Sets, Fuzzy Sets and Rough Sets

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Abstract

Rough set theory can be perceived as a new approach to vagueness and uncertainty. It seems to be of fundamental importance to AI and cognitive sciences, especially in the areas of machine learning, knowledge acquisition, decision analysis, knowledge discovery from databases, expert systems, inductive reasoning, pattern recognition, decision support systems and data mining.

In the paper basic concepts of rough set theory will be given and comparison with classical set and fuzzy sets briefly discussed.

Keywords: rough sets, fuzzy sets, vagueness

Introduction

The notion of a set is a fundamental concept for a whole contemporary mathematics, and set theory introduced by George Cantor in 1883 is, no doubt, a mile stone in development of modern mathematical thinking. It is well known that the concept of an infinite set has a very serious drawback – it leads to antinomies. This defect has rather philosophical than practical significance. In order to cure this fault several remedies have been proposed: axiomatisation (Zermelo-Freankel), theory of classes (von Nemann), type theory (Russell and Whitehead). Besides, instead of improving cantor's set theory another set theories have been proposed, e.g., mereology (Lesniewski) and alternative set theory (Vopenka).

Another important issue connected with a concept of the set is vagueness. Vague notions are notoriously used, e.g., in medicine, law, economy, and politics – and are intrinsically adhered to method of thinking and debates in those domains.

Vague concepts are characterized by a "boundary region", which consists of all elements which cannot be classified to the concept or its complement. For example, the concept of an odd (even) number is precise, because every number is either odd or even – whereas the concept of a beautiful women is vague, for some women we cannot decide, with certainty, whether they are beautiful or not. This approach is know in a philosophical literature as boundary-line approach to vagueness and is attributed to Gotlob Frege, who first formulate this idea in 1894.

Vagueness for many years attracted attention of philosophers and logicians. Recently, computer scientist also got interested in vagueness, for many computer applications, in particular referring to artificial intelligence, badly need use of vague notions and vague concepts based reasoning methods.

The most successful theoretical approach to vagueness is no doubt fuzzy set theory proposed by Zadeh. Basic idea of fuzzy set theory hinges on fuzzy membership function, which allows partial membership of elements to a set, i.e., it allows that elements can belong to a set to "a degree".

Rough set theory is another mathematical approach to vagueness. The theory has an overlap with many other theories, in particular with fuzzy sets. Many papers have been published on connections of rough set theory and many others similar approaches. In particular the relation between fuzzy set and rough sets have been pursued by many authors (e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 16, 18, 19, 21, 22, 23, 30].

Philosophy of Rough Sets

Rough set philosophy is based on the assumption that, in contrast to the classical set theory, we have some additional information (knowledge, data) about elements of a set. All elements with the same information are *indiscernible* (*similar*) in view of the available information and form blocks, which can be understood as elementary granules of knowledge. These granules are called *elementary sets* or *concepts*, and can be considered as elementary building blocks (atoms) of our knowledge about reality we are interested in. Elementary concepts can be combined into *compound concepts*, i.e., concepts that are uniquely defined in terms of elementary concepts. Any union of elementary sets is called a crisp set, and any other sets are referred to as *rough* (*vague*, *imprecise*). With every set X we can associate two crisp sets, called the *lower* and the *upper approximation* of X. The lower approximation of X is the union of all elementary set which are included in X, whereas the upper approximation of X is the union of all elementary set which have non-empty intersection with X. In other words the lower approximation of a set is the set of all elements that *surely* belongs to X, whereas the upper approximation of X is the set of all elements that *possibly* belong to X. The difference of the upper and the lower approximation of X is its *boundary region*. Obviously a set is rough if it has non empty boundary region whatsoever, otherwise the set is crisp. Elements of the boundary region cannot be classified, employing the available knowledge, either to the set or its complement. Approximations of sets are basic operation in rough set theory and are used as main tools to deal with vague and uncertain data.

Indiscernibility and Approximations of Sets

The starting point of rough set theory is the indiscernibility relation, generated by information about objects of interest. The indiscernibility relation is intended to express the fact that due to the lack of knowledge we are unable to discern some objects employing the available information. That means that, in general, we are unable to deal with single objects but we have to consider clusters (granules, atoms) of indiscernible objects, as fundamental concepts of knowledge. Below we present the above ideas formally.

Suppose we are given two finite, non-empty sets U and A, where U is the *universe*, and A – a set *attributes*. With every attribute $a \in A$ we associate a set V_a, of its *values*, called the *domain* of a. The pair $S = (U, A)$ will be called an *information system* or a *data table*. Any subset B of A determines a binary relation I_B on U, which will be called an *indiscernibility relation*, and is defined as follows: xI_By if and only if $a(x) = a(y)$ for every $a \in A$, where $a(x)$ denotes the value of attribute a for element x. Obviously I_B is an equivalence relation. The family of all equivalence classes of I_B, i.e., the partition determined by B, will be denoted by U/I_B, or simply U/B; an equivalence class of I_B, i.e., the block of the partition U/B, containing x will be denoted by $B(x)$.

If (x, y) belongs to I_B we will say that x and y are *B-indiscernible*. Equivalence classes of the relation I_B (or blocks of the partition U/B) are referred to as *B-elementary concepts* or *B-granules*. As mentioned previously in the rough set approach the elementary concepts are the basic building blocks (concepts) of our knowledge about reality.

The indiscernibility relation will be used next to define basic concepts of rough set theory. Let us define now the following two operations on sets

$$B_*(X) = \{x \in U : B(x) \subseteq X\},$$
$$B^*(X) = \{x \in U : B(x) \cap X \neq \varnothing\},$$

assigning to every subset X of the universe U two sets $B_*(X)$ and $B^*(X)$ called the B-lower and the *B-upper approximation* of X, respectively. The set

$$BN_B(X) = B^*(X) - B_*(X)$$

will be referred to as the *B-boundary* region of X.

If the boundary region of X is the empty set, i.e., $BN_B(X) = \varnothing$, then the set X is *crisp* (exact) with respect to B; in the opposite case, i.e., if $BN_B(X) \neq \varnothing$, the set X is referred to as *rough* (*inexact*) with respect to B.

Rough sets can be also defined using a *rough membership function*, defined as

$$\mu_X^B(x) = \frac{|X \cap B(x)|}{|B(x)|}, \text{ and } \mu_X^B(x) \in [0,1].$$

Value of the membership function $\mu_X(x)$ is kind of conditional probability, and can be interpreted as a degree of *certainty* to which x belongs to X (or $1 - \mu_X(x)$, as a degree of *uncertainty*).

Dependency of Attributes

Approximations of sets are strictly related with the concept of dependency (total or partial) of attributes.

Intuitively, a set of attributes D (*decision attributes*) *depends totally* on a set of attributes C (*condition attributes*), denoted $C \Rightarrow D$, if all values of attributes from D are uniquely determined by value of attribute form C. In other words, D depends totally on C, if there exists a functional dependency between values of D and C.

We would also need a more general concept of dependency of attributes, called the *partial dependency* of attributes. Partial dependency means that only some values of D are determined by values of C.

Formally dependency can be defined in the following way. Let D and C be subsets of A.

We will say that D depends on C in a degree $k\left(0 \le k \le 1\right)$, denoted $C \Rightarrow_k D$, if

$$k = \gamma\left(C, D\right) = \frac{\left|POS_C\left(D\right)\right|}{\left|U\right|}, \text{ where } POS_C\left(D\right) = \bigcup_{X \in U/D} C_*\left(X\right),$$

called a *positive region* of the partition U/D with respect to C, is the set of all elements of U that can be uniquely classified to blocks of the partition U/D, by means of C.

Obviously

$$\gamma\left(C, D\right) = \sum_{X \in U/D} \frac{\left|C_*\left(X\right)\right|}{\left|U\right|}.$$

If $k = 1$ we say that D *depends totally* on C, and if $k < 1$, we say that D *depends partially* (in a *degree* k) on C.

The coefficient k expresses the ratio of all elements of the universe, which can be properly classified to block of the partition U/D, employing attributes C and will be called the degree of the dependency, which can be also interpreted as a probability that $x \in U$ belongs to one of the decision classed determined by decision attributes.

Reduction of Attributes

We often face a question whether we can remove some data from a data table preserving its basic properties, that is – whether a table contains some superfluous data. This can be formulated as follows:

Let $C, D \subseteq A$, be sets of condition and decision attributes, respectively. We will say that $C' \subseteq C$ is a *D-reduct* (reduct with *respect* to *D*) of *C*, if C' is a minimal subset of *C* such that

$$\gamma\left(C, D\right) = \gamma\left(C', D\right).$$

Thus reduct enables us to make decisions employing minimal number of conditions.

Decision Rules and Certainty Factor

With every dependency $C \Rightarrow_k D$ we can associate a set of decision rules, specifying decisions that should be taken when certain condition are satisfied. In other words every dependency $C \Rightarrow_k D$ determines a set of formulas of the form: „*if ... then*". In order to express certainty of decision specified by a decision rule we will we define a *certainty factor* of the rule.

Let Φ and Ψ be logical formulas representing conditions and decisions, respectively and let $\Phi \rightarrow \Psi$ be a decision rule, where Φ_S denote the meaning of Φ in the system *S*, i.e., the set of all objects satisfying Φ in *S*, defined in a usual way.

With every decision rule $\Phi \rightarrow \Psi$ we associate a number, called the certainty factor of the rule, and defined as

$$\mu_S(\Phi, \Psi) = \frac{|\Phi_S \cap \Psi_S|}{|\Phi_S|}.$$

Of course $0 \leq \mu_S(\Phi, \Psi) \leq 1$; if the rule $\Phi \rightarrow \Psi$ is deterministic then $\mu_S(\Phi, \psi) = 1$, and for non-deterministic rules $\mu_S(\Phi, \Psi) < 1$.

With every decision rule $\Phi \rightarrow \Phi$ one can associate a rough deduction rule called *rough modus ponens*

$$\frac{\pi_S(\Phi);\ \mu_S(\Phi, \Psi)}{\pi_S(\Psi)},$$

where $\pi_S(\Phi) = \dfrac{|\Phi|_S}{|U|}$ and $\pi_S(\Psi) = \pi_S(\sim\Phi \wedge \Psi) + \pi_S(\Phi) \cdot \mu_S(\Phi, \Psi)$.

This rule allows to compute the probability of decisions in terms of probability of conditions and conditional probability of the decision rule.

Conclusion

Rough set theory is a new approach to imperfect knowledge. It overlaps with many other theories and approaches, nevertheless it can be consider in its own rights as a mature independent discipline. Particularly interesting is the relationship between rough sets and fuzzy sets. Both approaches are not competitive but complementary and they address different aspects of imperfect knowledge.

Rough set theory has been successfully applied in many real-life problems, in medicine, pharmacology, engineering, banking, financial and market analysis and others.

For basic ideas of rough set theory the reader is advised to consult the enclosed references. In particular in Polkowski and Skowron [20] one can find list of over one thousand publications on rough sets and their applications and information about rough set software for data analysis.

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