# On some Issues Connected with Indiscernibility


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## 1. Introduction

Similarity and indiscernibility have attracted the attention of philosophers and logicians for many years (cf. e.g. [11]). In recent years this topic has also become of great importance to AI researchers in the context of imprecise data analysis, computational linguistics, approximate reasoning and others – being closely connected with vagueness and uncertainty.

In this paper we would like to give some comments on this subjects from the rough sets perspective. The rough set theory [4], [6] is a new approach to vagueness and uncertainty. Although the proposed approach is somehow related to that offered by the fuzzy set theory [7] and the evidence theory [9], it can be viewed in its own rights.

The starting point of the rough set theory is the assumption that the universe we are dealing with is not accessible directly but through some information (knowledge) about its elements. Evidently, to some elements of the universe the same information can be associated and consequently the elements can be *indiscernible* in view of the available information. For example patients suffering from a certain disease, displaying the same symptoms are indiscernible in view of these symptoms.

Indiscernibility is usually meant to be an equivalence relation (reflexive, symmetric and transitive). However sometimes the transitivity condition is dropped and indiscernibility is thus understood as a tolerance relation. An interesting study of these problems can be found in two recent papers of Marcus [2], [3].

## 2. Rough sets and indiscernibility

In order to define a set we have to specify its membership function $\mu_X(x)$ saying whether the element $x$ belongs to the set $X$ or not. In the classical set theory the co-domain of the membership function is the set $\{0, 1\}$, which means that if $\mu_X(x) = 1$, then $x$ belongs to $X$ and if $\mu_X(x) = 0$, $x$ does not belong to $X$. For fuzzy sets the codomain of the membership function is the closed interval [0,1], which means that the element $x$ can belong to the set $X$ with a certain degree between zero and one.

Let us give now the definition of the membership function for rough sets (the rough membership function). The definition is based on an indiscernibility relation, hence beforehand we need some auxiliary notions.

Suppose we are given a finite not empty set $U$ called the *universe*, and let $I$ be a binary relation on $U$, called an *indiscernibility* relation. The pair $S = (U, I)$ will be referred to as an *indiscernibility* space.

If $I$ is reflexive and symmetric, i.e. $xIx$, for any $x \in U$ and $xIy$, implies $yIx$ for any $x, y \in U$, then $I$ is a *tolerance relation.* If $I$ is also transitive, i.e. $xIy$ and $yIz$ imply $xIz$, for any $x, y, z \in U$ then $I$ is an *equivalence relation.*

By $I(x)$ we mean the set of all $y \in U$ such that $yIx$. In the case of the equivalence relation we have $I(x) = [x]_I$, i.e. $I(x)$ is an equivalence class of the relation $I$ containing element $x$. If $xIy$, then $x, y$ are called *indiscernible* with respect to $I$ (*I-indiscernible*). In what follows we assume that indiscernibility can be either a tolerance or an equivalence relation.

The rough membership function can be easily defined employing the relation $I$ in the following way:

$$\mu_X^I(x) = \frac{|X \cap I(x)|}{|I(x)|}.$$

Obviously

$$\mu_X^I(x) \in [0, 1] \text{ for any } x \in U.$$

The rough membership function, can be used to define two basic operations on sets in the rough set theory, namely the *I-lower* and the *I-upper approximation* of *sets*, denoted by $I_*(X)$, $I^*(X)$, respectively, and defined as follows:

$$I_*(X) = \{x \in U : \mu(x) = 1\},$$

$$I^*(X) = \{x \in U : \mu(x) > 0\}.$$

Obviously the approximations have the following properties

$$I_*(X) = \{x \in U : I(x) \subseteq X\},$$

$$I^*(X) = \{x \in U : I(x) \cap X \neq \emptyset\}.$$

The difference between the upper and the lower approximation of $X$ will be called the *I-boundary region* of the set $X$ and is defined below

$$BN_I(X) = I^*(X) - I_*(X) = \{x \in U : 0 < \mu_X^I(x) < 1\}.$$

Thus the boundary region is the set of all objects which cannot be properly classified to the set or its complement, due to the indiscernibility of some objects of the universe.

If the boundary region of $X$ is the empty set, i.e. $BN_I(X) = \emptyset$, then the set $X$ will be called *crisp* (*exact*) with respect to $I$; otherwise, i.e. if $BN_I(X) \neq \emptyset$, the set $X$ will be referred to as *rough* (*inexact*) with respect to $I$. Hence the indiscernibility of objects of the universe gives rise to the concept of the rough set, i.e. the set with not clearly defined boundaries. For example the concept of an *odd* (*even*) *number* is precise, because every number is either odd or even – whereas the concept of *beautiful women* is rough, because for some women we cannot decide whether they are beautiful or not (there are boundary-line cases).

## 3. Indiscernibility of higher order

In this section we will extend the idea of indiscernibility to subsets of the universe (cf. [4]).

Suppose we are given an indiscernibility space $S = (U, I)$ and let $X, Y$ be two subsets of the universe $U$, i.e. $X, Y \subseteq U$. We will say that the sets $X, Y$ are *I-indiscernible*, in symbols $X \equiv_I Y$, if $I_*(X) = I_*(Y)$ and $I^*(X) = I^*(Y)$. The relation $\equiv_I$ is reflexive, symmetric and transitive, i.e. $\equiv_I$ is an equivalence relation, for both $I$ being a tolerance and equivalence relation. Hence any indicernibility space $S = (U, I)$ induces uniquely the indiscernibilty space $S' = (P(U), \equiv_I)$, where $P(U)$ denotes the powerset of $U$ and $\equiv_I \subseteq P(X) \times P(X)$ is the indiscernibility relation generated by $I$.

The above introduced definition can be extended inductively in the following way.

Let

$$\begin{array}{ll} \text{i)} & P^O(U) = U, \\ \text{ii)} & P^{n+1}(U) = P(P^n(U)), \end{array}$$

and

$$\begin{array}{ll} \text{iii)} & I^O = I, \\ \text{iv)} & I^{n+1} = \equiv_{I^n} . \end{array}$$

Consequently we can define an indiscernibility space $S^n = (P^n(U), I^n)$ called an *indiscernibility space of order* $n$.

Of course for the indiscernibility space $S^n = (P^n(U), I_n)$ we can define the membership function

$$\mu_X^n(x) = \frac{|X \cap I^n(x)|}{|I^n(x)|},$$

where $X \subseteq P^n(U)$ and $x \in P^n(U)$, as well as the approximations

$$I_*^n(X) = \{x \in P^n(U) : \mu_X^n(x) = 1\},$$

$$I^{n*}(X) = \{x \in P^n(U) : \mu_X^n(x) > 0\}.$$

The indiscernibility relation $I$ can be defined by the equality relation and an appropriate information function $Inf^n$.

**Proposition (Skowron).** *Let us assume*

*i)* $Inf_1(x) = \{(a, a(x)) : a \in A\}$ *for every* $x \in U$ *(the indiscernibility relation* $I$ *is defined by the set of attributes as follows:* $xIy$ *iff* $a(x) = a(y)$ *for any* $a \in A$*, (for notation see [6]),*

*ii)* $Inf^1(x) = \langle \bigcup\{\{Inf^O(x)\} : x \in A(X)\}, \bigcup\{\{Inf^O(x)\} : x \in A(X)\}\rangle$

*iii)* $Inf^{n+1}(x) = \{Inf^n(x) : x \in X\}$ *for any* $n \geq 1$ *and* $X \in P^{n+1}(U)$.

*Then we have*

$$xI^n y \text{ iff } Inf^n(x) = Inf^n(y) \text{ for any } x, y \in P^n(U) \text{ and } n \geq 0.$$

## 4. Conclusion

The above considerations have shown that the indiscernibility concept is a hereditary one, i.e. indiscernibility on elements of a set induces indiscernibility on elements of its power set, and so one. This property seems to be of some philosophical as well as practical significance and can be treated as a kind of theory of types for indiscernibility.

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