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ON ROUGH DERIVATIVES, ROUGH INTEGRALS AND ROUGH DIFFERENTIAL EQUATIONS

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Abstract

In this paper we define rough (discrete) lower and upper representation of real functions and define and investigate some properties of these representations, such as rough continuity, rough derivatives, rough integral and rough differential equations - which can be viewed as discrete counterparts of real functions. An illustrative example of the introduced concepts is given.
The presented approach can be used to synthesis and analysis of discrete dynamic system, in particular in control theory. It is also related to the qualitative reasoning methods

Keywords: Rough sets, Rough functions, Rough controllers

1 Introduction

Physical phenomena are usually described by differential equations. Solutions of these equations are real valued-functions, i.e. functions which are defined and valued on continuum of points. However due to limited accuracy of measurement and computations we unable to observe (measure) or compute (simulate) exactly the abstract solutions and consequently we deal rather with approximate then exact solutions and deal with discrete and not continues variables and functions.

In this paper we are going to give some remarks on the relationship between real and discrete functions based on the rough set philosophy. In particular we define rough (discrete) lower and upper representation of real functions and define and investigate some

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properties of these representations, such as rough continuity, rough derivatives, rough integral and rough differential equations - which can be viewed as discrete counterparts of real functions.

In particular we are interested how discretization of the real line effects basic properties of real functions, such as continuity, differentiability, etc. It turns out that some properties of real functions have counterparts in the case of discrete functions, but this is not always the case.

We define in this note rough (approximate) continuity, rough differentiability and rough integral of discrete functions, and give some of its basic properties - analogous to those of real functions.

The presented approach has some overlaps with other ideas developed in analysis in order to avoid the concept of infinity - in particular non-standard analysis [11], finistic analysis [4], infinitesimal analysis [2]. On the other hand "rough" analysis can be viewed as a theoretical basis for cell-to-cell mapping, used in discrete dynamical system analysis [3]. Last but not least the proposed philosophy can be seen as a generalization of qualitative reasoning [12], where three-valued $(+, 0, -$, i.e. increasing, not changing, decreasing) of qualitative derivatives are replaced by more general concept of multi-valued qualitative derivatives, so that expression like, for example, "slowly increasing", "fast increasing", "very fast increasing" etc. - instead of "increasing" only can be used.

2 Scale and Discretization

Let $[n] = \{0, 1, \ldots, n\}$ be a set of natural numbers. A strictly monotonic function $d : [n] \rightarrow \mathbf{R}$, i.e. such that for all $i, j \in [n]$, $i < j$ implies $d(i) < d(j)$ will be called a *scale*.

Any scale $d : [n] \rightarrow \mathbf{R}$ is really finite increasing sequence of reals $x_0, x_1, \ldots, x_n$, such that $x_i = d(i)$, for very $i \in [n]$ - thus it can be seen as a *discretization* of the closed interval $R_n = \langle d(0), d(n) \rangle = \langle x_0, x_n \rangle$.

Given a scale $d : [n] \rightarrow \mathbf{R}$ then one can define two functions

$$d_*(x) = max\{i \in [n] : x_i \leq x\},$$

$$d^*(x) = min\{i \in [n] : x_i \geq x\},$$

for every $x \in R_n$.

On the interval $R_n = \langle x_0, x_n \rangle$ we define an equivalence relation I_d, called the *indiscernibility* relation, and defined thus

$$x I_d y \text{ iff } d_*(x) = d_*(y) \text{ and } d^*(x) = d^*(x).$$

The family of all equivalence classes of the relation I_d, or the partition of the interval R_n, is given below

$$\{x_0\}, (x_0, x_1), \{x_1\}, (x_1, x_2), \{x_2\}, \ldots, (x_{n-1}, x_n), \{x_n\}$$

where each equivalence classe $[x]_d$ is an interval such that $[x]_d = (x_i, x_{i+1})$ whenever $x_i < x < x_{i+1}$, and $[x_i]_d = \{x_i\}$ for all $i \in [n]$.

If $x_i < x < x_{i+1}$, then $I_{*d}(x) = d(d_*(x)) = x_i$ and $I_{*d}(x) = d(d^*(x)) = x_{i+1}$, i.e. $I_d^*(x)$ and $I_d^*(x)$ are the ends of the interval $\langle x_i, x_{i+1} \rangle$; if $x = x_i$, then $I_{*d}(x) = I_d^*(x) = x_i$.

The ends of the interval $\langle x_i, x_{i+1} \rangle$ are called the *lower* and the *upper d-approximation* of x, respectively.

The above discussed ideas are illustrated in Fig. 1.

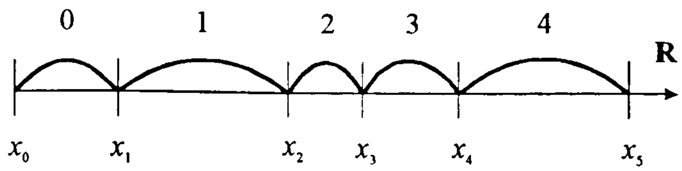

Fig. 1

3 Roughly Continuous Real Functions

Suppose we are given two scales $d : [n] \to \mathbf{R}$ and $e : [m] \to \mathbf{R}$, and let $f : R_n \to R_m$ be a function, where R_n, R_m denote the both side closed intervals $\langle x_0, x_n \rangle$, $\langle y_0, y_m \rangle$ respectively. We define its *lower rough representation* f_* with respect to d and e and its *upper rough representation* f^* with respect to d and e defined on $[n]$ and valued in $[m]$, as

$$f_*(i) = e_*(f(x_i))$$

$$f^*(i) = e^*(f(x_i))$$

for all $i \in [n]$ (see Fig. 2).

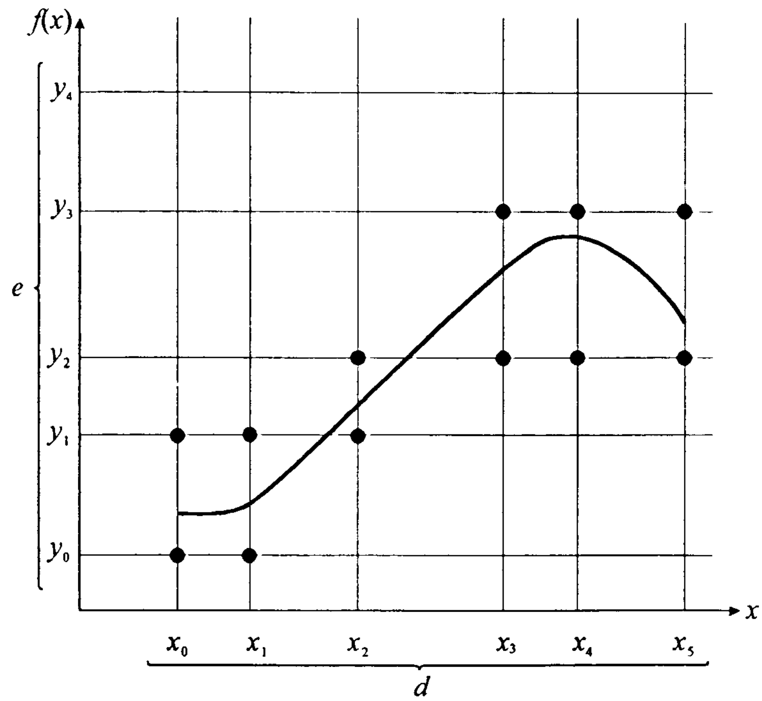

Fig. 2

Basic concept of analysis is that of continuity. Employing the idea of scaling one can define rough (approximate) continuity of real functions as shown below.

Suppose we are given scales $d : [n] \to \mathbf{R}$ and $e : [m] \to \mathbf{R}$ and a function $f : R_n \to R_m$. We say that f is roughly continuous with respect to d and e or roughly (d, e)-continuous iff for all $x, y \in R_n$, $xI_d y$ implies $f(x) I_e f(y)$, or equivalently $f([x]_d) \subseteq [f(x)]_e$, for every $x \in R_n$.

If a function f is roughly continuous it means that f cannot vary "to fast" (see Fig.3)

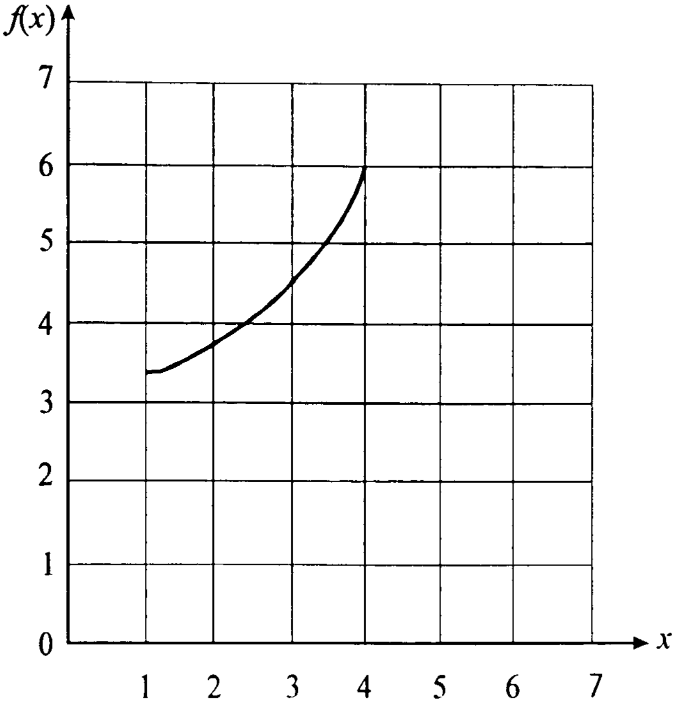

Fig. 3

but it can be not continuous in the classical sense, i.e. it can have points of discontinuity provided that they are "hidden" by the discretization (see Fig.4).

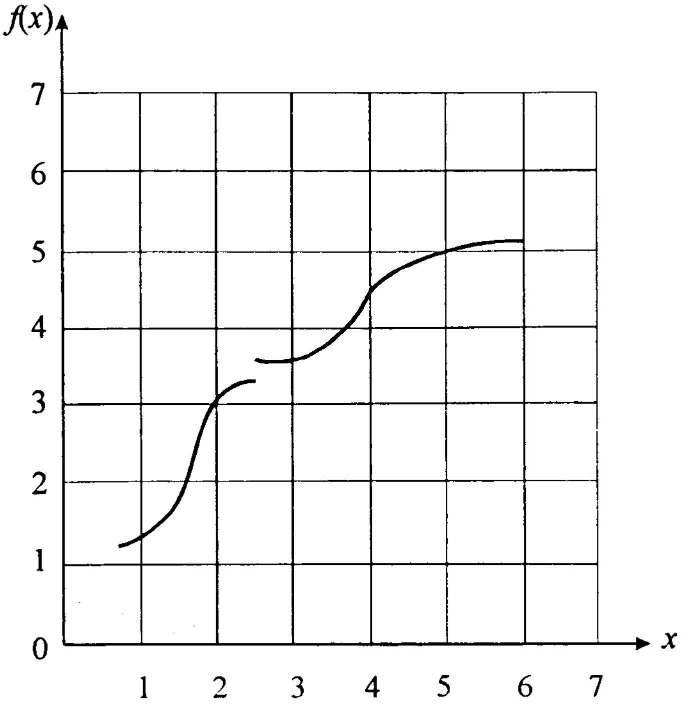

Fig. 4

Now we give important Darboux property for roughly continuous functions.

Supposes we are given two scales $d : [n] \to \mathbf{R}$ and $e : [m] \to \mathbf{R}$ and a function $f : R_n \to R_m$. We say that f has a *lower Darbuox property* with respect to d and e if for every $i \in [n-1]$ there exists $\alpha \in \{-1, 0, 1\}$ such that $f_*(i+1) = f_*(i) + \alpha$, where f_* is the lower representation of f with respect to d and e. Similarly one can define the upper Darboux property of f, by using the upper representation of f.

We will say that f has a *rough Darboux property* with respect to d and e if it has both the lower and the upper Darboux property with respect to d and e.

Proposition 1. Let $d : [n] \to \mathbf{R}$ and $e : [m] \to \mathbf{R}$ and let $f : R_n \to R_m$ be a function such that f is continuous in $d(i)$ by means of mathematical analysis. If f is roughly (d,e)-continuous then f has a rough Darboux property with respect to d and e.

Proposition 2. (Intermediate Value Theorem, [2]). Let $d : [n] \to \mathbf{R}$ and $e : [m] \to \mathbf{R}$ be scales and let $f : R_n \to R_m$ be a function. Then f has a Darboux property with respect to d and e iff for all $i, j \in [n], i \neq j$, and for every q between $f_*(i)$ and $f_*(j)$ there exist $p \in [n]$ between i and j for which $f_*(p) = q$.

This proposition is also valid for the upper rough representation of f.

The above properties are counterparts of the intermediate value property in classical analysis (Courant and John, 1965, p. 44).

The above propositions say that if a function f is roughly continuous then its lower as well upper representations cannot vary to "fast", i.e. pass from one value to another without passing through all intermediate values.

It is easily seen that if a function f is continuous, then f is not necessarily roughly continuous, and conversely, i.e. rough continuity of a function does not imply its continuity. It can be also observed that for every continuous function in an interval $\langle a, b \rangle$ one can find scales of $\langle a, b \rangle$ and $(f(a), f(b))$ such that f is roughly continuous with respect to these scales.

The rough continuity of a function is easily appreciated intuitively. Whether a function is roughly continuous or not depends on the scales of the domain and range of the function, i.e. it depends on how exactly we "see" the function through the scale.

4 Rough Derivatives and Rough Integrals

As mentioned in the introduction abstract description of physical systems requires real functions, whereas observations, measurements and computation of physical systems involves discrete functions, which describe systems with some approximation only. In what follows we are going to give some properties of discrete functions, defined and valued in the set of integers - mimicking some properties of real functions. It turns out that for this class of functions one can define concepts similar to that of real function, like continuity,derivatives, integrals, etc. These concepts display similar properties to that of real functions, and consequently discrete functions obtained as a result of measurements can be treated similarly to real functions.

We will start our consideration by defining rough (approximate) continuity for discrete functions.

A discrete function $f : [n] \to [m]$ is *roughly continuous* iff for all $i, j \in [n]$, $|i - j| = 1$ implies $|f(i) - f(j)| \leq 1$.

For a discrete function $f : [n] \to [m]$ we define the *rough derivative* f' as:

$$f'(i) = f(i+1) - f(i), \text{ for all } i \in [n-1].$$

We say that $f : [n] \to [m]$ has Darboux property if for every $i \in [n-1]$ we have that $f'(i) \in \{-1, 0, 1\}$. Thus for $f : [n] \to [m]$ having rough Darboux property and $i \in [n-1]$ the value $f'(i)$ is that $\alpha \in \{-1, 0, 1\}$ which makes $f(i+1) = f(i) + \alpha$.

Proposition 3. A discrete function $f : [n] \to [m]$ is roughly continuous iff f has Darboux property.

Thus the intermediate value property is also valid for roughly continuous discrete functions as shown by the following proposition.

Proposition 4. A discrete function $f : [n] \to [m]$ has a Darboux property iff for all $i, j \in [n], i \neq j$, and for every q between $f(i)$ and $f(j)$ there exist $p \in [n]$ between i and j for which $f(p) = q$.

Now we are going to define two basic concepts in our approach to discrete functions, namely the rough derivative and the rough integral. It turns out that they display similar properties to "classical" derivatives and integrals. Hence they can used as basic tools to deal with discrete functions obtained, for example, as a result of measurements or observations, i.e. discrete functions given in tabular form.

Directly from the definition of the rough derivative for discrete functions, we obtain the following counterpart of the well known theorem of differential calculus.

Proposition 5. Let f and g be discrete function with domain $[n]$ and range $[m]$ respectively. Than for $f + g, fg$ and f/g we have

a) $(f+g)'(i) = f'(i) + g'(i)$,

b) $(fg)'(i) = f'(i)g(i) + f(i)g'(i) + f'(i)g'(i)$,

c) $(f/g)'(i) = (f'(i)g(i) - f(i)g'(i))/(g^2(i) + g(i)g'(i))$.

From the definition of the rough derivative of discrete function and the Proposition 5 we get the following properties.

Obviously the rough derivative of a constant discrete function is equal to zero.

If $f(i) = i + k$, where k is an integer constant, then $f'(i) = 1$.

If $f(i) = ki$, then $f'(i) = k$.

If $f(i) = k^i$, then $f'(i) = (k-1)k^i$; for $k = 2$ we have $f'(i) = 2^i$.

If $f(i) = i^k$, then $f'(i) = \sum_{j=0}^{k} \binom{k}{j} i^{k-j} - i^k$;

In particular, if $k = 2$ we get $f'(i) = 2i + 1$; for $k = 3$ we have $f'(i) = 3i^2 + 3i + 1$, etc.

Some important properties are not valid for discrete functions, even for discrete roughly continuous functions, as shown by the following two propositions.

Proposition 6. Assume that a discrete function $f : [n] \to [m]$ has a maximum (minimum) at $i \in (n)$, where $(n) = \{1, 2, \ldots, n-1\}$. Then not necessarily $f'(i) = 0$ (see Fig. 5).

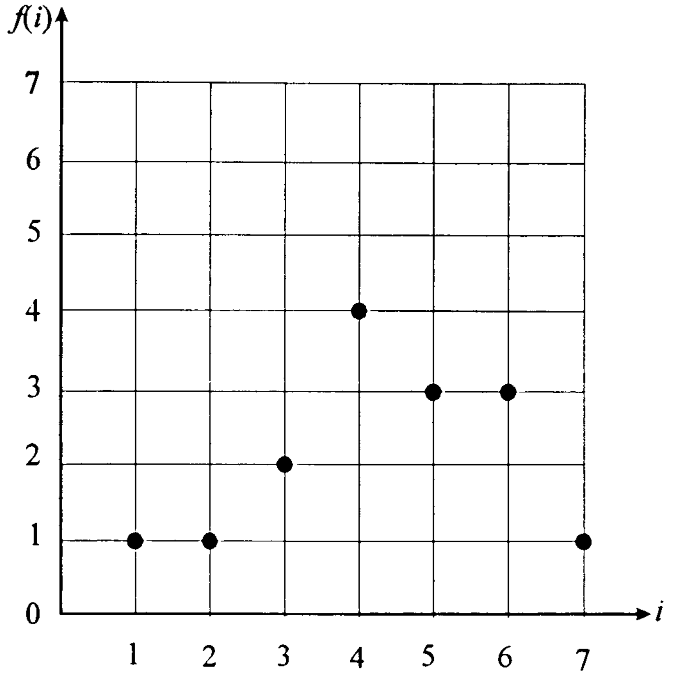

Fig. 5

The Rolle's theorem does not hold for discrete functions, as shown by the proposition below.

Proposition 7. Let $f : [n] \to [m]$ be a discrete, function, such that $f(0) = f(n) = 0$. Then not necessarily there exists $i \in (n)$ such that $f'(i) = 0$.

We say that a discrete function f is *roughly smooth* if its first derivative is roughly continuous. It can be easily seen that for roughly smooth functions the above two propositions are valid, provided that they are slightly modified. Detailed discussion of this problem is left to the reader.

Higher order derivatives can be also defined in the same manner, as shown in the example below.

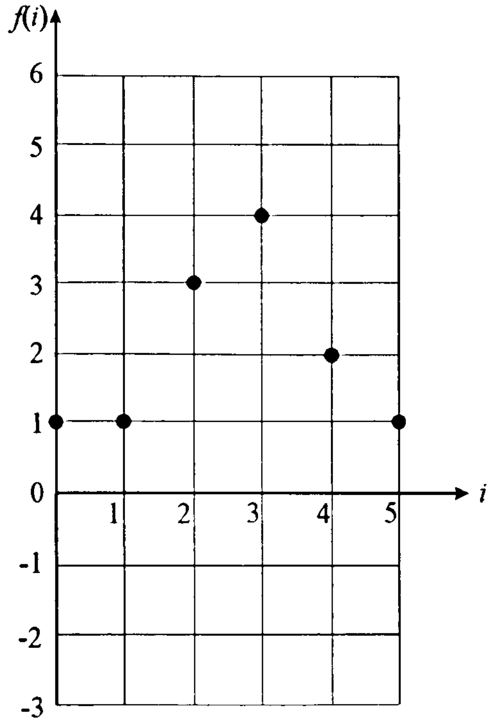

Fig. 6

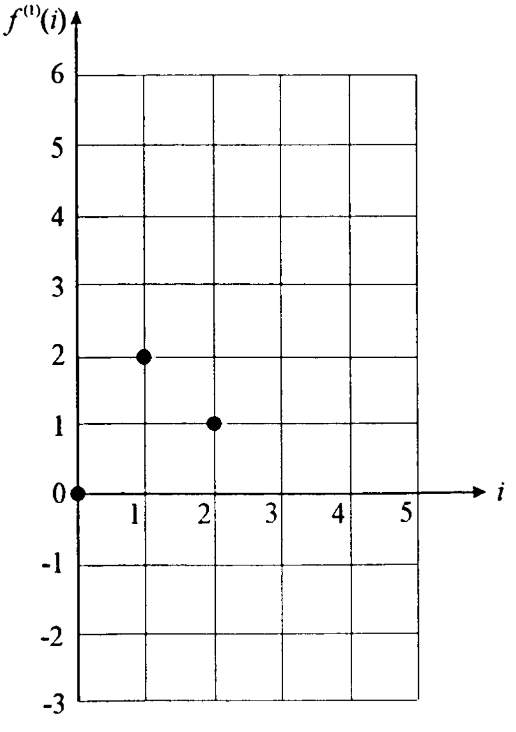

Fig. 7

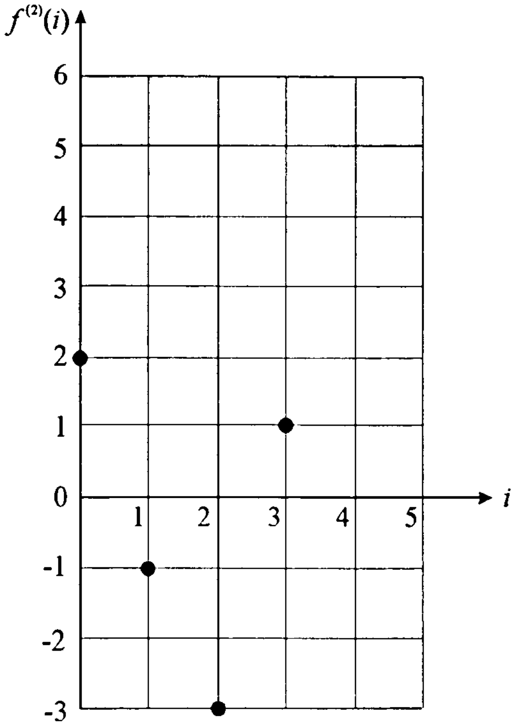

Fig. 8

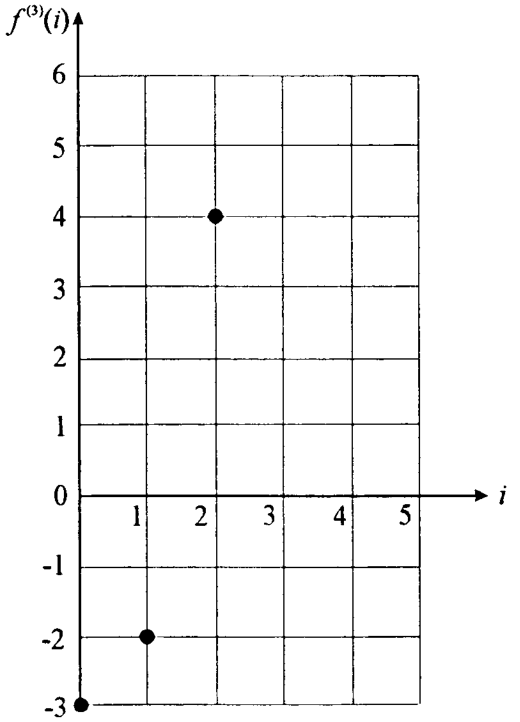

Fig. 9

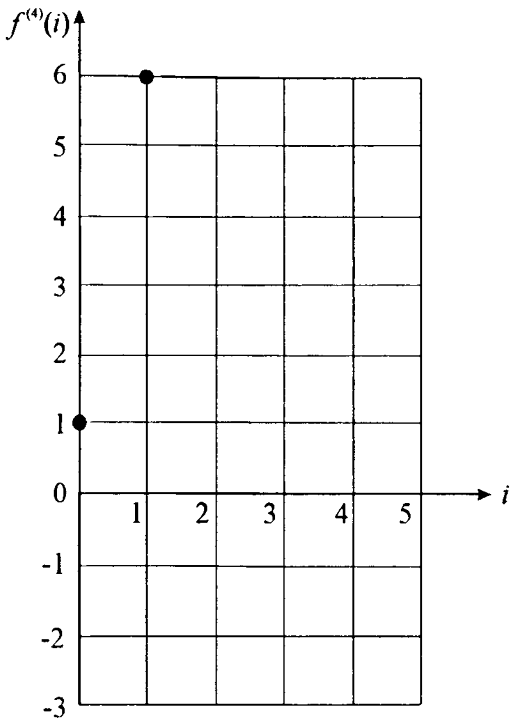

Fig. 10

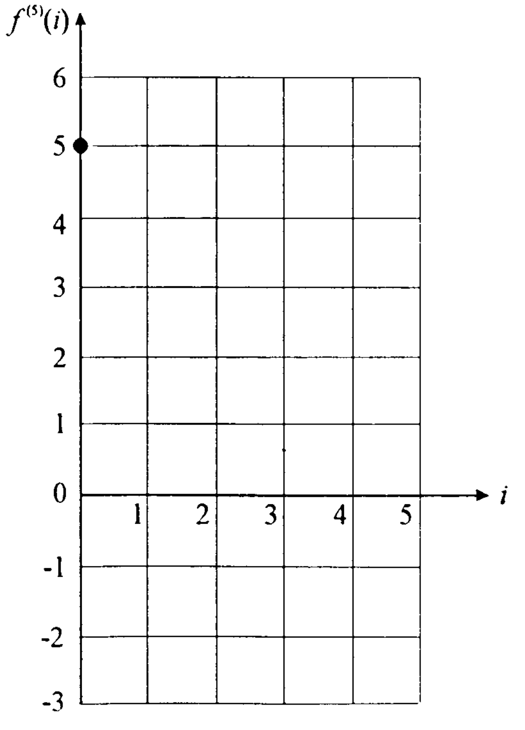

Fig. 11

Notice that f is a discrete function $f : [n] \longrightarrow [m]$ defined on $n+1$ points, i.e. on the set $\{0, 1, \ldots, n\}$, and its k-th derivative $f^{(k)} : [n-k] \longrightarrow [m]$ is defined on $n-k+1$ points. Thus each discrete function $f : [n] \longrightarrow [m]$ has at most derivatives up to the n-th order. Consequently each discrete function $f : [n] \longrightarrow [m]$ is uniquely defined by the set of the following initial conditions $f^{(n)}(0), f^{(n-1)}(0), \ldots, f^{(1)}(0), f^{(0)}(0)$, where $f^{(0)}(0) = f(0)$.

Next we define integration of discrete functions.

Let $f : [n] \longrightarrow [m]$ be a discrete function. By a *rough integral* of f we mean the function

$$\int_{j=0}^{i} f(j)\Delta(j) = \sum_{j=0}^{i} f(j)\Delta(j),$$

where $\Delta(j) = (j+1) - j = 1$.

The following important property holds.

Proposition 8.

$$\int_{j=0}^{i} f'(j)\Delta(j) = f(i) + k,$$

where k is a integer constant.

In other words

$$f(i) = f(0) + \sum_{j=0}^{i-1} f'(j),$$

or in recursive form

$$f(i+1) = f(i) + f'(i).$$

with the intitial condition

$$f(0) = k.$$

This proposition can be used for solving rough differential equations, and will be discused in the next section.

5 Rough Differential Equations

Starting from the notion of rough derivative for discrete functions one can define a concept of differential equation for discrete functions, called in what follows rough differential equation, [3]. Rough differential equation, together with initial condition can be solved inductively by employing Proposition 8, which gives the relationship between initial condition, rough derivative and the solution.

Ordinary 1-st order differential equation is shown below

$$f'(x) = \Phi(x, f(x)),$$

where Φ is a real valued function on the Cartesian product of reals.

Similarly one can define a *rough differential equation*, for discrete functions as

$$(*) \qquad f'(i) = \Phi(i, f(i)),$$

where Φ is an integer valued function defined on the Cartesian product $[n] \times [m]$.

Because $f'(i) = f(i+1) - f(i)$, the rough differential equation can be presented as

$$f(i+1) = \Phi(i, f(i)) + f(i),$$

which together with an initial condition

$$f(0) = j_0, j_0 \in [m]$$

defines uniquely the solution of the rough differential equation $(*)$.

Example

Consider a very simple rough differential equation given by the formula

$$(**) \qquad f'(i) = 4i + 1$$

with the initial condition $f(0) = 2$.

By employing Proposition 5 one can easily show that the solution of this equation has the form

$$f(i) = f(0) + 2i^2 - i.$$

We can also solve this equation by using Proposition 8. Suppose we are given the rough differential equation $(**)$ in tabular form, and we do not know its analytical presentation. In this case, by Proposition 8 we have

$$f(i+1) = f(i) + f'(i)$$

with $f(0) = 2$ which yields

$$f(0) = 2$$

$$f(1) = f(0) + f'(0) = 3$$

$$f(2) = f(1) + f'(1) = 8$$

$$f(3) = f(2) + f'(2) = 17$$

$$f(4) = f(3) + f'(3) = 30$$

$$f(5) = f(4) + f'(4) = 47$$

etc.

Thus we have two ways of solving rough differential equations. The first one is similar to that used in analysis, and it boils down to symbolic manipulation on formulas, whereas the second is suitable to functions presented in tabular form.

6 Conclusion

The basic concepts of the presented approach to experimental data analysis is based on concepts of rough, continuity of discrete functions (data), rough derivatives and rough integrals of discrete functions valued and defined on integers. This leads to rough differential equations which can be solved by using symbolic formula manipulations or by using recursive procedures applied to functions presented in tabular form. Similar procedure can be applied also for systems of rough differential equations.

The presented approach can be used to synthesis and analysis of discrete dynamic system, in particular in control theory. To this end the concept of rough stability must be defined and investigated. This issue will be discussed in separate paper.

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