BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 32, No. 9-10, 1984

PROBABILITY THEORY

Rough Probability

by

Zdzisław PAWLAK

Presented on January 24, 1984

Summary. In this paper we define the notion of partial observability considered in statistical models in terms of "rough" sets. With each event we associate "inner" and "outer" probability, or an interval whose end points are inner and outer probability respectively. The interval is called the rough probability of the event. Some elementary properties of rough probabilities are given.

1. Introduction. Pleszczyńska and Dąbrowska [2] introduced partial observability in statistical models.

We propose in this paper somewhat different formulation of this notion based on the concept of the rough set (see Pawlak [3]) which can be of interest in situations when the "exact" probability of some events is not known, but only the interval, to which this probability belongs.

As a starting point of our considerations we introduce the notion of a stochastic approximation space S and the inner and outer probability $\underline{P}_S(X)$, $\bar{P}_S(X)$ of the event X in the approximation space S is defined. In fact the inner and outer probability of an event X is the probability of the interior and closure of X respectively, in the topological space generated by the approximation space S. Some elementary properties of inner and outer probabilities are given and the notion of the rough (approximate) probability of an event X is defined, as $P_S^*(X) = \langle \underline{P}_S(X), \bar{P}_S(S) \rangle$, which is to be understood as an interval to which the probability of X belongs. Elementary properties of the rough probability are given.

The inner and outer probabilities considered in this paper may be viewed as a special case of lower and upper probabilities introduced by Dempster [1] and a special case of Shafer's belief theory (see Shafer [4]).

2. Approximation space, approximation of sets. In this section we recall after Pawlak [3] basic notions concerning the concept of a rough set, used as a starting point of our considerations.

Let U be a certain set, and let R be an equivalence relation on U. The pair $A = (U, R)$ will be called an *approximation space*, and R will be referred to as an *indiscernibility* relation. If $x, y \in U$ and $R(x, y)$ we say that x and y are indistinguishable in A.

Equivalence classes of the relation R and the empty set will be called *elementary sets* (atoms) in A or in short elementary sets (atoms) if A is understood.

Every union of elementary sets in A will be called a *composed set* in A, or in short a composed set, if A is known.

The family of all composed sets in A is denoted by Com (A).

Obviously Com (A) is a Boolean algebra, i.e. the family of all composed sets is closed under intersection, union and complement of set.

Let X be a certain subset of U. The least composed set in A containing X will be called the *best upper approximation* of X in A, in symbols $\bar{A}(X)$; the greatest composed set in A contained in X will be called the *best lower approximation* of X in A, and will be denoted by $\underline{A}(X)$.
In other words

$$\underline{A}(X) = \{x \in U : [x]_R \subset X\}$$

and

$$\bar{A}(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}$$

where $[x]_R$ denotes the equivalence class of relation R containing x.

3. Properties of approximations. One can easily check that the approximation space $A = (U, R)$ defines uniquely the topological space $T_A = (U, \mathrm{Com}(A))$, and Com (A) is the family of all open and closed sets in T_A, and U/R is a base for T_A.

From the definition of approximations follows that $\underline{A}(X)$ and $\bar{A}(X)$ are interior and closure of X in the topological space T_A respectively.

Thus for every $X, Y \subset U$ and every approximation space $A = (U, R)$ the following properties of approximations are valid:

(A1) $\underline{A}(X) \subset X \subset \bar{A}(X)$,
(A2) $\underline{A}(U) = \bar{A}(U) = U$,
(A3) $\underline{A}(\emptyset) = \bar{A}(\emptyset) = \emptyset$,
(A4) $\bar{A}(X \cup Y) = \bar{A}(X) \cup \bar{A}(Y)$,
(A5) $\underline{A}(X \cup Y) \supset \underline{A}(X) \cup \underline{A}(Y)$,
(A6) $\bar{A}(X \cap Y) \subset \bar{A}(X) \cap \bar{A}(Y)$,
(A7) $\underline{A}(X \cap Y) = \underline{A}(X) \cap \underline{A}(Y)$,
(A8) $\bar{A}(-X) = -\underline{A}(X)$,
(A9) $\underline{A}(-X) = -\bar{A}(X)$.

Moreover we have

(A10) $\bar{A}(\bar{A}(X)) = \underline{A}(\bar{A}(X)) = \bar{A}(X)$,

(A11) $\underline{A}(\underline{A}(X)) = \bar{A}(\underline{A}(X)) = \underline{A}(X)$,

and

(A12) If $X \subset Y$, then $\bar{A}(X) \subset \bar{A}(Y)$ and $\underline{A}(X) \subset \underline{A}(Y)$.

Let us also notice that $\underline{A}(X) = \bar{A}(X)$ if and only if X is a composed set in A.

Let $A = (U, R)$ and $A' = (U, R')$ be approximation spaces. If $R' \subset R$ we say 7hat the space A' is *finer* than the space A or that the space A is *coarser* than the space A'.

If A' is finer than A, then the following is true:

(A13) $\underline{A}(X) \supset \underline{A}'(X)$,

(A14) $\bar{A}(X) \subset \bar{A}'(X)$,

for every $X \subset U$.

4. Observable and unobservable sets. Let $A = (U, R)$ be an approximation space.

If we assume that we are able to observe elementary sets only and their unions, i.e., composed sets, than we can classify subsets of the approximation space $A = (U, R)$ in the following way:

1) If $\underline{A}(X) = \bar{A}(X)$ then X will be called *observable* in A, otherwise set X is *unobservable* in A.

Let $X \subset U$ and let X be unobservable in A. We introduce the following four categories of unobservable sets:

2) If $\underline{A}(X) \neq \emptyset$ and $\bar{A}(X) \neq U$, then we shall call set X *roughly observable* in A.

3) If $\bar{A}(X) = U$, set X will be called *externally unobservable* in A.

4) If $\underline{A}(X) = \emptyset$ set X will be called *internally unobservable* in A.

5) If X is both externally and internally unobservable in A then set X will be called *totally unobservable* in A.

Let us now give an intuitive motivation for the above classification.

If set X is roughly observable in A, that is to mean that we are able to observe set X with a certain approximation only, i.e., to observe only lower and upper approximations of X.

If set X is externally unobservable in A means that we can not exclude any element $x \in U$ being possibly a member of X.

If set X is internally unobservable in A it means that we are unable to say for sure that any $x \in U$ is a member of X.

Finally, if set X is totally unobservable in A it means that we cannot exclude any element $x \in U$ being possibly a member of X and we cannot also say for sure that any $x \in U$ is a member of X.

5. Stochastic approximation space, inner and outer probability. The purpose of this section is to give probabilistic interpretation of the notions given in previous sections.

Let $A = (U, R)$ be an approximation space. Any subset of U will be called an *event* in A. In particular one-element event is called *primitive* in A and elementary sets in A are called *elementary* (or atomic) events in A. Observable sets in A are called *observable* events in A.

By a stochastic approximation space we mean an ordered triple

$$S = (U, R, P)$$

where $A = (U, R)$ is an approximation space, called an underlying space, and P is a probability measure defined an observable sets in A.

Evidently $P(\emptyset) = 0$, $P(U) = 1$, and if $X = \bigcup_{i=1}^{n} X_i$ is an observable set in A and X_i are atomic sets in A, then $P(X) = \sum_{i=1}^{n} P(X_i)$.

Our aim is to evaluate probability of unobservable events. (We recall that we are not given probabilities of primitive events).

In order to investigate the problem we introduce inner and outer probability of an event in the stochastic approximation space $S = (U, R, P)$ denoted as $\underline{P}_S(X)$ and $\bar{P}_S(X)$ respectively and defined as follows:

$$\underline{P}_S(X) = P(\underline{A}(X))$$
$$\bar{P}_S(X) = P(\bar{A}(X)),$$

where $A = (U, R)$ is the underlying approximation space of $S = (U, R, P)$.

From the definition of the probability measure and properties of approximations we get the following properties of inner and outer probabilities:

(B1) If X is observable in A then $\underline{P}_S(X) = \bar{P}(X) = P_S(X)$,
(B2) $\underline{P}_S(X) \leqslant P(X) \leqslant \bar{P}_S(X)$,
(B3) $\underline{P}(\emptyset) = \bar{P}_S(\emptyset) = 0$,
(B4) $\underline{P}_S(U) = \bar{P}_S(U) = 1$,
(B5) $\underline{P}_S(-X) = (1 - \bar{P}_S(X))$,
(B6) $\bar{P}_S(-X) = (1 - \underline{P}_S(X))$,
(B7) $\underline{P}_S(X \cup Y) \geqslant \underline{P}_S(X) + \underline{P}_S(Y)$ provided $\underline{A}(X) \cap \underline{A}(Y) = \emptyset$,
(B8) $\bar{P}_S(X \cup Y) = \bar{P}_S(X) + \bar{P}_S(Y)$ provided $\bar{A}(X) \cap \bar{A}(Y) = \emptyset$,
(B9) $\underline{P}_S(X \cap Y) = \underline{P}_S(X) \cdot \underline{P}_S(Y)$ provided $\underline{A}(X)$ and $\underline{A}(Y)$ are stochastically independent,
(B10) $\bar{P}_S(X \cap Y) \leqslant \bar{P}_S(X) \cdot \bar{P}_S(Y)$ provided $\bar{A}(X)$ and $\bar{A}(Y)$ are stochastically independent.

In general case we have

(B11) $\underline{P}_S(X \cup Y) \geqslant \underline{P}_S(X) + \underline{P}_S(Y) - \underline{P}_S(X \cap Y)$,
(B12) $\bar{P}_S(X \cup Y) \leqslant \bar{P}_S(X) + \bar{P}_S(Y) - \bar{P}_S(X \cap Y)$.

Let $S=(U,R,P)$ and $S'=(U,R',P')$ be two stochastic approximation spaces; if $A'=(U,R')$ is finer than $A=(U,R)$, we shall say that also S' is finer than S.

Obviously the following is true:

(B13) $\underline{P}_{S'}(X)\geqslant \underline{P}_S(X)$,

(B14) $\bar{P}_{S'}(X)\leqslant \bar{P}_S(X)$.

6. **Rough probability.** With every event X in a stochastic approximation space $S=(U,R,P)$ we associate the interval $P_S^*(X)\subset\langle 0,1\rangle$ defined as

$$P_S^*(X)=\langle \underline{P}_S(X), \bar{P}_S(X)\rangle,$$

and $P_S(X)$ will be called *rough probability* of X in S.

Thus $P_S^*(X)$ is the interval to which the probability of the unobservable event X belongs.

Of course if X is an observable event in S then

$$\underline{P}_S(X)=\bar{P}_S(X)=P(X)$$

and

$$P_S^*(X)=\langle P(X), P(X)\rangle$$

or simply

$$P_S^*(X)=P(X),$$

i.e. $P_S^*(X)$ reduces to one point.

Certainly $P_S^*(X)$ has the following properties:

(C1) $P_S^*(\emptyset)=0$,

(C2) $P_S^*(U)=1$,

(C3) $P_S^*(-X)=\langle 1-\bar{P}_S(X), 1-\underline{P}_S(X)\rangle$,

(C4) $P_S^*(X\cup Y)\subset\langle \underline{P}_S(X)+\underline{P}_S(Y), \bar{P}_S(X)+\bar{P}_S(X)\rangle$,

(C5) $P_S^*(X\cap Y)\subset \underline{P}_S(X)\cdot\underline{P}_S(Y), \bar{P}_S(X)\cdot\bar{P}_S(Y)\rangle$.

Obviously we have the following properties:

(a) If X is externally unobservable in A, then

$$P_S^*(X)=\langle \underline{P}_S(X), 1\rangle,$$

(b) If X is internally unobservable in A then

$$P_S^*(X)=\langle 0, \bar{P}_S(X)\rangle$$

(c) If X is totally unobservable in S, then

$$P_S^*(X)=\langle 0,1\rangle.$$

In other words: if the event X is observable in A, then we can give exact probability $P(X)$: if X is roughly observable in A then we can give the interval $P_S^*(X)$ to which the probability of event X belongs; if X is externally unobservable in A then we can give only lower bound of the probability of X; if X is internally unobservable in A, then we can give only upper bound of the probability of X, if X is totally unobservable in A then we cannot give any bounds for the probability of X.

Moreover we have the following property:

If S' is finer than S, then

$$P^*_{S'}(X) \subset P^*_S(X)$$

for every $X \subset U$.

7. Uncertainty measure. In order to describe to what extent the probability of an event $X \subset U$ can be evaluated in the given stochastic approximation space S we introduce uncertainty measure $\eta_S(X)$ defines as below:

$$\eta_S(X) = \bar{P}_S(X) - \underline{P}_S(X),$$

which is simply the length of the interval $P^*_S(X)$.

By simple calculation one can show the followings:

(D1) $\eta_S(-X) = \eta_S(X)$,

(D2) $\eta_S(X \cup Y) \leqslant \eta_S(X) + \eta_S(Y)$,

(D3) $\eta_S(X \cap Y) \leqslant \eta_S(X) \cdot \eta_S(Y)$,

(D4) If S' is finer than S, then $\eta_{S'}(X) \leqslant \eta_S(X)$.

Thanks are due to Dr. M. von Rimscha and Dr. S. T. Wierzchoń for critical remarks.

INSTITUTE OF COMPUTER SCIENCE, POLISH ACADEMY OF SCIENCES, PKIN, 00-901 WARSAW
(INSTYTUT PODSTAW INFORMATYKI, PAN)
COMPUTER SCIENCE DEPARTMENT, UNIVERSITY OF NORTH CAROLINA, CHARLOTTE, N.C. 28223, (USA)

REFERENCES

[1] A. P. Dempster, *Upper and lower probabilities induced by a multi-valued mapping*, Ann. Math. Statist., **38** (1967), 325–339.

[2] E. Pleszczyńska, D. Dąbrowska, *On partial observability in statistical models*, Math. Operationforsch. Statist. Ser. Statistics, **11** (1980), 49–59.

[3] Z. Pawlak, *Rough sets*, International Journal of Information and Computer Sciences, **11** (1982), 341–356.

[4] G. Shafer, *A mathematical theory of evidence*, Princeton University Press, 1976.

З. Павляк, **Приближенная вероятность**

В настоящей работе излагается концепция частичной наблюдаемости, рассматриваемои в статистических моделях в терминологии приблизительных множеств. С любым случаем связана внутренняя и внешняя вероятность или интервал, концы которого являются соответственно внутренней в вышней вероятностью. Этот интервал называется приближенной вероятностью случая. Исследуются некоторые основные свойства приближенной вероятности.