An Inquiry into Vaguenes and Uncertainty

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1 Introduction

We are going to present in this paper some remarks on two fundamental concepts of modern philosophy, logic and AI - vagueness and uncertainty.

Vagueness and uncertainty have been studied for many years by philosophers and logicians (cf. e.g. [14, 15, 1, 2, 3]. Recently, computers scientists, in particular researcher interested in AI, contributed essentially to this area of research. The most important contributions seemingly are the fuzzy set theory (cf. [22]) and the theory of evidence (cf. [16]).

This paper concerns with another approach to vagueness and uncertainty offered by the rough set theory (cf. [9, 10]). Although the proposed approach is somehow related to that offered by the fuzzy set theory (cf. [11]) and the evidence theory, (cf. [18]) it can be viewed in its own rights.

The rough set theory bears on the assumption that we have initially some information (knowledge) about elements of the universe we are interested in. Evidently to some elements of the universe the same information can be associated and consequently the elements can be *similar* or *indiscernible* in view of the available information. Similarity is assumed to be a reflexive and symmetric relation, whereas the idiscernibility relation - also transitive. Thus similarity is a tolerance relation and indiscernibility is an equivalence relation.

The concepts of similarity and indiscernibility attracted attention of philosophers and logicians for many years (cf. e.g. [21, 7, 8]), nevertheless these concepts are still far of being understood fully.

2 Vaguenes and the Boundary Region

The idea of vagueness is usually connected with the so called "boundary-line" approach first formulated by Frege (cf. [4]), who writes:

"The concept must have a sharp boundary. To the concept without a sharp boundary there would correspond an area that had not a sharp boundary-line all around ".

Thus according to Frege "the concept without a sharp boundary", i.e. vague concept, must have boundary-line examples which cannot be classified, on the basis of available information, neither to the concept nor to its complement. For example the concept of an *odd (even) number* is precise, because every number is either odd or even - whereas the concept of a *beautiful women* is vague, because for some women we cannot decide whether they are beautiful or not (there are boundary-line cases).

In the rough set approach vagueness is due to the lack of information about some elements of the universe. If with some elements the same information is associated, in view of this information these elements are indiscernible. For example if some patients suffering from a certain disease display the same symptoms, they are indiscernible with respect to the information about them. It turns out that the indiscernibility leads to the boundary-line cases, i.e. some elements cannot be classified to the concept or its complement in view of the information available and thus form the boundary-line cases.

Now let us present these ideas more formally.

Suppose we are given a finite not empty set U called the *universe*, and let I be a binary relation on U. By $I(x)$ we mean the set of all $y \in U$ such that yIx. If I is reflexive and symmetric, i.e.

$$xIx, \text{ for every } x \in U,$$

$$xIy, \text{ implies } yIx \text{ for every } x, y \in U,$$

then I is a tolerance relation. If I is also transitive, i.e. xIy and yIz implies xIz, then I is an equivalence relation. In this case $I(x) = [x]_(I)$, i.e. $I(x)$ is an equivalence class of the relation I containing element x. If I is a tolerance relation and xIy, then x, y are called *similar* with respects to I (*I-similar*), whereas if I is an equivalence relation and xIy, then x, y are referred to as *indiscernible* with respect to I (*I-indiscernible*).

Let us define now two following operations on sets

$$I_*(X) = \{x \in U : I(x) \subseteq X\},$$

$$I^*(X) = \{x \in U : I(x) \cap X \neq \emptyset\},$$

assigning to every subset X of the universe U two sets $I_*(X)$ and $I^*(X)$ called the *I-lower* and the *I-upper approximation* of X respectively. The set

$$BN_I(X) = I^*(X) - I_*(X)$$

will be referred to as the *I-boundary region* of X.

If the boundary region of X is the empty set, i.e. $BN_I(X) = \emptyset$, then the set X will be called *crisp* (*exact*) with respect to I; in the opposite case, i.e. if $BN_I(X) \neq \emptyset$, the set X will be referred to as *rough* (*inexact*) with respect to I.

Thus rough sets seems to be a natural mathematical model of vague concepts.

3 Some Prpoerties of Vagueness

One can easily show the following properties:

1) $I_*(X) \subseteq X \subseteq I^*(X)$,
2) $I_*(\emptyset) = I^*(\emptyset) = \emptyset, I_*(U) = I^*(U) = U$,
3) $I^*(X \cup Y) = I^*(X) \cup I^*(Y)$,
4) $I_*(X \cap Y) = I_*(X) \cap I_*(Y)$,
5) $X \subseteq Y$ implies $I_*(X) \subseteq I_*(Y)$ and $I^*(X) \subseteq I^*(Y)$,
6) $I_*(X \cup Y) \supseteq I_*(X) \cup I_*(Y)$,
7) $I^*(X \cap Y) \subseteq I^*(X) \cap I^*(Y)$,
8) $I_*(-X) = -I^*(X)$,
9) $I^*(-X) = -I_*(X)$,
10) $I_*(I_*(X)) = I^*(I_*(X)) = I_*(X)$,
11) $I^*(I^*(X)) = I_*(I^*(X)) = I^*(X)$.

It is easily seen that the lower and the upper approximation of a set are interior and closure operations in a topology generated by the indiscernibility relation. Thus vagueness is related to some topological properties of inexact concepts. This seem to be quite natural, for topology can be seen as a mathematical model of "closeness".

4 Topological Classification of Vagueness

It turns out that the above considerations give rise to the following four basic classes of rough sets, i.e. four classes of vagueness:

a) $I_*(X) \neq \emptyset$ and $I^*(X) \neq U$, iff X is *roughly I-observable*,
b) $I_*(X) = \emptyset$ and $I^*(X) \neq U$, iff X is *internally I-unobservable*,
c) $I_*(X) \neq \emptyset$ and $I_*(X) = U$, iff X is *externally I-unobservable*,
d) $I_*(X) = \emptyset$ and $I^*(X) = U$, iff X is *totally I-unobservable*.

The intuitive meaning of this classification is the following.

If set X is roughly *I-observable*, this means that we are able to decide for some elements of U whether they belong to X or $-X$.

If X is internally I-unobservable, this means that we are able to decide whether some elements of U belong to $-X$, but we are unable to decide for any element of U, whether it belongs to X or not.

If X is externally I-unobservable, this means that we are able to decide for some elements of U whether they belong to X, but we are unable to decide, for any element of U whether it belongs to $-X$ or not.

If X is totally I-unobservable, we are unable to decide for any element of U whether it belongs to X or $-X$.

That means, that the set X is roughly observable if there are some elements in the universe which can be positively classified, to the set X. This definition also implies that there are some other elements which can be classified without any ambiguity as being outside the set X.

External I-unobservability of a set refers to the situation when positive classification is possible for some elements, but it is impossible to determine that an element does not belong to X.

5 Numarical Characterization of Vagueness

Vagueness can be also characterized numerically by defining the following coefficient

$$\alpha_I(X) = \frac{|I_*(X)|}{|I^*(X)|},$$

where $|X|$ denotes the cardinality of the set X.

Obviously $0 \leq \alpha_I(X) \leq 1$. If $\alpha_I(X) = 1$, the set X is *crisp* with respect to I (the concept X is *precise* with respect to I), and otherwise, if $\alpha_I(X) < 1$, the set X is *rough* with respect to I (the concept X is vague with respect to X).

6 Uncertainty and the Membership Function

A vague concept has a boundary-line cases, i.e. elements of the universe which cannot be with *certainty* classified as elements of the concept. Hence *uncertainty* is related to the question of *membership* of elements to a set. Therefore in order to discuss the problem of uncertainty from the rough set perspective we have to define the membership function related to the rough set concept (rough membership function), and investigate its properties.

The rough membership function can be easily defined employing the relation I in the following way:

$$\mu_X^I(x) = \frac{|X \cap I(x)|}{|I(x)|}$$

Obviously

$$\mu_X^I(x) \in [0, 1].$$

The rough membership function, can be used to define the approximations and the boundary region of a set, as shown below:

$$I_*(X) = \{x \in U : \mu_X^I(x) = 1\},$$

$$I^*(X) = \{x \in U : \mu_X^I(x) > 0\},$$

$$BN_I(X) = \{x \in U : 0 < \mu_X^I(x) < 1\}.$$

Thus there exists a strict connection between vagueness and uncertainty. As we mentioned above vagueness is related to sets (concepts), whereas uncertainty is related to elements of sets, and the rough set approach shows clear connection between the two concepts, namely vagueness is defined in terms of uncertainty.

It can be shown (cf. [19]) that the rough membership function has the following properties:

a) $\mu_X^I(x) = 1$ iff $x \in I_*(X)$,

b) $\mu_X^I(x) = 0$ iff $x \in U - I^*(X)$,
c) $0 < \mu_X^I(x) < 1$ iff $x \in BN_I(X)$,
d) If $I = \{(x,x) : x \in U\}$, then $\mu_X^I(x)$ is the characteristic function of X,
e) If xIy, then $\mu_X^I(x) = \mu_X^I(y)$ provided I is an equivalence relation,
f) $\mu_{U-X}^I(x) = 1 - \mu_X^I(x)$ for any $x \in U$,
g) $\mu_{X \cup Y}(x) \geq \max(\mu_X^I(x), \mu_Y^I(x))$ for any $x \in U$,
h) $\mu_{X \cap Y}^I(x) \leq \min(\mu_X^I(x), \mu_Y^I(x))$ for any $x \in U$,
i) If $\mathbf{X}$ is a family of pair wise disjoint sets of U, then $\mu_{\cup \mathbf{X}}^I(x) = \sum_{X \in \mathbf{X}} \mu_X^I(x)$ for any $x \in U$, provided that I is an equivalence relation.

The above properties show clearly the difference between fuzzy and rough memberships. In particular properties g) and h) show that the rough membership can be regarded as a generalization of of fuzzy membership, for the max and the min operations for union and intersection of sets respectively for fuzzy sets are special cases of that for rough sets.

7 Conclusion

The rough set theory seems to be well suited as a mathematical model of vagueness and uncertainty. Vagueness is a property of sets (concepts) and is strictly related to the existence to the boundary region of a set, whereas uncertainty is a property of elements of sets and is related to the rough membership function. In the rough set approach both concepts are closely related and are due to the indiscernibility caused by insufficient information about the world we are interested in.

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