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ROUGH REAL FUNCTIONS AND ROUGH CONTROLLERS

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1 INTRODUCTION

This paper is an extension of articles Pawlak (1987), where some ideas concerning rough functions were outlined. The concept of the rough function is based on the rough set theory (Pawlak, 1991) and is needed in many applications, where experimental data are processes, in particular as a theoretical basis for rough controllers (Czogala et al., 1994, Mrozek and Plonka, 1994).

The presented approach is somehow related to nonstandard analysis (Robinson, 1970), measurement theory (Orlowska and Pawlak, 1984) and cell-to-cell mapping (Hsu, 1980) but these aspects of rough functions will be not considered here.

In recent years we witness rapid grow of development and applications of fuzzy controllers. The philosophy behind fuzzy control is that instead of describing, as in the case of classical control theory, the process being controlled in terms of mathematical equations - we describe the behavior of human controller in terms of fuzzy decision rules, i.e. rules that involve rather qualitative then quantitative variables and can be seen as a common-sense model of the controlled process, similarly as in qualitative physics physical phenomena are described in terms of qualitative variables instead of mathematical equations.

The idea of rough (approximate) control steams yet from another philosophical background. It is based on the assumption that the controlled process is observed and data about the process are registered. The data are then used to generate the control algorithms, which can be afterwards optimized. Both, the generation of the control algorithm from observation, as well the optimization of the algorithm can be based on the rough set theory, which seems to be very well suited for this kind of tasks. The control algorithms obtained in this way are objective and can be viewed as an intermediate approach between classical and fuzzy approach to control systems.

In some cases the observation can be postponed and control algorithm can be obtained directly from the knowledgeable expert, similarly as in the fuzzy set approach. In this case the control algorithm can be also simplified using the rough set theory approach.

In general we assume that a rough controller can be seen as an implementation of rough (approximate) function, i.e. function obtained as a result of physical measurements with predetermined accuracy, depending on assumed scale.

The aim of this paper is to give basic ideas concerning rough functions, which are meant to be used as a theoretical basis for rough controllers synthesis and analysis. The presented ideas can be also applied to other problems – in general to discrete dynamic systems, and will be discussed in further papers.

2 BASIC OF THE ROUGH SET CONCEPT

Basic ideas of the rough set theory can be found in Pawlak (1991). In this section we will give only those notions which are necessary to define concepts used in this paper.

Let U be a finite, nonempty set called the *universe*, and let I be an equivalence relation on U, called an *indiscernibility relation.* By $I(x)$ we mean the set of all y such that xIy, i.e. $I(x) = [x]_I$, i.e.- is an equivalence class of the relation I containing element x. The indiscernibility relation is meant to capture the fact that often we have limited information about elements of the universe and consequently we are unable to discern them in view of the available information. Thus I represents our lack of knowledge about U.

We will define now two basic operations on sets in the rough set theory, called the *I-lower* and the *I-upper approximation*, and defined respectively as follows:

$$I_*(X) = \{x \in U : I(x) \subseteq X\},$$

$$I^*(X) = \{x \in U : I(x) \cap X \neq \emptyset\}.$$

The difference between the upper and the lower approximation will be called the *I-boundary* of X and will be denoted by $BN_I(X)$, i.e.

$$BN_I(X) = I^*(X) - I_*(X).$$

If $I^*(X) = I_*(X)$ we say the the set is *I-exact* otherwise the set X is *I-rough.* Thus rough sets are sets with unsharp boundaries.

Usually in order to define a set we use the membership function. The membership function for rough sets is defined by employing the equivalence relation I as follows:

$$\mu_X^I(x) = \frac{card(X \cap I(x))}{card(I(x))}.$$

Obviously

$$\mu_X^I(x) \in [0, 1].$$

The value of the membership function expresses the degree to which the element x belongs to the set X in view of the indiscernibility relation I.

The above assumed membership function, can be used to define the two previously defined approximations of sets, as shown below:

$$I_*(X) = \{x \in U : \mu_X^I(x) = 1\},$$

$$I^*(X) = \{x \in U : \mu_X^I(x) > 0\}.$$

3 ROUGH SETS ON THE REAL LINE

In this section we reformulate the concepts of approximations and the rough membership function referring to the set of reals, which will be needed to formulate basic properties of rough real functions.

Let $\mathbf{R}$ be the set of reals and let (a,b) be an open interval. By a *discretization* of the interval (a,b) we mean a sequence $S = \{x_0, x_1, \ldots, x_n\}$ of reals such that $a = x_0 < x_1 < \ldots < x_n = b$. Besides, we assume that $0 \in S$. The ordered pair

$A = (\mathbf{R}, S)$ will be referred to as the *approximation space* generated by S or simple as S-*approximation space.* Every discretization S induces the partition $\pi(S) = \{\{x_0\}, (x_0 x_1), \{x_1\}, (x_1, x_2), \{x_2\}, (x_2, x_3),$
$\{x_3\}, \ldots, \{x_{n-1}\}, (x_{n-1}, x_n), \{x_n\}\}$ on (a, b). By $S(x)$ (or $[x]_S$) we will denote block of the partition $\pi(S)$ containing x. In particular, if $x \in S$ then $S(x) = \{x\}$. If $S(x) = (x_i, x_{i+1})$, then by $S_*(x)$ and $S^*(x)$ we will denote the left and the right ends of the interval $S(x)$ respectively, i.e. $S_*(x) = x_i$ and $S^*(x) = x_{i+1}$. The *closure* of $S(x)$ will be denoted by $S'(x)$.

In what follows we will be interested in approximating intervals $(0, x) = Q(x)$ for any $x \in [a, b]$.

Suppose we are given an approximation space $A = (\mathbf{R}, S)$. By the S-*lower* and the S-*upper* approximation of $Q(x)$, denoted by $Q_S(x)$ and $Q^S(x)$ respectively, we mean sets defined below:

$$Q_S(x) = \{y \in \mathbf{R} : S(y) \subseteq Q(x)\} = Q(S_*(x))$$

$$Q^S(x) = \{y \in \mathbf{R} : S(y) \cap Q(x) \neq \emptyset\} = Q(S^*(x)).$$

The above definitions of approximations of the interval $(0, x)$ can be also understood as approximations of the real number x which are simple the ends of the interval $S(x)$.

In other words given any real number x and a discretization S, by the S-lower and the S-upper approximation of x we mean the numbers $S_*(x)$ and $S^*(x)$ respectively.

We will say that the number x is exact in $A = (\mathbf{R}, S)$ if $S_*(x) = S^*(x)$, otherwise the number x is *inexact* (*rough*) in $A = (\mathbf{R}, S)$. Of course x is exact iff $x \in S$.

Any discretization S can be interpreted as a scale (e.g. km, in, etc.), by means of which reals from $\mathbf{R}$ are measured with some approximation due to the scale S.

The introduced idea of the rough set on the real line corresponds exactly to those defined for arbitrary sets and can be seen as a special case of the general definition.

Now we give the definition of the next basic notion in the rough set approach - the rough membership function – referring to the real line (Pawlak and Skowron, 1993).

The rough membership function for set on the real line has the form

$$\mu_{Q(x)}(y) = \frac{\Delta(Q(x) \cap S(y))}{\Delta(S(y))},$$

where $\Delta(X) = Sup|x - y|, x, y \in X$.

Assuming that $x = y$, we get

$$\mu_{Q(x)}(y) = \mu(y),$$

which can be understood as an error of measurement of x in the scale S.

Remark

We can also assume that the discretizatin S induces partition $\pi(S) = \{(-\infty, x_0), \{x_0\}, (x_0 x_1), \{x_1\}, (x_1, x_2), \{x_2\}, (x_2, x_3), \{x_3\}, \ldots, \{x_{n-1}\}, (x_{n-1}, x_n), \{x_n\}, (x_n, +\infty)\}$ on $\mathbf{R}$. In this case for $x > b$ the upper approximation of x is $S^*(x) = +\infty$, and similarly for $x < a$, we have $S^*(x) = -\infty$. However for the sake of simplicity we will not consider this case here. □

4 ROUGH SEQUENCIES AND ROUGH FUNCTIONS

Let $A = (\mathbf{R}, S)$ be an approximation space and let $\{a_n\}$ be an infinite sequence of real numbers.

A sequence $\{a_n\}$ is *roughly convergent* in $A = (\mathbf{R}, S)$, (*S-convergent*), if there exists i such that for every $j > i$ $S(a_j) = S(a_i)$; $S_*(a_i)$ and $S^*(a_i)$ are referred to as the *rough lower* and the *rough upper limit* (*S-upper, S-lower limit*) of the sequence $\{a_n\}$. Any roughly convergent sequence will be called *rough Cauchy sequence.*

A sequence $\{a_n\}$ is *roughly monotonically increasing* (*decreasing*) in $A = (\mathbf{R}, S)$, (*S-increasing* (*S-decreasing*)), if $S(a_n) = S(a_{n+1})$ or $a_n < a_{n+1} (a_n > a_{n+1})$ and $S(a_n) \neq S(a_{n+1})$.

A sequence $\{a_n\}$ is *roughly periodic* in $A = (\mathbf{R}, S)$ (*S-periodic*), if there exists k such that $S(a_n) = S(a_{n+k})$. The number k is called the period of $\{a_n\}$.

A sequence $\{a_n\}$ is *roughly constant* in $A = (\mathbf{R}, S)$ (*S-constant*), if $S(a_n) = S(a_{n+1})$.

Suppose we are given a real function $f : X \to Y$ and discretizations $S = \{x_0, x_1, \ldots, x_n\}$ and $P = \{y_0, y_1, \ldots, y_m\}$ on X and Y respectively. If f is continuous in every $x \in S$, we with say that f is *S-continuous*. Let f be a S-continuous function, and let $N(x_i) = i$.

Function $F_f : \{n\} \to \{n\}$, such that $F_f(N(x_i)) = N(P_* f(x_i))$ will be called *rough representation* of f (or *(S, P)-representation* of f).

The function F_f can be used to define some properties of real functions.

A function f is *roughly monotonically increasing* (*decreasing*) if $F_f(i + 1) = f(i) + \alpha$, where α is a non-negative integer, (α is non-positive integer), for every $i = 0, 1, 2, \ldots n - 1$.

A function f is *roughly periodic* if there exist k such that $F_f(i) = F_f(i + k)$ for every $i = 0, 1, \ldots, n - 1$.

A function f is *roughly constant* if $F_f(i) = F_f(i+1)$, for every $i = 0, 1, \ldots, n-1$.

Many other basic concepts concerning functions can be expressed also in the rough function setting.

By the *P-lower approximation* of f we understand the function $f_* : X \to Y$ such that

$$f_*(x) = P_*(f(x)), \text{for every} x \in X.$$

Similarly the *P-upper approximation* of f is defined as

$$f^*(x) = P^*(f(x)), \text{for every} x \in X.$$

We say that a function f is *exact* in x iff $f_*(x) = f^*(x)$; otherwise the function f is *inexact* (*rough*) in x. The number $f^*(x) - f_*(x)$ is the *error of approximation* of f in x.

Finally in many applications we need the fix-point properties of functions.

We say that $x \in S$ is a *rough fix-point* (*rough equilibrium point*) of a real function f if $F_f(N(x)) = N(P_*(f(x))$.

Now we give a definition of a very important concept, the rough continuity of real function.

Suppose we are given a real function $f : X \to Y$, where both X and Y are sets of reals and S, P are discretizations of X and Y respectively.

A function f is (*roughly continuous*) (*S,P*)*-continuous* in x if

$$S(x) \subseteq P(f(x)).$$

In other words a function f is roughly continuous in x iff for every $y \in S(x), f(y) \in P(f(x))$.

The intuitive meaning of this definition is obvious. Whether the function is roughly continuous or not depends on the information we have about the function, i.e. it depends on how exactly we "see" the function through the discretization of X and Y.

Obviously a function f is roughly continuous iff $F_f(i + 1) = F_f(i) + \alpha$, where $\alpha \in \{-1, 0, +1\}$ for every $i = 0, 1, \ldots, n - 1\}$.

Remark

Particularly interesting is the relationship between dependency of attributes in information systems and the rough continuity of functions

Let $\mathbf{S} = (U, A)$, be an *information system*, (Pawlak, 1991), where U is a finite set of *objects*, called the *universe* and A is a finite set of attributes. With every attribute $a \in A$ a set of *values* of attribute a, called *domain* of a, is associated and is denoted by V_a . Every attribute $a \in A$ can be seen as a function $a : U \to V$, which to every object $x \in U$ assigns a value of the attribute a. Any subset of attributes $B \subseteq A$ determines the equivalence relation $IND(B) = \{x, y \in U : a(x) = a(y) \text{for every} a \in A\}$. Let $B, C \subseteq A$. We will say that the set of attributes C *depends* on the set of attributes B, in symbols $B \to C$, iff $IND(B) \subseteq IND(C)$. If $B \to C$ then there exists a *dependency function* $f_{B,C} : V_{b_1} \times V_{b_2} \times \ldots \times V_{b_n} \to V_{c_1} \times V_{c_2} \times \ldots \times V_{c_m}$, such that $f_{B,C}(v_1, v_2, \ldots, v_n) = (w_1, w_2, \ldots, w_m)$, iff $\sigma(v_1) \cap \sigma(v_2) \cap, \ldots, \cap \sigma(v_n) \subseteq$

$\sigma(w_1) \cap \sigma(w_2) \cap, \ldots, \cap \sigma(w_m)$, where $v_1 \in V_b, w_j \in V_{c_j}, \sigma(v) = \{x \in U : a(v) = x\}$ and $v \in V_a$. The dependency function $B \to C$, where $B = \{b_1, b_2, \ldots, b_n\}$ and $C = \{c_1, c_2, \ldots, c_m\}$ assigns uniquely to every n-tuple of values of attributes from B the m-tuple of values of attributes from C.

There exists the following important relationship. $B \to C$ iff $f_{B,C}$ is (B,C)-roughly continuous. □

5 CONLUSIONS

Rough function concept is meant to be used as a theoretical basis for rough controllers. Basic definitions concerning rough functions were given and some basic properties of these functions investigated.

Applications of the above discussed ideas will be presented in the forthcoming papers.

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