ON SOME ISSUES CONNECTED WITH ROUGHLY CONTINUOUS FUNCTIONS *

Zdzislaw Pawlak
Institute of Computer Science
Warsaw University of Technology
ul. Nowowiejska 15/19, 00 665 Warsaw, Poland
e-mail: zpw@ii.pw.edu.pl

and

Institute of Theoretical and Applied Informatics
Polish Academy of Sciences
ul. Baltycka 5, 44 000 Gliwice, Poland

1 Introduction

Physical phenomena are usually described by real-valued functions, i.e. functions which are defined and valued on continuum of points. However due to limited accuracy of physical measurements, in reality we are faced rather with discrete then continuous variables - representing, intervals related to the accuracy of measurement. Consequently as a mathematical tools for description of physical systems we should use discrete rather than real-valued mappings - which can be understood as mappings obtained as a result of measurements rather then abstract definitions.

In this paper we are going to discuss the concept of rough continuity introduced in Pawlak, 1987 and investigated in Pawlak, 1995 and Obtulowicz, 1995.

In particular we are interested how discretization of the real line effects basic properties of real functions, such as continuity, differentiability, etc. It turns out that some properties of real functions have counterparts in the case of discrete functions, but this is not always the case.

We define in this note rough (approximate) continuity, rough differentiability and rough integral of discrete functions, and give some of its basic properties - analogous to those of real functions.

The presented approch is somehow related to qualitative reasoning methods, developed extensively in AI and other areas.

2 Discretization

Let $[n] = \{0, 1, \ldots, n\}$ be a set of natural numbers. A strictly monotonic function $d : [n] \rightarrow \mathbf{R}$, i.e. such that for all $i, j \in [n]$, $i < j$ implies $d(i) < d(j)$ will be called a *scale*.

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Any scale $d : [n] \to \mathbf{R}$ is really finite increasing sequence of reals $x_0, x_1, \ldots, x_n$, such that $x_i = d(i)$, for very $i \in [n]$ - can be seen as a *discretization* of the closed interval $R_n = \langle d(0), d(n) \rangle = \langle x_0, x_n \rangle$.

Given a scale $d : [n] \to \mathbf{R}$ then one can define two functions

$$d_*(x) = max\{i \in [n] : x_i \leq x\},$$

$$d^*(x) = min\{i \in [n] : x_i \geq x\},$$

for every $x \in R_n$ On the interval $R_n = \langle x_0, x_n \rangle$ we define an equivalence relation I_d, called the *indiscernibility* relation, and defined thus

$$xI_dy \text{ iff } d_*(x) = d_*(y) \text{ and } d^*(x) = d^*(x).$$

The family of all equivalence classes of the relation I_d, or the partition of the interval R_n, is given below

$$\{x_0\}, (x_0, x_1), \{x_1\}, (x_1, x_2), \{x_2\}, \ldots, (x_{n-1}, x_n), \{x_n\}$$

where each equivalence classe $[x]_d$ is an interval such that $[x]_d = (x_i, x_{i+1}$ whenever $x_i < x < x_{i+1}$, and $[x_i]_d = \{x_i\}$ for all $i \in [n]$.

If $x_i < x < x_{i+1}$, then $I_{d^*}(x) = d(d_*(x)) = x_i$ and $I^*_d(x) = d(d^*(x)) = x_{i+1}$, i.e. $I_{d^*}(x)$ and $I^*_d(x)$ are ends of the interval $\langle x_i, x_{i+1} \rangle$; if $x = x_i$, then $I_{d^*}(x) = I^*_d(x) = x_i$.

End of the interval $\langle x_i, x_{i+1} \rangle$ are called the *lower* and the *upper d-approximation* of x respectively.

Employing the idea of discretization we can easily formulate many basic concepts of analysis in the rough approach setting. Some examples are given below.

Let $d : [n] \to \mathbf{R}$ be a scale and let $\{a_n\}$ be an infinite sequence of reals.

A sequence $\{a_n\}$ is *roughly convergent* with respect to d (*d-convergent*), if there exists i such that for every $j > i$, $a_j \in R_n$ and $[a_j]_d = [a_i]_d$; $I_{d*}(a_i)$ and $I^*_d(a_i)$ are referred to as the *rough lower* and the *rough upper limit* (*d-upper, d-lower limit*) of the sequence $\{a_n\}$. Any roughly convergent sequence will be called *rough Cauchy sequence.*

A sequence $\{a_n\}$ is *roughly monotonically increasing* (*decreasing*) with respect to d (*briefly d-increasing* (*d-decreasing*)), if $d_*(a_n) < d_*(a_{n+1})$ and $d^*(a_n) < d^*(a_{n+1})$ ($d_*(a_n) > d_*(a_{n+1})$ and $d^*(a_n) > d^*(a_{n+1})$).

Obviously, $\{a_n\}$ is the Cauchy sequence iff $\{a_n\}$ is roughly monotonically increasing or decreasing.

A sequence $\{a_n\}$ is *roughly periodic* with respect to d (*d-periodic*), if there exists k such that $[a_n]_d = [a_{n+k}]_d$. The number k is the *period* of $\{a_n\}$.

3 Roughly Continuous Functions

Suppose we are given two scales $d : [n] \to \mathbf{R}$ and $e : [m] \to \mathbf{R}$, and let $f : R_n \to R_m$ be a function, where R_n, R_m denote the both side closed intervals $\langle x_0, x_n \rangle$, $\langle y_0, y_m \rangle$ respectively. We define its *lower rough representation* f_* with respect to d and e and its *upper rough representation* f^* with respect to d and e defined on $[n]$ and valued in $[m]$, as

$$f_*(i) = e_*(f(x_i))$$

$$f^*(i) = e^*(f(x_i))$$

for all $i \in [n]$.

Suppose we are given scales $d : [n] \rightarrow \mathbf{R}$ and $e : [m] \rightarrow \mathbf{R}$ and a function $f : R_n \rightarrow R_m$. We say that f is *roughly*, or (d,e)*-continuous* function iff for all $x, y \in R_n$, xI_dy implies $f(x)I_ef(y)$, or equivalently $f([x]_d) \subseteq [f(x)]_e$, for every $x \in R_n$.

Now we give important Darboux property (Obtulowicz, 1995 Courant, 1965) for roughly continuous functions.

Supposes we are given two scales $d : [n] \rightarrow \mathbf{R}$ and $e : [m] \rightarrow \mathbf{R}$ and a function $f : R_n \rightarrow R_m$. We say that f has a *lower Darbuox property* with respect to d and e if for every $i \in [n-1]$ there exists $\alpha \in \{-1, 0, 1\}$ such that $f_*(i+1) = f_*(i) + \alpha$, where f_* is the lower representation of f with respect to d and e. Similarly one can define the upper Darboux property of f, by using the upper representation of f.

We will say that f has a *rough Darboux property* with respect to d and e if it has both the lower and the upper Darboux property with respect to d and e.

Proposition 1. (Obtulowicz, 1995, Pawlak, 1995). Let $d : [n] \rightarrow \mathbf{R}$ and $e : [m] \rightarrow \mathbf{R}$ be scales and let $f : R_n \rightarrow R_m$ be a function such that for all $i \in [n]$ the function is continuous in x_i in classical sense. Then f is roughly (d,e)-continuous iff f has a rough Darboux property with respect to d and e.

The above proposition says that a function f is roughly continuous is equivalent that its lower as well upper representations cannot vary to "fast", i.e. pass from one value to another without passing through all intermediate values.

Proposition 2. (Obtulowicz, 1995). Let $d : [n] \rightarrow \mathbf{R}$ and $e : [m] \rightarrow \mathbf{R}$ be scales and let $f : R_n \rightarrow R_m$ be a function. Then f has a Darboux property with respect to d and e iff for all $i, j \in [n], i \neq j$, and for every q between $f_*(i)$ and $f_*(j)$ there exist $p \in [n]$ between i and j for which $f_*(p) = q$.

This proposition is also valid for the upper rough representation of f.

The above properties are counterparts of the intermediate value property in classical analysis given by the following theorem (Courant and John, 1965, p. 44).

Intermediate Value Theorem. Consider a function $f(x)$ continuous at every point of an interval. Let a and b be any two points of the interval and let η be any number between $f(a)$ and $f(b)$. Then there exists a value v between a and b for which $f(v) = \eta$.

It is easily seen that if a function f is continuous, then f is not necessarily roughly continuous, and conversely, i.e. rough continuity of a function does not imply its continuity. It can be also observed that for every continuous function in an interval (a, b) one can find scales of (a, b) and $(f(a), f(b))$ such that f is roughly continuous with respect to these scales.

The rough continuity of a function is easily appreciated intuitively. Whether a function is roughly continuous or not depends on the scales of the domain and range of the function, i.e. it depends on how exactly we "see" the function through the scale.

The above considerations on roughly continuous real functions can also be applied to functions with discrete domain and range - which will be called *discrete functions.*

A discrete function $f : [n] \rightarrow [m]$ is *roughly continuous* iff for all $i, j \in [n]$, $|i - j| = 1$ implies $|f(i) - f(j)| \leq 1$.

For a discrete function $f : [n] \to [m]$ we adopt the notation:

$$\Delta_f(i) = f(i+1) - f(i) \text{ for all } i \in [n-1].$$

Then we say that $f : [n] \to [m]$ has Darboux property if for every $i \in [n-1]$ we have that $\Delta_f(i) \in \{-1, 0, 1\}$. Thus for $f : [n] \to [m]$ having rough Darboux property and $i \in [n-1]$ the value $\Delta_f(i)$ is that $\alpha \in \{-1, 0, 1\}$ which makes $f(i+1) = f(i) + \alpha$.

Proposition 3. A discrete function $f : [n] \to [m]$ is roughly continuous iff f has Darboux property.

Proposition 4. A discrete function $f : [n] \to [m]$ has a Darboux property iff for all $i, j \in [n], i \neq j$, and for every q between $f(i)$ and $f(j)$ there exist $p \in [n]$ between i and j for which $f(p) = q$.

Thus the intermediate value property is also valid for roughly continuous discrete functions.

4 Rough Derivatives and Rough Integrals

One can also define a very important concept in our approach - the rough derivative of a real and discrete functions. In confirmity with the classic definition of the derivative, the rough derivative concept should reflect the basic properties of the classic definition.

Suppose we are given two scales $d : [n] \to \mathbf{R}$ and $e : [m] \to \mathbf{R}$ and a function $f : R_n \to R_m$. The *rough lower derivative* of f is defined as

$$f'_*(x) = \frac{f_*(d^*(x)) - f_*(d_*(x))}{d^*(x) - d_*(x)} = \frac{f_*(i+1) - f_*(i)}{(i+1) - i}$$

for $i = d_*(x)$.

If f is roughly continuous with respect to d and e, then $f'(x) \in \{-1, 0, 1\}$.

Similar definition can be given for the *rough upper derivative* of f.

$$f^{*\prime}(x) = \frac{f^*(d^*(x)) - f^*(d_*(x))}{d^*(x) - d_*(x)} = \frac{f^*(i+1) - f^*(i)}{(i+1) - i}$$

for $i = d_*(x)$.

The concept of a derivative can be also extended for discrete functions.

Let $f : [n] \to [m]$ be a discrete function. Then one defines in natural way the *rough derivative* f' of f as shown below

$$f'(i) = \frac{f(i+1) - f(i)}{(i+1) - i} = \Delta_f(i).$$

Thus obviously a discrete function f has Darboux property iff $f'(i) \in \{-1, 0, 1\}$ for every $i \in [n-1]$.

Higher order derivatives can be also defined in the same manner.

Some important properties are not valid for discrete functions, even for discrete roughly continuous functions, as shown by the following two propositions

Proposition 5. Assume that a discrete function $f:[n]\rightarrow[m]$ has a maximum (minimum) at $i\in(n)$, where $(n)=\{1,2,\ldots,n-1\}$. Then not necessarily $f'(i)=0$.

The Rolle's theorem does not hold for discrete functions, as shown by the proposition below.

Proposition 6. Let $f:[n]-[m]$ be a discrete, function, such that $f(0)=f(n)=0$. Then not necessarily there exists $i\in(n)$ such that $f'(i)=0$.

Next we define integration of discrete functions.

Let $f:[n]\rightarrow[m]$ be a discrete function. By a *rough integral* of f we mean the function

$$\int_{j=0}^{i} f(j)\Delta(j)=\sum_{j=0}^{i} f(j)\Delta(j),$$

where $\Delta(j)=(j+1)-1=1$.

The following important property holds.

Proposition 8.

$$\int_{j=0}^{i} f'(j)\Delta(j)=f(i)+k,$$

where k is a integer constant.

In other words

$$f(i)=f(0)+\sum_{j=0}^{i-1} f'(j).$$

5 Conclusion

The basic concept of the real function theory is that of continuity. The idea of continuity is intuitively best expressed by the Intermediate Value Theorem. But it turned out that the basic thought of this theorem is preserved if instead of real functions, discrete functions are considered. Thus the concept of continuity may not be necessarily attributed to real functions only, for discrete functions it displays similar nature.

The rough (approximate) continuity defined in this note shares some properties of "classical" continuity, however not all of them are preserved in this case. Thus classical continuity and rough continuity are different conceps, despite of the fact that both formal definitions are similar.

References

[1] **Courant, R and John, F. (1965).** Introduction to Calculus and Analysis, 1, Interscience Publishers, A Division of John Wiley and Sons, Inc. New York, London, Sydney

[2] **Obtulowicz, A. (1995).** Some Remarks on Rough Real Functions. *ICS WUT Reports*, 9/95

[3] **Pawlak, Z. (1987).** Rough Functions, *Bull. PAS, Tech. Ser.* Vol. 35, No.5-6, pp. 249-251.

[4] **Pawlak, Z. (1994).** Rough Sets, Rough Real Functions, *ICS WUT Repart*, 50/94

[5] **Pawlak, Z. (1994).** Rough Real Functions and Rough Controllers. *ICS WUT Raports,* 1/95