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ON A REPRESENTATION OF ROUGH SETS BY MEANS OF INFORMATION SYSTEMS

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A b s t a c t: Rought sets are investigated as a tool for expressing uncertainty of the relation "to be an element of". We give some representation theorems for rough sets expressed in terms of information systems.

K e y w o r d s: approximation space, upper approximation, lower approximation, rough top equality, rough bottom equality, rough equality, upper rough set, lower rough set, information system.

1. ROUGH SETS

Let U be a finite set, R an equivalence on U. Then the ordered pair $A = (U,R)$ is said to be an approximation space. For any $X \subseteq U$, we put $\overline{X}^A = \bigcup\{C \in U/R;\ C \cap X \neq \emptyset\}$, $\underline{X}_A = \bigcup\{C \in U/R;\ C \subseteq X\}$. The set $\overline{X}^A$ is said to be the upper approximation and the set $\underline{X}_A$ is called the lower approximation of the set X in A. For any $X \subseteq U$, $Y \subseteq U$, we put $X \underset{A}{\simeq} Y$ if and only if $\overline{X}^A = \overline{Y}^A$; the sets X, Y are then said to be roughly top equal. Similarly, we put $X \underset{A}{\eqsim} Y$ for $X \subseteq U$, $Y \subseteq U$ if and only if $\underline{X}_A = \underline{Y}_A$; the sets X, Y are then said to be roughly bottom equal. Finally, we put $X \underset{A}{\approx} Y$ for $X \subseteq U$, $Y \subseteq U$ if and only if $\overline{X}^A = \overline{Y}^A$, $\underline{X}_A = \underline{Y}_A$; the sets X, Y are then said to be roughly equal.

For any set X, we denote by $B(X)$ the set of all subsets of X. It is a semilattice $(B(X), \cup)$ with respect to the operation $\cup$ of union and also a semilattice $(B(X), \cap)$ with respect to the operation $\cap$ of intersection. Similarly, $E(X)$ denotes the set of all equivalences on X; it is a complete lattice with respect to the relation of inclusion.

By A3 and A4 of 2.2 in [3], we obtain

1. Lemma. If $A = (U,R)$ is an approximation space, then $\underset{A}{\simeq}$ is a congruence on the semilattice $(B(U), \cup)$ and $\underset{A}{\eqsim}$ is a congruence on the semilattice $(B(U), \cap)$.

No congruence on a semilattice $(B(U), \cup)$ can be expressed in the form $\underset{A}{\simeq}$. This is demonstrated by the following

2. Example. Let $U = \{a,b\}$, let $\equiv$ be a congruence on $(B(U), \cup)$ whose blocks are $\{\emptyset, \{a\}\}, \{\{b\}, U\}$. We have $E(U) = \{id_U, U\times U\}$. If $R = id_U$, then $\overline{\{b\}}^A = \{b\} \neq U = \overline{U}^A$ and

hence $\underset{A}{\sim} \neq \equiv$. If $R = U \times U$, then $\overline{\{a\}}^{A} = U = \overline{U}^{A}$, which implies $\underset{A}{\simeq} \neq \equiv$ as well. $\square$

Elements of $B(U)/\underset{A}{\sim}$ are called upper rough sets, elements of $B(U)/\underset{A}{\simeq}$ lower rough sets, and elements of $B\,U\,/_{A}$ are said to be rough sets.

2. INFORMATION SYSTEMS

Let $S = \langle X, T, V, \varrho \rangle$ be an information system, i.e., X, T, V are finite sets and ϱ is a mapping of $X \times T$ into V . For any $t \in T$, we put $\overset{\vee}{t}^{S} = \{(x,y) \in X \times X \ ; \ \varrho(x,t) = \varrho(y,t)\}$. Clearly, $\overset{\vee}{t}^{S} \in E(X)$. For any $Z \subseteq T$, we put $\widetilde{Z}^{S} = \inf_{E(X)}\{\overset{\vee}{t}^{S} \ ; \ t \in Z\}$. Clearly, $\widetilde{Z}^{S} \in E(X)$ for any $Z \subseteq T$, $\widetilde{Z}^{S} = \bigcap_{t \in Z} \overset{\vee}{t}^{S}$ for $Z \neq \emptyset$ and $\widetilde{\emptyset}^{S} = X \times X$.

Information systems are able to represent congruences on semilattices of the form $(B(T), \cup)$. More exactly

1. Theorem. Let T be a finite nonempty set and $\equiv$ a congruence on the semilattice $(B(T), \cup)$. Then there exists an information system $S = \langle X, T, V, \varrho \rangle$ such that $\equiv \ = \ker \widetilde{\ }^{S}$.

This theorem is proved as 2.4 in [1]. The proof consists in constructing $S = \langle X, T, V, \varrho \rangle$ with the above mentioned property. We repeat the construction of S and sketch the proof that it has the above-mentioned property.

Construction of S . Let $\equiv$ be a congruence on $(B(T), \cup)$. For any $M \in B(T)$, we put $\Gamma M = \equiv M \cup \{\equiv M\}$ where $\equiv M$ is the block of $\equiv$ containing M . Further, we put $X = \bigcup_{M \in B(T)} \Gamma M$. Clearly, $\{\Gamma M; \ M \in B(T)\}$ is a decomposition of X whose blocks have at least two elements.

For any $t \in T$, we define an equivalence $\breve{t}$ on X such that $X/\breve{t} = \{ \bigcup_{t \in M \in B(T)} \Gamma M \} \cup \{ \{x\};\ x \in X - \bigcup_{t \in M \in B(T)} \Gamma M \}$. Furthermore, we set $V = \bigcup_{t \in T} X/\breve{t}$; for any $x \in X$ and $t \in T$, we define $\varrho(x,t)$ to be the block of $\breve{t}$ containing x. Then $S = \langle X,T,V,\varrho \rangle$ is an information system such that $\breve{t}^S = \breve{t}$ for any $t \in T$.

Sketch of proof. The constructed objects have the following properties.

(A) If $M \in B(T)$, $N \in B(T)$, then $\Gamma M = \Gamma N$ implies that $\equiv M = \equiv N$.

(B) For any $M_0 \in B(T)$ we have $X/\widetilde{M}_0^S = \{ \bigcup_{M_0 \subseteq M \in B(T)} \Gamma M \} \cup \{ \{x\};\ x \in X - \bigcup_{M_0 \subseteq M \in B(T)} \Gamma M \}$.

(C) If $M_0 \in B(T)$, $N_0 \in B(T)$, and $M_0 \equiv N_0$, then $\widetilde{M}_0^S = \widetilde{N}_0^S$.

(D) If $M_0 \in B(T)$, $N_0 \in B(T)$, and $M_0^S = N_0^S$, then $M_0 \equiv N_0$.

(E) If $M_0 \in B(T)$, $N_0 \in B(T)$, then $M_0 \equiv N_0$ is equivalent with $\widetilde{M}_0^S = \widetilde{N}_0^S$, i.e., with $(M_0,N_0) \in \ker \sim^S$. □

3. DUAL INFORMATION SYSTEMS

Let $S = \langle X,T,V,\varrho \rangle$ be an information system. We put $D(S) = \langle T,X,V,\sigma \rangle$ where $\sigma(t,x) = \varrho(x,t)$ for any $(t,x) \in T \times X$. Then $D(S)$ is an information system that is said to be dual to S. For any $x \in X$, we put $\underset{\smile}{x}_S = \breve{x}^{D(S)}$ and for any $Z \subseteq X$, we set $\underset{\sim}{Z}_S = \tilde{Z}^{D(S)}$.

Hence, for $Z \subseteq X$, $Y \subseteq X$, we have $\underset{\sim}{Z}_S = \underset{\sim}{Y}_S$ if and only if $\tilde{Z}^{D(S)} = \tilde{Y}^{D(S)}$. Thus, $\ker_{\sim S} = \ker^{\sim D(S)}$.

We now formulate our representation theorem for upper rough sets.

1. Theorem. Let $A = (U,R)$ be an approximation space. Then there exists an information system $S = \langle U,T,V,\varrho \rangle$ such that $\underset{A}{\sim} = \ker_{\sim S}$.

Proof. By 1.1, $\underset{A}{\sim}$ is a congruence on the semilattice $(B(U), \cup)$. By 2.1, there exists an information system $P = \langle X,U,V,\varrho \rangle$ such that $\underset{A}{\sim} = \ker^{\sim P}$. If we put $S = D(P)$, we obtain $P = D(S)$ and hence $\underset{A}{\sim} = \ker^{\sim D(S)} = \ker_{\sim S}$. $\square$

4. EXAMPLE

We describe the construction of an information system representing upper rough sets of a given approximation space.

Let $U = \{a,b,c\}$, $U/R = \{\{a\},\{b,c\}\}$, $A = (U,R)$. Then $\overline{\emptyset}^{A} = \emptyset$, $\overline{\{a\}}^{A} = \{a\}$, $\overline{\{b\}}^{A} = \overline{\{c\}}^{A} = \overline{\{b,c\}}^{A} = \{b,c\}$, $\overline{\{a,b\}}^{A} = \overline{\{a,c\}}^{A} = \overline{U}^{A} = U$. Hence, blocks of $\underset{A}{\sim}$ are: $\{\emptyset\} = 0$, $\{\{a\}\} = 1$, $\{\{b\},\{c\},\{b,c\}\} = 2$, $\{\{a,b\},\{a,c\}, U\} = 3$. Thus, blocks of Γ are: $\{\emptyset, 0\}$, $\{\{a\}, 1\}$, $\{\{b\},\{c\},\{b,c\}, 2\}$, $\{\{a,b\},\{a,c\}, U, 3\}$.

Let $S = \langle U,T,V,\varrho \rangle$ be the required information system. Then nontrivial blocks of $\underset{\smile}{a}_S$, $\underset{\smile}{b}_S$, $\underset{\smile}{c}_S$ are, respectively, P, Q, Q wgere $P = \{\{a\},\{a,b\},\{a,c\}, U, 1, 3\}$, $Q = \{\{b\},\{c\},\{b,c\},\{a,b\},\{a,c\}, U, 2, 3\}$.
Thus $T = \{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}, U,0,1,2,3\}$. The function ϱ is given by the following table:

	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	U	0	1	2	3
a	$\{\emptyset\}$	P	$\{\{b\}\}$	$\{\{c\}\}$	P	P	$\{\{b,c\}\}$	P	$\{0\}$	P	2	P
b	$\{\emptyset\}$	$\{\{a\}\}$	Q	Q	Q	Q	Q	Q	$\{0\}$	1	Q	Q
c	$\{\emptyset\}$	$\{\{a\}\}$	Q	Q	Q	Q	Q	Q	$\{0\}$	1	Q	Q

This implies that $\underset{\sim}{\emptyset}_S = T \times T$, $\underset{\sim}{\{a\}}_S = \underset{\smile}{a}_S$, $\underset{\sim}{\{b\}}_S = \underset{\smile}{b}_S$, $\underset{\sim}{\{c\}}_S = \underset{\smile}{c}_S$.

Nontrivial blocks of $\underset{\sim}{\{a,b\}}_S$, $\underset{\sim}{\{a,c\}}_S$, $\underset{\sim}{\{b,c\}}_S$, $\underset{\sim}{U}_S$ are respectively, $P \cap Q$, $P \cap Q$, Q, $P \cap Q$. Thus, blocks of $\ker_{\sim S}$ are: $\{\emptyset\}$, $\{\{a\}\}$, $\{\{b\}, \{c\}, \{b,c\}\}$, $\{\{a,b\}, \{a,c\}, U\}$.
We see that $\underset{X}{\cong} = \ker_{\sim S}$.

5. REPRESENTATION THEOREMS FOR LOWER ROUGH SETS AND ROUGH SETS

Let U be a finite set. For any $X \subseteq U$, we put $\mathrm{Co}\, X = U - X$.
Since $\mathrm{Co}\, X \cup \mathrm{Co}\, Y = \mathrm{Co}(X \cap Y)$ and $\mathrm{Co}\, X \cap \mathrm{Co}\, Y = \mathrm{Co}(X \cup Y)$, we see that Co is an isomorphism of the semilattice $(B(U), \cap)$ onto $(B(U), \cup)$ and also an isomorphism of the semilattice $(B(U), \cup)$ onto $(B(U), \cap)$. This implies that for any congruence $\equiv$ on $(B(U), \cup)$ and/or $(B(U), \cap)$ respectively, the relation $\Theta = \{(X,Y) \in B(U) \times B(U);\ (\mathrm{Co}\, X, \mathrm{Co}\, Y) \in \equiv\}$ is a congruence on $(B(U), \cap)$, $(B(U), \cup)$, respectively.

Thus, 2.1 entails

1. Theorem. Let T be a finite nonempty set and $\equiv$ a congruence on the semilattice $(B(T), \cap)$. Then there exists an information system $S = \langle X, T, V, \varrho \rangle$ such that $\equiv\, = \ker(\sim^S \circ \mathrm{Co})$.

Indeed, if $X \in B(T)$, $Y \in B(T)$, then $(X,Y) \in \equiv$ is equivalent with $(\mathrm{Co}\, X, \mathrm{Co}\, Y) \in \Theta$ where $\Theta = \{(X,Y) \in B(T) \times B(T);$ $(\mathrm{Co}\, X, \mathrm{Co}\, Y) \in \equiv\}$.
We have seen that Θ is a congruence on $(B(T), \cup)$. By 2.1, there exists an information system $S = \langle X, T, V, \varrho \rangle$ such that $\ker^{\sim S} = \Theta$. Clearly, $(X,Y) \in \ker(\sim^S \circ \mathrm{Co})$ means that $(\mathrm{Co}\, X, \mathrm{Co}\, Y) \in \ker^{\sim S} = \Theta$, which is equivalent to $(X,Y) \in \equiv$.
This implies the assertion.

Particularly, if $\equiv = \underset{A}{\approx}$ for an approximation space $A = (U,R)$, then

$\Theta = \{(X,Y) \in B(U) \times B(U)\, ;\ (\text{Co } X, \text{Co } Y) \in \underset{A}{\approx}\} = \{(X,Y) \in B(U) \times B(U)\, ;$

$\underline{\text{Co } X}_A = \underline{\text{Co } Y}_A\} = \{(X,Y) \in B(U) \times B(U)\, ;\ \overline{X}^A = \overline{Y}^A\} = \underset{A}{\simeq}$ by A9 of 1.3 in [3].

Thus, 3.1 implies

2. Theorem. Let $A = (U,R)$ be an approximation space. Then there exists an information system $S = \langle U,T,V,\mathcal{S}\rangle$ such that $\underset{A}{\approx} = \ker(\sim_S \circ \text{Co})$.

Indeed, if $X \in B(U)$, $Y \in B(U)$, then $(X,Y) \in \underset{A}{\approx}$ is equivalent to $(\text{Co},X,\text{Co } X) \in \underset{A}{\simeq}$ and, thus, $(X,Y) \in \underset{A}{\approx}$ is equivalent to $\underset{\sim}{\text{Co}}\, X_S = \underset{\sim}{\text{Co}}\, Y_S$ by 3.1 and hence $\underset{A}{\approx} = \ker(\sim_S \circ \text{Co})$.

Combining 2 with 3.1, we obtain

3. Theorem. Let $A = (U,R)$ be an approximation space. Then there exist two information systems $S_1 = \langle U,T_1,V_1,\mathcal{S}_1\rangle$ $S_2 = \langle U,T_2,V_2,\mathcal{S}_2\rangle$ such that $\underset{A}{\approx} = \ker_{\sim S_1} \cap \ker(\sim_{S_2} \circ \text{Co})$.

This is a consequence of the fact that $\underset{A}{\approx} = \underset{A}{\approx} \cap \underset{A}{\simeq}$.

6. CONCLUDING REMARKS

(A) By 1.1, the relation $\underset{A}{\simeq}$ is a congruence on the semi-lattice $(B(U), \cup)$ for any approximation space $A = (U,R)$. By 1.2, no congruence on $(B(U), \cup)$ can be expressed in the form $\underset{A}{\simeq}$ for a suitable approximation space $A = (U,R)$. Thus, we have the following

1. Problem. Characterize all upper rough equalities among

all congruences on the semilattice $(B(U), \cap)$.

Similarly

2. Problem. Characterize all lower rough equalities among all congruences on the semilattice $(B(U), \cap)$.

(B) There are two kinds of relationship between approximation spaces and information systems. By 3.1, to any approximation space $A = (U,R)$ there exists an information system $S = \langle U,T,V,\varrho \rangle$ such that $\underset{A}{\simeq} = \ker_{\sim S}$. On the other hand, for any information system $S = \langle U,T,V,\varrho \rangle$, $\tilde{T}^S$ is an equivalence on U and therefore $(U, \tilde{T}^S)$ is an approximation space.

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