structures. □

This proposition indicates that consistent $F_{\mathbb{P}}$-coalgebras with lax cohomomorphisms are just another formulation of the category $\widehat{\mathbb{P}}$. Moreover, by Proposition 3.2, condition on a $\mathbb{P}$-sorted function to be a coalgebra morphism is equivalent to $\mathbb{P}$-openness of this function (treated as a natural transformation) in $\widehat{\mathbb{P}}$. Presheaves seem to be better suited since we don't need to impose any consistency requirements on them. On the other hand, the notion of bisimulation (and bisimilarity) for coalgebras arises more naturally with coalgebra morphisms, and we avoid formulating any explicit condition on morphisms to represent a bisimulation, like the property of openness. Common for both approaches is that we start by choosing a path category $\mathbb{P}$, which stands for all possible observations to be performed on objects.

The isomorphism agrees with functors $\mathcal{Y}_{\mathbb{P}}^{\mathbb{M}}$ and $\mathcal{B}eh_{\mathbb{P}}^{\mathbb{M}}$, i.e., $\mathcal{Y}_{\mathbb{P}}^{\mathbb{M}}; \mathcal{B}eh_{\mathbb{P}}^{\widehat{\mathbb{P}}} = \mathcal{B}eh_{\mathbb{P}}^{\mathbb{M}}$. From this it follows that for any morphism f in $\mathbb{M}$,

$$f \text{ is } \mathbb{P}\text{-open in } \mathbb{M} \iff \mathcal{Y}_{\mathbb{P}}^{\mathbb{M}}(f) \text{ is open in } \widehat{\mathbb{P}}. \tag{4}$$

This coincidence was observed already in [8] in the case when $\mathbb{P}$ is a full and dense subcategory of $\mathbb{M}$ – as we see here, this assumption is not necessary.

Our coalgebraic characterization of presheaves with open morphisms is in contrast with a standard result (cf. for instance [12]), stating that the category of presheaves over $\mathbb{P}^{op}$ (with all natural transformations) is equivalent to the category of coalgebras of the comonad in $Set^{|\mathbb{P}|}$ defined by the rule:

$$\{X_P\}_{P\in|\mathbb{P}|} \longmapsto \{\prod_{Q\in|\mathbb{P}|} X_Q^{Hom_{\mathbb{P}}(Q,P)}\}_{P\in|\mathbb{P}|} \;\cong\; \{\prod_{m\in\biguplus_{Q\in|\mathbb{P}|} Hom_{\mathbb{P}}(Q,P)} X_{\mathrm{codomain}(m)}\}_{P\in|\mathbb{P}|}.$$

(compare this with (2) in Section 3: $Hom_{\mathbb{P}}(P,Q)$ is replaced here by $Hom_{\mathbb{P}}(Q,P)$). Equivalently, presheaves can be presented as algebras of the monad defined by a dual construction:

$$\{X_P\}_{P\in|\mathbb{P}|} \longmapsto \{\coprod_{Q\in|\mathbb{P}|} X_Q^{Hom_{\mathbb{P}}(P,Q)}\}_{P\in|\mathbb{P}|} \;\cong\; \{\coprod_{m\in\biguplus_{Q\in|\mathbb{P}|} Hom_{\mathbb{P}}(P,Q)} X_{\mathrm{codomain}(m)}\}_{P\in|\mathbb{P}|}.$$

This time the original direction of arrows is restored ($Hom_{\mathbb{P}}(P,Q)$), but the product is replaced by coproduct.

These two (co)algebraic presentations of presheaves seem to be more elegant, since they do not require imposing any additional conditions, like our consistency condition. On the other hand, their serious disadvantage is that (co)algebra morphisms correspond to all morphisms of presheaves, not only to those which are open. Our approach is better suited for behavioural semantics, since it relates directly open maps and coalgebra morphisms.

6. Bisimulations

As usual in theory of coalgebras, by a *bisimulation* we mean a relation represented by a span of coalgebra morphisms [16]. For $F_{\mathbb{P}}$, bisimulations have an explicit formulation, in analogy to strong bisimulations of transition systems:

Definition 6.1 *A* bisimulation *between two coalgebras $C = \langle S, t\rangle$ and $C' = \langle S', t'\rangle$ is any $\mathbb{P}$-sorted relation $R = \{R_P\}_{P\in|\mathbb{P}|} \subseteq S \times S'$ such that, if $(\alpha,\beta) \in R_P$ and $m : P \to Q$ in $\mathbb{P}$, then*

- *whenever $\alpha \xrightarrow{m} \alpha'$ in C, there exists some $\beta' \in S'$ with $\beta \xrightarrow{m} \beta'$ in C' and $(\alpha',\beta') \in R_Q$,*
- *whenever $\beta \xrightarrow{m} \beta'$ in C', there exists some $\alpha' \in S$ with $\alpha \xrightarrow{m} \alpha'$ in C and $(\alpha',\beta') \in R_Q$,*

One can easily see that each bisimulation, being a $|\mathbb{P}|$-sorted set, has a coalgebra structure (transitions are all $(\alpha,\beta) \xrightarrow{m} (\alpha',\beta')$ with $m : P \to Q$, when $(\alpha,\beta) \in R_P$, $(\alpha',\beta') \in R_Q$,

$\alpha \xrightarrow{m} \alpha'$ and $\beta \xrightarrow{m} \beta'$) and together with two projections forms a span of coalgebra maps. We call a bisimulation R *consistent* whenever this coalgebra is so.

In [8] a generalization of bisimulation was studied for an arbitrary category $\mathbb{M}$ (with a path subcategory $\mathbb{P}$), called there a *path bisimulation*. Similarly as there, we assume in this section that $\mathbb{P}$ and $\mathbb{M}$ have a common initial object.

Definition 6.2 ([8]) *A* path bisimulation *wrt. $\mathbb{P}$ between two objects X, Y in $\mathbb{M}$ is a ($|\mathbb{P}|$-sorted) set R of pairs of computations in X and Y, respectively, with a common domain, such that:*

- *R contains the pair of unique computations ι_X, ι_Y from the initial object,*
- *for $(\alpha, \beta) \in R$ and a morphism m in $\mathbb{P}$,*
  - *if $\alpha = m; \alpha'$, there exists some β' satisfying $\beta = m; \beta'$ and $(\alpha', \beta') \in R$.*
  - *if $\beta = m; \beta'$, there exists some α' satisfying $\alpha = m; \alpha'$ and $(\alpha', \beta') \in R$.*

Intending to clarify a relationship between a path bisimulation and its coalgebraic counterpart, we derive the following equivalence:

Proposition 6.3 *For any two objects X, Y in $\mathbb{M}$, a ($|\mathbb{P}|$-sorted) relation R containing the pair (ι_X, ι_Y) is a path bisimulation between X and Y iff it is a coalgebraic bisimulation between $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}(X)$ and $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}(Y)$.*

Since a path bisimulation is in general too weak to force the existence of a span of open maps, in [8] some strengthening of path bisimulation was proposed – it is called *strong* if it further satisfies:

- whenever $(\alpha, \beta) \in R$, with $\alpha : Q \to X$, $\beta : Q \to Y$ and $m : P \to Q$ in $\mathbb{P}$, then $(m; \alpha, m; \beta) \in R$.

Proposition 6.4 *A path bisimulation is strong iff it is consistent.*

In other words, strong path bisimulations are precisely those path bisimulations which, equipped with a coalgebra structure and seen as a coalgebra, correspond to a presheaf.

Equivalently, one could only require reachability in Proposition 6.4 instead of consistency, since the other two condition from Definition 3.3 follow.

As a corollary, the following statements are equivalent:

- X and Y in $\mathbb{M}$ are strong path bisimilar,
- $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}(X)$ and $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}(Y)$ are related by a nonempty consistent bisimulation,
- $\mathcal{Y}^{\mathbb{M}}_{\mathbb{P}}(X)$ and $\mathcal{Y}^{\mathbb{M}}_{\mathbb{P}}(X)$ are related by a span of open maps in the category of non-empty (or rooted) presheaves.

For $\mathbb{M}$ being the category of non-empty (or rooted) presheaves we conclude by Propositions 6.4 and 5.1 that strong path bisimilarity and open-map bisimilarity coincide – a fact proved already in [8]. It is also the case in many concrete models, like event structures or transition systems, but not in general.

7. Conclusions and future research

We stated a relationship between bisimulations defined by open-maps and by coalgebra morphisms. A central role in our development is played by categories of presheaves, being at an intermediate position and linking both approaches: they are uniform models to study open maps as well as they correspond to consistent coalgebras, with open maps corresponding precisely to coalgebra morphisms.

By Proposition 3.2, spans of $\mathbb{P}$-open maps in $\mathbb{M}$ are mapped to spans of coalgebra morphisms, in the image subcategory $\mathcal{B}eh_{\mathbb{P}}^{\mathbb{M}}(\mathbb{M})$, being usually a proper subcategory of $\mathcal{C}oalg_{\mathbb{P}}^{lax}$. To be able to relate both bisimilarities, it is an important issue to characterize this image subcategory, or equivalently to find a necessary and sufficient condition on a consistent coalgebra (or a presheaf) to correspond to some object from $\mathbb{M}$. Such characterizations exist for transition systems [8] and event structures [17]. We have found analogous characterizations for standard and partial algebras, to be reported in a separate paper. These results are in close analogy to Lawvere's functorial presentation of models of algebraic theories [11] and to a representation theorem for locally finitely presentable categories [1]. An interesting question is whether there exists any general characterization of a subcategory of presheaves $\widehat{\mathbb{P}}$ meaningful as a model of computation, for a suitably chosen path category $\mathbb{P}$.

Another direction of further research is motivated by the following intuition about a behaviour-realization relationship. For an arbitrary cocomplete $\mathbb{M}$ and a small path subcategory $\mathbb{P}$, consider a category $\int_{\mathbb{P}}^{\widehat{\mathbb{P}}} F$ of computations of a presheaf F (it is also called *category of elements* of F). F induces an object in $\mathbb{M}$, as a colimit of the diagram given by the projection functor Δ_F, taking (P, α) to P:

$$\mathrm{R}_{\mathbb{P}}^{\mathbb{M}}(F) = \mathrm{colim}\big(\big(\int_{\mathbb{P}}^{\widehat{\mathbb{P}}} F\big) \xrightarrow{\Delta_F} \mathbb{P} \hookrightarrow \mathbb{M}\big). \tag{5}$$

$\mathrm{R}_{\mathbb{P}}^{\mathbb{M}}(F)$ can be intuitively understood as a reconstruction of an object from a behaviour F. By universal properties of colimits, this construction can be extended to a functor $\mathrm{R}_{\mathbb{P}}^{\mathbb{M}} : \widehat{\mathbb{P}} \to \mathbb{M}$ (cf. [12], I.5); moreover the functors $\mathrm{R}_{\mathbb{P}}^{\mathbb{M}}$ and $\mathcal{Y}_{\mathbb{P}}^{\mathbb{M}}$ form an adjunction:

$$\widehat{\mathbb{P}} \underset{\mathcal{Y}_{\mathbb{P}}^{\mathbb{M}}}{\overset{\mathrm{R}_{\mathbb{P}}^{\mathbb{M}}}{\rightleftarrows}} \mathbb{M}$$

with the right adjoint giving a *behaviour* of an object from $\mathbb{M}$ while the left adjoint returning a canonical *realization* of a behaviour. It is desirable to closer investigate some concrete examples of this general situation as well as to compare it to other adjunction-based approaches investigating a relationship between behaviour and its implementation, for instance to [6].

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