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Rough Logic

by

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Summary. In this note we define the concept of approximate truth, and outline the corresponding logic, called the rough (approximate) logic, in short *R*-logic. The starting point of our considerations is the notion of a rough set introduced by the author earlier, as mathematical tool to deal with the imprecise information.

The concept of approximate truth has called attention of many researchers in philosophy, logic and recently in computer science. The latest were mainly motivated by artificial intelligence research in expert system, approximate reasoning methods and other.

The aim of this note is to provide a theoretical framework to this kind of problems, where truth is not known exactly but only with a certain approximation.

1. Introduction. The need to speak of partial truth has been recognized long time ago. Relevant literature can be found in many books and papers concerning the subject. We refer here only to the paper of Hilpinen [1], because our approach is somewhat similar to that presented in [1].

Starting point of our considerations is the concept of a rough set and an approximation space introduced in [3]. Next, a logic with the approximation space is associated, in which five logical values are assumed: truth, falsity, rough truth, rough falsity and rough inconsistency. Some elementary properties of this logic are shown. The rough (approximate) truth and falsity represent our partial knowledge about the world and with the increase of our knowledge the roughly true (or false) formulas tend to be more true (or false) and approach the truth and falsity closer and closer. Thus the truth and falsity is the limit of our partial knowledge. Hence the rough truth and rough falsity can be seen as an inductive truth and falsity, and truth and falsity are of deductive nature.

Another kind of logical investigations associated with rough sets and approximate reasoning can be found in [2, 4, 5] but our approach is different to that considered in those papers, because we are mainly interested here in the concept of partial truth.

2. Approximation space, rough sets. By an approximation space we mean an ordered pair $A = (U, R)$, where U is a set called the universe, and $R \subseteq U \times U$ is a binary relation over U, called the indiscernibility relation. We assume that R is an equivalence relation. Equivalence classes of R are refered as elementary sets in A. Any finite union of elementary sets in A is called R-discernible set (or set discernible in A). Indiscernible sets in A are called rough sets in A.

The indiscernible relation R represents our knowledge about the universe U, in the approximation space A. The finer the indiscernibility relation is, the more accurate knowledge we have about the universe and we are able to discern more exactly the elements of the universe. In practicall application the indiscernibility relation is determined by the accuracy of observation, measurement or description of phenomena we are interested in.

The basic conceptions on which our approach is based are those of a lower and upper approximation of $X \subseteq U$.

By the lower approximation of X in A ($\underline{R}X$) we mean the set

$$\underline{R}X = \{x \in U : [x]_R \subseteq X\},$$

and by the upper approximation of X in A ($\bar{R}X$) we mean the set

$$\bar{R}X = \{x \in U : [x]_R \cap X \neq \emptyset\},$$

where $[x]_R$ denotes the equivalence class of R, containing the element x.

The boundary of X in A is the set

$$BN_R(X) = \bar{R}X - \underline{R}X.$$

Obviously, set X is discernible in A if $\bar{R}X = \underline{R}X$; otherwise set X is indiscernible in A (rough in A).

Intuitively speaking, if our knowledge about the universe is not precise enough, we are unable to discern object in the universe and consequently we cannot discern some subset of the universe, or in other words – if we observe a subset X of the universe with some approximation determined by the indiscernibility relation R, in general we are unable to observe the subset exactly, but only with a certain approximation. The boundary of the set X is the region which cannot be recognized because of the default in our knowledge.

Let us notice that the idea of "imprecise" sets can be also expressed employing two membership relations $\underline{\in}$, $\bar{\in}$ defined as follows:

$$x \underline{\in}_R X \text{ iff } x \in \underline{R}X \quad \text{and} \quad x \bar{\in}_R X \text{ iff } x \in \bar{R}X,$$

which reads: "x surely belongs to X" and "x possibly belongs to X" – respectively. The modal logic flavor is easily seen from the definition.

Each approximation space $A = (U, R)$ defines uniquely the topological space $T_A = (U, \mathrm{DIS}(R))$, where $\mathrm{DIS}(R)$ is the family of all R-discernible sets in A – and it is the family of all open and closed sets (the topology of T_A) in T_A. Thus, the upper and lower approximations are interior and closure in T_A, respectively, and have hence the following properties:

1) $\underline{R}X \subseteq V \subseteq \bar{R}X$
2) $\underline{R}U = \bar{R}U = U$; $\underline{R}\emptyset = \bar{R}\emptyset = \emptyset$
3) $\underline{R}(X \cup Y) \supseteq \underline{R}X \cup \underline{R}Y$
4) $\bar{R}(X \cup Y) = \bar{R}X \cup \bar{R}Y$
5) $\underline{R}(X \cap Y) = \underline{R}X \cap \underline{R}Y$
6) $\bar{R}(X \cap Y) \subseteq \bar{R}X \cap \bar{R}Y$
7) $\underline{R}X = -\bar{R}(-X)$
8) $\bar{R}X = -\underline{R}(-X)$.

In the topology generated by an approximation space we have additional properties:

9) $\underline{R}\underline{R}X = \bar{R}\underline{R}X = \underline{R}X$
10) $\bar{R}\bar{R}X = \underline{R}\bar{R}X = \bar{R}X$.

Sets in the approximation space $A = (U, R)$ can be classified from the topological point of view in the following way:

1) if $\underline{R}X = \bar{R}X$, X is discernible in A
2) if $\underline{R}X \neq \bar{R}X$, X is indiscernible in A (rough in A).

Rough sets can be classified into the following four classes:

2a) if $\underline{R}X \neq \emptyset$ and $\bar{R}X \neq U$, X is roughly discernible in A (RDIS)
2b) if $\underline{R}X = \emptyset$, X is internally indiscernible in A (IIND)
2c) if $\bar{R}X = U$, X is externally indiscernible in A (EIND)
2d) if $\underline{R}X = \emptyset$ and $\bar{R}X = U$, X is totally indiscernible in A (TIND).

It is obvious that:

a) if $X \subseteq U$ is discernible (roughly discernible or totally indiscernible) in A, so is $-X$;

b) if $X \subseteq U$ is internally (externally) indiscernible in A, then $-X$ is externally (internally) indiscernible in A.

3. Rough logic. We assume in R-logic the following notations: p, q, r etc., are propositional variables, T and F stand for truth and falsity. The basic propositional connectives are $(-)$, $(\vee)$, and $(\wedge)$. Lower case greek letters φ, ψ etc., denote formulas of R-logic, and are defined in usual way. The semantics of R-logic is defined by means of interpretation function $(\|)$ in a standard way, i.e. the function $(\|)$ assigns to each formula φ in R-logic its meaning in the universe U, in the following way:

1) $|T| = U$
2) $|F| = \emptyset$
3) $|-\varphi| = U - |\varphi|$

4) $|\varphi \vee \psi| = |\varphi| \cup |\psi|$

5) $|\varphi \wedge \psi| = |\varphi| \cap |\psi|$.

By a model of R-logic we mean a system $M = (U, R, \|\|)$, where (U, R) is an approximation space and $(\|\|)$ is an interpretation function.

We say that φ is *true* on $x \in U$ in M, if $x \in |\varphi|$, and we write $\models_x \varphi$, or $\mathrm{val}_x(\varphi) = T$; otherwise φ is *false* on x in M.

If $|\varphi| = U$ we say that formula φ is *true* in M; otherwise formula φ is *false* in M, and we write $\models_M(\varphi)$ (or $\mathrm{val}_M(\varphi) = T$), and $\not\models_M(\varphi) = F$ (or $\mathrm{val}_M(\varphi) = F$), respectively.

Of course formula φ is *true* in M, if and only if φ is true on every $x \in U$, and φ is false in M, if and only if φ is false for some $x \in U$.

Formula φ is *surely true* on $x \in U$ in M ($\models_{x,\underline{R}} \varphi$), if $x \in \underline{R}\,|\varphi|$ (or $x \underline{\in}_R |\varphi|$); formula φ is *possibly true* on $x \in U$ in M ($\models_{x,\bar{R}} \varphi$), if $x \in \bar{R}\,|\varphi|$ (or $x \bar{\in}_R |\varphi|$).

Fact 1.

a) $\models_{x,\underline{R}} \varphi$ iff $[x]_R \subseteq |\varphi|$

b) $\models_{x,\bar{R}} \varphi$ iff $[x]_R \cap |\varphi| \neq \emptyset$.

If $|\varphi|$ is EIND, we say that φ is roughly true in M, and we write $\models_R \varphi$ (or $\mathrm{val}_R(\varphi) = RT$).

Obviously if φ is *roughly true* in M, then φ is possibly true in M on every $x \in U$.

If φ is roughly true in M, then $-\varphi$ will be called roughly false in M, and we shall write $\not\models_R \varphi$ (or $\mathrm{val}_R(\varphi) = RF$).

Fact 2. Formula φ is roughly false in M, if and only if $|\varphi|$ is IIND.

Formula φ is *roughly inconsistent* in M, if $|\varphi|$ is roughly true and roughly false in M. The following fact is obvious:

Fact 3. Formula φ is roughly inconsistent in M if and only if $|\varphi|$ is TIND.

4. Degree of truth. In the R-logic we can introduce degree of truth in the following way:

If φ and ψ are roughly true in M, then we shall say that φ is *more true* than ψ in M (or ψ is *less true* than φ), symbolically $\models_R \varphi > F_R \psi$ (or $\not\models_R \varphi < \not\models_R \psi$), if $\varphi \supset \psi$.

If φ and ψ are roughly false in M and $\varphi \subset \psi$, then φ will be called *more false* than ψ.

Of course, if φ and ψ are both roughly true in M, and φ is *more true* than ψ, then $-\varphi$ is *more false* than $-\psi$.

Fact 4. If φ and ψ are both roughly true in M, then $\varphi \vee \psi$ is not less true than either φ or ψ, and $\varphi \wedge \psi$ is not more true than either φ or ψ.

Let $\varphi_1, \varphi_2, \ldots, \varphi_n \ldots$ be an infinite sequence of formulas in R-logic, such that φ_{i+1} is more true then φ_i, for every i $(1 \leqslant i \leqslant n)$, i.e. $\models_R \varphi_i < \models_R \varphi_{i+1}$. The following facts are obvious

Fact 5.

a) $$\bar{R} \bigcup_{i=1}^{\infty} |\varphi_i| = \bigcup_{i=1}^{\infty} \bar{R}\, |\varphi_i| = U,$$

b) $$\underline{R} \bigcap_{i=1}^{\infty} |-\varphi_i| = \bigcap_{i=1}^{\infty} \underline{R}\, |-\varphi_i| = \emptyset.$$

Fact 6. The following conditions are equivalent

c) $$\bigcup_{i=1}^{\infty} \underline{R}\, |\varphi_i| = U,$$

d) $$\bigcap_{i=1}^{\infty} \bar{R}\, |-\varphi_i| = \emptyset.$$

Instead of (c) and (d) we can write

e) $$\limsup_{i \to \infty} \operatorname{val}(\varphi_i) = T,$$

f) $$\liminf_{i \to \infty} \operatorname{val}(-\varphi_i) = F.$$

The intuitive meaning of the above properties may be explained in the following way: if φ_i is roughly true in M, we can also say that φ_i is inductively true in M, i.e. true – according to our present state of knowledge. More knowledge alow us to replace formula φ_i by more true formula φ_{i+1}, etc. – eventually the process leads to the true formula φ, which is the limit of roughly true formulas φ_i. The limit formula φ can be called deductively true. Thus, the process of acquiring knowledge leads from partial (inductive) truth, to total (deductive) truth.

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З. Павляк, **Приближенная логика**

В настоящей работе рассматривается понятие приближенной правды и дается набросок соответственной логики. Исходной точкой предлагаемых рассуждений является прежде введенное автором понятие „приближенного множества" (rough set) как математического инструмента для анализа приближенной информации.

Понятие приближенной правды уже давно вызывало интерес философов, логиков, а в последнее время и информатиков. Для последних это понятие интересно по поводу проведения исследовательских работ по искусственному интеллекту, а особенно по экспертным системам, по методам автоматизации приближенных рассуждений и другим.

Цель настоящей работы — дать набросок теоретической концепции, позволяющей изучать проблемы, когда не является известной вся правда, а лишь некоторое ее приближение.