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Expressive power of knowledge representation systems

EWA ORŁOWSKA AND ZDZISŁAW PAWLAK

Institute of Computer Science, Polish Academy of Sciences, P.O. Box 22, 00-901 Warsaw, PKiN, Poland

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In this article we attempt to clarify some aspects of expressive power of knowledge representation systems. We show that information about objects provided by a system is given up to an indiscernibility relation determined by the system and hence it is incomplete in a sense. We discuss the influence of this kind of incompleteness on definability of concepts in terms of knowledge given by a system. We consider indiscernibility relations as a tool for representing expressive power of systems, and develop a logic in which properties of knowledge representation systems related to definability can be expressed and proved. We present a complete set of axioms and inference rules for the logic.

1. Introduction

Methods of knowledge representation are related to many artificial intelligence tasks such as natural language processing, theorem proving, development of information retrieval systems, problem-solving systems and expert consulting systems. Techniques for modelling and representation of knowledge have come from several of the application areas and there are many ways of dealing with these problems (Bobrow, 1977; Bobrow & Winograd, 1977; Brachman & Smith, 1980; McDermott, 1978; Goguen, 1969; Hájek & Havránek, 1978; Newell, 1982; Nilsson, 1982; Tou, 1980; Vopenka, 1979; Winograd, 1975; Zadeh, 1965).

Our object in this article is to draw attention to some aspects of knowledge representation which are common to many methods and to discuss expressive power of knowledge representation systems from the point of view of these common features.

Any knowledge representation system provides information about some parts of the perceivable reality. We assume that is the process of perception we distinguish entities (objects) and their properties. Properties of objects are perceived through assignment of some characteristics (attributes) and their values to the objects. In this way we establish a universe of discourse (a problem domain) consisting of objects and elementary information items providing a characterization of these objects in terms of attributes and attribute values. In general, information about objects obtained in this way is not sufficient to characterize objects uniquely; that is, we are not able to distinguish all the objects by means of the admitted attributes and their values. This means that objects are recognized up to an indiscernibility relation determined by elementary information items. Any two objects are indiscernible whenever they assume the same values for all the attributes under consideration.

Next, we form concepts; that is, we agregate some objects into sets. Information about a concept is composed from information about objects which are instances of

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the concept. Since objects are not necessarily distinguishable, information characterizing a concept may be ambiguous to some extent. For example, let a universe of discourse consist of some dogs, cats, horses and bears, and let these aniamals be characterized by means of colour. That is, our primary knowledge is given, for example, by the following table:

	Animal	Colour
A1	Bear	Black
A2	Bear	Black
A3	Dog	Brown
A4	Cat	Black
A5	Horse	Black
A6	Horse	Black
A7	Horse	Brown

Let animals A1, A3, A6 and A7 be recognized to be instances of concept "dangerous animal". Information represented by the given table does not enable us to characterize this concept precisely. We cannot say that an animal from universe of discourse {A1, . . . , A7) is dangerous iff it is a black bear, a brown dog, a black horse or a brown horse, because A2 and A5 are also instances of the concept defined by this information. This means that knowledge represented by the table is incomplete in a sense.

In this article we offer an approach to the interpretation of the expressive power of knowledge representation systems based on the fact that in general a system determines an indiscernibility relation it its universe of discourse. As a result, implicit information which can be inferred from explicit information provided by the system is not necessarily precise, it is determined with some tolerance. The limits of this tolerance are chosen to be a measure of the expressive power of systems.

In the first part of the article we present formal counterparts of a notion of knowledge representation system and we discuss the problem of definability of concepts in the system. In the second part, a language and a logic are introduced enabling us to express and to prove facts concerning expressive power of systems.

The article is a revised version of an earlier report Orłowska & Pawlak (1981), and it includes some definitions and facts presented in Konrad, Orłowska & Pawlak (1981), Orłowska (1982) and Pawlak (1981, 1982, 1983).

2. Knowledge representation system

The basic component of a system is a non-empty set OB of objects, e.g. books or human beings. Knowledge about objects is expressed through assignment of some characteristic features to the objects, e.g. human beings can be characterized by sex and age, books by title and author's name, etc. These features are represented by attributes and values of attributes. Thus a non-empty set AT of attributes, and for each $a \in \mathrm{AT}$ a set VAL_a of values of attribute a are the components of the system. Moreover, we assume that a function f assigning attribute values to objects is given. The formal definition of knowledge representation system is as follows. A knowledge representation system is a quadruple

$$S=(OB, AT, \{VAL_a\}_{a\in AT}, f),$$

where OB is a non-empty set, whose elements are called objects, AT is a non-empty set, whose elements are called attributes, VAL_a is a non-empty set, whose elements are called values of attribute a and f is a total function (information function) from set OB × AT into set $VAL=\bigcup_{a\in AT} VAL_a$ such that $f(o, a)\in VAL_a$ for each $o\in OB$ and each $a\in AT$.

Consider the following system.

EXAMPLE 2.1

$$OB=\{o1, o2, o3, o4, o5, o6\},$$

$$AT=\{\text{sex}, \text{age}\},$$

$$VAL_{\text{sex}}=\{\text{male}, \text{female}\},$$

$$VAL_{\text{age}}=\{\text{young}, \text{medium}, \text{old}\}.$$

Function f is defined by the table:

	Sex	Age
$o1$	Male	Young
$o2$	Male	Medium
$o3$	Female	Old
$o4$	Male	Medium
$o5$	Female	Old
$o6$	Female	Young

3. Definable sets of objects

Usually it is the case that information about objects provided by attributes and attribute values is not sufficient for distinguishing all the objects of a system. In general, we cannot tell one object from all others by means of values of an information function. In example 2.1, objects $o1$, $o2$ and $o4$ cannot be distinguished by attribute sex, objects $o3$ and $o5$ cannot be distinguished by attributes sex and age. To deal with such cases we introduce family $\{\tilde{A}\}_{A\subseteq AT}$ of equivalence relations on set OB defined as follows:

$$(o1, o_2)\in\tilde{A} \quad \text{iff } f(o1, a)=f(o2, a) \quad \text{for all } a\in A$$

$$\text{if A is the empty set then } \tilde{\varnothing}=OB\times OB$$

Relation $\widetilde{AT}$ determined by all the attributes of system S is called indiscernibility determined by S, and it will be denoted by ind (S).

EXAMPLE 3.1

Consider system S given in example 2.1. We have

$$(o1, o2) \in \widetilde{\text{sex}},$$

$$(o1, o6) \in \widetilde{\text{age}},$$

$$(o2, o4), (o3, o5) \in \text{ind}\,(S).$$

Observe that for any $A \subseteq AT$ we have $\tilde{A} = \bigcap_{a \in A} \tilde{a}$. Equivalence classes of relation ind (S) are called elementary sets in S. Relation ind(S) determines a classification of objects of system S according to information about these objects provided by the system. Objects belonging to the elementary sets are undistinguishable with respect to this information.

EXAMPLE 3.2

The elementary sets in the system from example 2.1 are

$$\{o1\}, \{o2, o4\}, \{o3, o5\}, \{o6\}.$$

This means that knowledge provided by the given system enables us to distinguish the following sets of individuals:

young males,
males of medium age,
old females,
young females,
all sets which can be obtained from the
sets given above by using set theoretical operations.

We say that set $X \subseteq OB$ is definable in system S (S-definable) iff X is the empty set or the union of some of the elementary sets of relation ind (S).

EXAMPLE 3.3

Set $X = \{o1, o2, o4, o6\}$ is definable in system S from example 2.1. Sets $Y = \{o1, o3\}$ and $Z = \{o2, o5\}$ are not definable in this system. It follows that set X consists of young males, males of medium age and young females. For sets Y and Z we are able to given a characterization of this kind. We cannot say that Y coincides with a set of young males and old females, since $o5$ does not belong to Y. Similarly, it is not true that set Z coincides with a set of males of medium age and old females, since $o4$ and $o3$ do not belong to Z.

Observe that in any system set OB of this system and the empty set are definable. Moreover, the family of definable sets is closed under union, intersection, and complement. Hence we have the following theorems:

FACT 3.1

(a) Family DEF (S) of all the sets definable in system S is a Boolean algebra.
(b) Elementary sets are atoms in algebra DEF (S).

A system S is said to be selective iff each elementary set consists of exactly one object.

FACT 3.2

The following conditions are equivalent:

(a) system S is selective and
(b) any set $X \subseteq OB$ is definable in S.

Given a definable set $X \subseteq OB$, knowledge provided by the system enables us to decide for any object $o \in OB$ whether it belongs to X or not. However, if set X is not definable in system S, and as a consequence it is not uniquely characterized by knowledge of the system, we are not able to answer a membership question precisely. For example, on the base of knowledge provided by the system from example 2.1 we cannot establish whether $o5$ belongs to set Y from example 3.3, because $o3$ and $o5$ are indistinguishable in this system. To deal with such cases we introduce notions of approximations of sets of objects.

4. Approximations of sets of objects

Let a system $S=(OB, AT, VAL, f)$ be given, we define a pair of operations in set OB of objects, namely the operation of lower approximation and upper approximation of a set. These operations enable us to assign a pair of definable sets to any subset X of set OB. For a set X which is definable in the system its approximations coincide with X, and for a nondefinable set X its approximations are, roughly speaking, close enough to X. They determine limits of tolerance for deciding whether objects belong to X or not.

An upper approximation $\bar{S}X$ of set X in system S is the least set which is definable in S and includes set X.

A lower approximation ($\underline{S}X$ of set X in system S is the greatest set which is definable in S and is included in X.

The following facts follow immediately from the given definitions.

FACT 4.1

(a) $\bar{S}X=\{o \in OB$: there is an $o' \in OB$ such that $(o, o') \in ind(S)$ and $o' \in X\}$,
(b) $\underline{S}X=\{o \in OB$: for all $o' \in OB$ if $(o, o') \in ind(S)$ then $o' \in X\}$.

FACT 4.2

The following conditions are equivalent:

(a) a set $X \subseteq OB$ is definable in system S and
(b) $\underline{S}X=X=\bar{S}X$.

EXAMPLE 3.1

Let us consider the system from example 2.1 and the following sets of objects

$$X=\{o1, o2\}, \qquad Y=\{o1, o3, o6\}, \qquad Z=\{o2, o4, o6\}.$$

We have

$$\bar{S}X=\{o1, o3, o4\}, \qquad \underline{S}X=\{o1\},$$

$$\bar{S}Y=\{o1, o3, o5, o6\}, \qquad \underline{S}Y=\{o2, o6\},$$

$$\bar{S}Z=\underline{S}Z=\{o2, o4, o6\}=Z.$$

In the following we list some properties of the operations of lower and upper approximation.

FACT 4.3

(a) $\underline{S}(X \cap Y) = \underline{S}X \cap \underline{S}Y$,
(b) $\underline{S}X \subseteq X$,
(c) $\underline{S}\underline{S}X = \underline{S}X$,
(d) $\underline{S}OB = OB$.

FACT 4.4

(a) $\bar{S}(X \cup Y), = \bar{S}X \cup \bar{S}Y$,
(b) $X \subseteq \bar{S}X$,
(c) $\bar{S}\bar{S}X = \bar{S}X$,
(d) $\bar{S}\varnothing = \varnothing$.

It follows that algebra P(OB) of all the subsets of set OB with additional operations $\bar{S}$ and $\underline{S}$ is a topological field of sets, where $\bar{S}$ is a closure operation and $\underline{S}$ is an interior operation.

FACT 4.5

(a) $\bar{S}X = -\underline{S}(-X)$,
(b) $\underline{S}X = -\bar{S}(-X)$,
(c) if $X \subseteq Y$ then $\bar{S}X \subseteq \bar{S}Y$ and $\underline{S}X \subseteq \underline{S}Y$.

Thus operations $\underline{S}$ and $\bar{S}$ are dual and monotonic with respect to inclusion.

5. Rough definability

Given a system $S = (OB, AT, VAL, f)$ and a set $X \subseteq OB$, for any object $o \in OB$ we say that:

o is an S-positive instance of X iff $o \in \underline{S}X$,
o is an S-negative instance of X iff $o \in OB - \bar{S}X$ and
o is an S-borderline instance of X iff $o \in \bar{S}X - \underline{S}X$.

It follows that if o is a positive instance of X then knowledge provided by system S enables us to state that o definitely belongs to X. For negative instances of X we know that they definitely do not belong to X. Borderline instances of X represent a doubtful region, they possibly belong to X but we cannot decide if for certain in virtue of knowledge given in the system.

We say that (Pawlak, 1984):

a set X is roughly definable in a system S iff $\underline{S}X \neq \varnothing$ and $\bar{S}X \neq OB$.

Thus, for roughly definable sets a membership question can be decided approximately. However, if lower approximation $\underline{S}X$ is empty then there are no S-positive instances of X and hence none of the objects can be recognized to be surely an element of X. Similarly, if upper approximation $\bar{S}X$ equals set OB then there are no negative instances of X and hence none of the objects can be definitely excluded from X.

We say that:

a set X is internally nondefinable in a system S iff $\underline{S}X=\varnothing$,
a set X is externally nondefinable in a system S iff $\bar{S}X=OB$ and
a set X is totally nondefinable in a system S iff X is internally nondefinable and externally nondefinable in S

FACT 5.1

(a) A set X is internally nondefinable in a system S iff none of the objects is an S-positive instance of X,

(b) A set X is externally nondefinable in S iff none of the objects is an S-negative instance of X.

EXAMPLE 5.1

Consider system S from example 2.1 and the following sets:

$$X=\{o2, o3, o4\}, \qquad Y=\{o2, o3\}, \qquad Z=\{o1, o2, o3, o6\}.$$

We have

$$\underline{S}X=\{o2, o4\}, \qquad \bar{S}X=\{o2, o3, o4, o5\},$$

$$\underline{S}Y=\varnothing, \qquad \bar{S}Y=\{o2, o3, o4, o5\},$$

$$\underline{S}Z=\{o1, o6\}, \qquad \bar{S}Z=OB.$$

It follows that objects $o2$ and $o4$ are the positive instances of X, objects $o1$ and $o6$ are the negative instances of X and objects $o3$ and $o5$ are borderline instances of X. Set Y is internally nondefinable and set Z is externally nondefinable in system S.

Observe, that if an indiscernibility ind (S) generates a one-element elementary set then there are no totally nondefinable objects in system S.

6. Comparing knowledge representation systems

It can be seen from the previous considerations that expressive power of knowledge representation systems is closely related to their ability for defining sets of objects. In this section we consider a family $\mathcal{S}=\{S_i\}_{i\in I}$ of knowledge representation systems of the form

$$S_i=(OB, AT_i, VAL_i, f_i),$$

where set OB is the same for all the systems and I is a nonempty set of indices.

We say that

a system $S_1\in\mathcal{S}$ is more expressive than a system $S_2\in\mathcal{S}(S_1\leq S_2)$iff ind $(S_1)\subseteq$ ind (S_2).

This means that if $S_1\leq S_2$ then the indiscernibility relation of system S_1 provides a finer partition of set OB into elementary sets than the indiscernibility relation of system S_1. It follows that approximations of sets of objects in system S_1 are closer to these sets than their approximations in system S_2, namely the following theorems hold.

FACT 6.1

The following conditions are equivalent:

(a) $S_1 \leq S_2$ and
(b) $\bar{S}_1 X \subseteq \bar{S}_2 X$ for any $X \subseteq OB$.

Proof

Let $[o]_i$, for $i = 1, 2$, denote the equivalence class with respect to relation ind (S_i) determined by object $o \in OB$. If ind $(S_1) \subseteq$ ind(S_2) then for any $o \in OB$ we have $[o]_1 \subseteq [o]_2$, and hence condition (b) holds. Let us now suppose that for any set $X \subseteq OB$ we have $\bar{S}_1 X \subseteq \bar{S}_2 X$ and not $S_1 \leq S_2$. Hence there is a pair (o, o') of objects such that $(o, o') \in$ ind (S_1) and $(o, o') \notin$ ind (S_2). Consider set $\{o\}$. We have $o' \in \bar{S}_1\{o\}$ and $o' \notin S_2\{o\}$, which contradicts condition (b).

FACT 6.2

The following conditions are equivalent:

(a) $S_1 \leq S_2$ and
(b) $\underline{S}_2 X \subseteq \underline{S}_1 X$ for any $X \subseteq OB$.

A proof follows from 4.5 and 6.1.

In the following we list some properties of relation $\leq$.

FACT 6.3

(a) If $AT_1 \subseteq AT_2$ then $S_2 \leq S_1$.
(b) If $S_1 \leq S_2$ then algebra DEF (S_2) is a subalgebra of algebra DEF (S_1).
(c) Relation $\leq$ is a partial order in any family $\mathcal{S}$ of systems.
(d) Selective systems are minimal elements in any family $\mathcal{S}$ ordered by relation $\leq$.

EXAMPLE 6.1

Consider systems S_1 and S_2 such that

$$OB_1 = OB_2 = \{o1, o2, o3, o4, o5\},$$

$$AT_1 = \{a, b\}, \qquad AT_2 = \{a, c, d\},$$

$$VAL_a = \{p1, p2\}, \qquad VAL_b = \{q1, q2\},$$

$$VAL_c = \{r1, r2\}, \qquad VAL_d = \{s1, s2\}.$$

		a	b
f_1	$o1$	$p1$	$q1$
	$o2$	$p2$	$q2$
	$o3$	$p1$	$q1$
	$o4$	$p2$	$q1$
	$o5$	$p2$	$q1$

		a	c	d
f_2	$o1$	$p1$	$r1$	$s1$
	$o2$	$p2$	$r2$	$s2$
	$o3$	$p1$	$r2$	$s1$
	$o4$	$p2$	$r2$	$s1$
	$o5$	$p2$	$r2$	$s1$

The indiscernibility relations of these systems generate the following elementary sets:

$$\text{ind}\,(S_1): \{o1, o3\}, \{o2\}, \{o4, o5\};$$
$$\text{ind}\,(S_2): \{o1\}, \{o2\}, \{o3\}, \{o4, o5\}.$$

We clearly have $S_2 \leq S_1$. Consider set $X = \{o1, o4\}$ and its approximations in the given systems:

$$\bar{S}_1 X = \{o1, o3, o4, o5\}, \qquad \underline{S}_1 X = \emptyset,$$
$$\bar{S}_2 X = \{o1, o4, o5\}, \qquad \underline{S}_2 X = \{o1\}.$$

7. A formalized language

The logic considered in the following sections is intended to provide a formal method for comparing an expressive power of knowledge representation systems. The expressive power of a system is represented by the indiscernibility relation of the system. A system S_1 is considered to be more expressive than a system S_2 iff indiscernibility relation ind (S_1) is included in indiscernibility relation ind (S_2). We define a formalized language which enables us to express facts concerning sets of objects in knowledge representation systems. We also give a deductive structure to the language and hence we are able to recognize valid facts or to infer facts from given facts. In particular we can axiomatize a class of selective systems.

Expressions of the logic are intended to represent sets of objects. They are built up from atomic expressions, i.e. variables by means of operations corresponding to set-theoretical operations and operations of upper and lower approximation. To define formulae of the logic we use symbols from the following non-empty at most denumerable and pairwise disjoint sets:

set VAROB of variables representing sets of objects,
set CONREL of constants representing indiscernibility relations,
set $\{\neg, \vee, \wedge, \rightarrow, \leftrightarrow\}$ of propositional operations of negation, disjunction, conjunction, implication and equivalence,
set $\{\_\_, \overline{}\}$ of equations of lower approximation and upper approximation and
set FOR of all formulae of the logic is the least set satisfying the conditions:

VAROB $\subseteq$ FOR,
if A, B $\in$ FOR then $\neg$A, A $\vee$ B, A $\wedge$ B, A $\rightarrow$ B, A $\leftrightarrow$ B $\in$ FOR,
if R $\in$ CONREL and A $\in$ FOR then $\underline{R}$A, $\bar{R}$A $\in$ FOR.

Formulae of the form A, A $\vee$ B, and A $\wedge$ B are intended to represent complement, union and intersection of sets of objects represented by A and B, repsectively. Expression A $\rightarrow$ B represents the union of complement of a set corresponding to A and set corresponding to B. Expression A $\leftrightarrow$ B represents the intersection of sets of objects determined by A $\rightarrow$ B and B $\rightarrow$ A. Finally, expressions $\underline{R}$A and $\bar{R}$A represent the lower and upper approximation, respectively, of a set corresponding to A with respect to an indiscernibility relation R.

8. Semantics of the language

We define a meaning of formulae of the given language by means of notions of model and satisfiability of formulae in a model. By a model we mean a triple

$$M = (OB, m, v),$$

where OB is a non-empty set of objects, m is a meaning function which assigns equivalence relations on set OB to constants from set CONREL and v is a function from set VAROB into set P(OB) of all the subsets of set OB.

By induction with respect to a structure of a formula we define the notion of satisfiability of formulae in a model. We say that a formula A is satisfied in a model M by an object $o \in OB$ (M, o sat A) iff the following conditions are satisfied:

M, o sat p iff $o \in v(p)$ for $p \in VAROB$,
M, o set $\neg$ A iff not M, o sat A,
M, o sat $A \vee B$ iff M, o sat A or M, o sat B,
M, o sat $A \wedge B$ iff M, o sat A and M, o sat B,
M, o sat $A \rightarrow B$ iff not M, o sat A or M, o sat B,
M, o sat $A \leftrightarrow B$ iff M, o sat $A \rightarrow B$ and M, o sat $B \rightarrow A$,
M, o sat $\underline{R}A$ iff for all $o' \in OB$ if $(o, o') \in m(R)$ then, M, o sat A and
M, o sat $\overline{R}A$ iff there is an $o' \in OB$ such that $(o, o') \in m(R)$ and M, o' sat A.

We say that a set T of formulae is satisfied in a model M by an object o (M, o sat T) iff for each formula $A \in T$ we have M, o sat A. A set T is satisfiable iff there is a model M and an object o such that M, o sat T.

According to the given semantics to each formula A of the language there is associated the set of those objects which satisfy the formula in a model; we call this set extension of formula in model

$$\text{ext}_M A = \{o \in OB: M, o \text{ sat } A\}.$$

Extensions of compound formulae depend on the extensions of their components in the following way

FACT 8.1

(a) $\text{ext}_M\, p = v(p)$ for $p \in VAROB$,
(b) $\text{ext}_M \neg A = -\text{ext}_M A$,
(c) $\text{ext}_M A \vee B = \text{ext}_M A \cup \text{ext}_M B$,
(d) $\text{ext}_M A \wedge B = \text{ext}_M A \cap \text{ext}_M B$,
(e) $\text{ext}_M A \rightarrow B = -\text{ext}_M A \cup \text{ext}_M B$,
(f) $\text{ext}_M A \leftrightarrow B = \text{ext}_M (A \rightarrow B) \cap \text{ext}_M (B \rightarrow A)$,
(g) $\text{ext}_M \underline{R}A = \underline{m(R)}\, \text{ext}_M A$ and
(h) $\text{ext}_M \overline{R}A = \overline{m(R)}\, \text{ext}_M (A)$.

We say that a formula A is true in a model M ($\models_M A$) iff $\text{ext}_M A = OB$. A formula A is valid ($\models A$) iff it is true in every model. A formula A is a semantical consequence of a set T of formulas ($T \models A$) iff M, o sat A whenever M, o sat T for every model M and for every object o from the set of objects of M.

9. Properties of systems expressible in the language

In this section we show how formulae of the given language can be used to express properties of sets of objects and properties of knowledge representation systems.

FACT 9.1

(a) $\models_M A \to B$ iff $\text{ext}_M A \subset \text{ext}_M B$,
(b) $\models_M A \leftrightarrow B$ iff $\text{ext}_M A = \text{ext}_M B$,
(c) $\models_M \bar{R}A \to \underline{R}A$ iff $\text{ext}_M A$ is definable in a system S such that ind (S) $= m(R)$,
(d) $\models_M \underline{R}A$ iff $\text{ext}_M A$ is internally nondefinable in a system S such that ind (S) $= m(R)$,
(e) $\models_M \bar{R}A$ iff $\text{ext}_M A$ is externally nondefinable in a system S such that ind (S) $= m(R)$ and
(f) $\models_M \neg(\bar{R}A \to \underline{R}A)$ iff $\text{ext}_M A$ is totally nondefinable in a system S such that ind (S) $= m(R)$.

The proof follows immediately from the definition of satisfiability.

In the next lemma we list some properties of a knowledge representation system related to a model. Let a model $M = (OB, m, v)$ be given and let S be a system such that ind (S) $= m(R)$ for a certain $R \in$ CONREL.

FACT 9.2

(A) $\models_M \bar{R}A \to \underline{R}A$ for every $A \in$ FOR iff system S is selective,
(b) $\models_M \bar{R}(A \wedge B) \wedge \bar{R}(A \wedge \neg B)$ for every A, $B \in$ FOR iff equivalence classes of ind (S) have at least two elements and
(c) $\models_M \bar{R}(A \wedge B) \wedge \bar{R}(A \wedge \neg B) \to \underline{R}(A)$ for every A, $B \in$ FOR iff each equivalence class of ind (S) has exactly two elements.

Proof

The formula in condition (a) says that for any A the upper approximation of a set corresponding to A is included in its lower approximation. By 4.3(b) and 4.4(b) condition (a) holds. The formula in condition (b) says that in model M for any object $o \in OB$ there are objects o_1 and o_2 such that $o_1 \in \text{ext}_M A$, $o_1 \in \text{ext}_M B$, $(o, o_1) \in m(R)$, $o_2 \in \text{ext}_M A$, $o_2 \in \text{ext}_M B$, and $(o, o_2) \in m(R)$. $m(R)$ is an equivalence relation and we possibly have $o_1 = o$ or $o_2 = o$ but not $o_1 = o_2$, since o_1 and o_2 are separated by $\text{ext}_M B$. Hence condition (b) holds. In the formula from condition (c) the left-hand side of the implication guarantees the existence of an object o satisfying condition (b). The formula on the right-hand side of this implication says that this object is the only one satisfying this condition.

It is easy to see that in the similar way we can define formulae expressing the fact in a system related to a model each elementary set has at least or exactly n elements, for $n \geq 1$.

In the following we list some formulae which express relations between knowledge representation systems. Let a model $M = (OB, m, v)$ be given and let S_1, S_2, be the systems such that ind $(S_1) = m(R_i)$, for $i = 1, 2, 3$ for some constants R_1, R_2 and R_3.

FACT 9.3

(a) $\models_M \bar{R}_1 A \to \bar{R}_2 A$ for every $A \in$ FOR iff $S_1 \leq S_2$,
(b) $\models_M \underline{R}_2 A \to \underline{R}_1 A$ for every $A \in$ FOR iff $S_1 \leq S_2$,

(c) $\models_M (\underline{R}_1A \to \underline{R}_3A) \wedge (\underline{R}_2A \to \underline{R}_3A)$ for every $A \in FOR$ and $\models_M (\underline{R}_1A \to \underline{R}A) \wedge (\underline{R}_2A \to \underline{R}A) \to (\underline{R}_3A \to \underline{R}A)$ for every $A \in FOR$ and every $R \in CONREL$ iff $\text{ind}(S_3) = \text{ind}(S_1) \cap \text{ind}(S_2)$ and

(d) $\models_M (\underline{R}_3A \to \underline{R}_1A) \wedge (\underline{R}_3A \to \underline{R}_2A)$ for every $A \in FOR$ and $\models_M (\underline{R}A \to R_1A) \wedge (\underline{R}A \to \underline{R}_2A) \to (\underline{R}A \to \underline{R}_3A)$ for every $A \in FOR$ and for every $R \in CONREL$ iff $\text{ind}(S_3) = (\text{ind}(S_1) \cup \text{ind}(S_2))^*$.

Proof

The formula in condition (a) says that for any A the upper approximation of $\text{ext}_M A$ in system S_1 is included in its upper approximation in system S_2. By 6.1, condition (a) holds. Similarly, condition (b) follows from (6.2). The first formula in condition (c) says that relation $\text{ind}(S_3)$ is included both in $\text{ind}(S_1)$ and $\text{ind}(S_2)$. The second formula says that $\text{ind}(S_3)$ is the greatest relation with that property and hence condition (c) holds. The formulae given in condition (d) say that $\text{ind}(S_3)$ is the least relation containing both $\text{ind}(S_1)$ and $\text{ind}(S_2)$. Since these relations are equivalences, $\text{ind}(S_3)$ is the transitive closure of $\text{ind}(S_1) \cup \text{ind}(S_2)$. Thus condition (d) holds.

10. Deductive system for the language

We give a deductive structure to the language in the usual way, first specifying a recursive set of axioms and inference rules, and then defining a theorem to be any formula obtainable from the axioms by repeated application of the rules.

Observe that the following conditions are satisfied.

FACT 10.1

(a) $\text{ext}_M (A \vee B) = \text{ext}_M (\neg A \to B)$,
(b) $\text{ext}_M (A \wedge B) = \text{ext}_M \neg(A \to \neg B)$ and
(c) $\text{ext}_M \bar{R}A = \text{ext}_M \neg(\underline{R} \neg A)$.

Due to 10.1 and 9.1(f) it is sufficient to define the deductive system for the language based on operations of negation, implication and lower approximation.

AXIOMS

A1. All formulae having the form of a tautology of the classical propositional logic.
A2. $\underline{R}(A \to B) \to (\underline{R}A \to \underline{R}B)$.
A3. $\underline{R}A \to A$.
A4. $A \to \underline{R} \neg \underline{R} \neg A$.
A5. $\underline{R}A \to \underline{R}\underline{R}A$.

Rules of inference

$$R1 \ \frac{A, A \to B}{B}, \qquad R2 \ \frac{A}{\underline{R}A}.$$

This axiomatization corresponds very closely to the axiomatization for modal logic S5 (Gabbay, 1976); however, a difference consists in considering a family of equivalence relations in the language and in models.

A proof of a formula A from a set T of formulae is a finite sequence of formulae each of which is either an axiom or an element of set T or else is obtainable from earlier formulae by a rule of inference, and A is the last formula in the sequence. A formula A is derivable from a set T ($T \vdash A$) whenever there is a proof of A from T. A formula A is a theorem ($\vdash A$) iff there is a proof of A merely from axioms. A set T is consistent if a formula of the form $A \wedge \neg A$ is not derivable from T.

FACT 10.2 (SOUNDNESS THEOREM)

(a) $\vdash A$ implies $\models A$,
(b) $T \vdash A$ implies $T \models A$ and
(c) T satisfiable implies T consistent.

The proof can be easily obtained by checking that all the axioms are valid and the rules preserve validity.

11. Completeness theorem

We prove the completeness theorem for the logic by using the Rasiowa & Sikorski (1970) method adopted for modal logics by Mirkowska (1983).

Let T be a consistent set of formulae. We define relation $\approx$ in set FOR as:

$$A \approx B \quad \text{iff } T \vdash A \leftrightarrow B.$$

FACT 11.1

(a) Relation $\approx$ is an equivalence relation.
(b) Relation $\approx$ is a congruence with respect to operations $\neg$, $\vee$, and $\wedge$.
(c) If $A \approx B$ then $\underline{R}A \approx \underline{R}B$ for any $R \in$ CORNREL.

Let $FOR/_{\approx}$ denote the set of all the equivalence classes of relation $\approx$, and let [A] denote the equivalence class determined by a formula A. We consider the algebra

$$AFOR = (FOR/_{\approx}, -, \cap, \cup, 1, 0),$$

where

$$-[A] = [\neg A],$$

$$[A] \cup [B] = [A \vee B], \qquad [A] \cap [B] = [A \wedge B],$$

$$1 = [A \vee \neg A], \qquad 0 = [A \wedge \neg A].$$

FACT 11.2

(a) Algebra AFOR is a non-degenerate Boolean algebra,
(b) $[A] \leq [B]$ iff $T \vdash A \rightarrow B$,
(c) $T \vdash A$ iff $[A] = 1$ and
(d) $[\neg A] \neq 0$ iff not $T \vdash A$.

Proofs of these two facts are similar to that presented in Rasiowa & Sikorski (1970).

Let $\mathcal{F}$ be the family of all the maximal filters in algebra AFOR. Set $\mathcal{F}$ is non-empty since the algebra is non-degenerate. We define a canonical model as:

$$M_0 = (OB_0, m_0, v_0),$$

where $OB_0 = \mathcal{F}$,

$m_0(R) = \{(F_1, F_2) \in \mathcal{F} \times \mathcal{F}$: for any formula A if $[\underline{R}A] \in F_1$ then $[A] \in F_2\}$,

$v_0(p) = \{F \in \mathcal{F}: [p] \in F\}$.

FACT 11.3

For any $R \in$ CONREL $m_0(R)$ is an equivalence relation.

Proof

By axiom A2 and 11.2(b) we have $[\underline{R}A] \subseteq [A]$. Hence if $[\underline{R}A] \in F$ then $[A] \in F$, and so relation $m_0(R)$ is reflexive. Let us now assume that $(F_1, F_2) \in m_0(R)$, $[\underline{R}A] \in F_2$ and suppose that $[A] \notin F_1$. Hence, F_1 is a maximal filter, we have $[\neg A] \in F_1$. By axiom A4 we have $[\underline{R}\neg\underline{R}A] \in F_1$. Thus $[\neg\underline{R}A] \in F_2$, a contradiction. Hence relation $m_0(R)$ is symmetric. Let us now assume that $(F_1, F_2) \in m_0(R)$, $(F_2, F_3) \in m_0(R)$, $[\underline{R}A] \in F_1$, and suppose that $[A] \notin F_3$. By Axiom A5 we have $[\underline{R}\underline{R}A] \in F_1$, and hence $[\underline{R}A] \in F_3$. It follows that $[A] \in F_3$, a contradiction. Hence relation $m_0(R)$ is transitive.

FACT 11.4

The following conditions are equivalent:
(a) M_0, F sat A and
(b) $[A] \in F$.

Proof

The proof is by inducation with respect to a structure of a formula.

Case 1. A is $p \in$ *VAROB*

We have M_0, F sat p iff $F \in v_0(p)$ iff $[p] \in F$.

Case 2. A is $\neg B$

Condition M_0, F sat $\neg B$ is equivalent to not M_0, F sat B. By the induction hypothesis we have $[B] \notin F$. Since F is a maximal filter, we have $[\neg B] \in F$.

Case 3. A is $B \to C$

Condition M_0, F sat $B \to C$ is equivalent to not M_0, F sat B or M_0, F sat C. By the induction hypothesis we have $[B] \notin F$ or $[C] \in F$. Since F is a maximal filter, we have $[\neg B] \in F$ or $[C] \in F$. Since F is a prime filter, we have $[\neg B] \cup [C] \in F$, and hence $[B \to C] \in F$.

Case 4. A is $\underline{R}B$

Let us assume that M_0, F sat $\underline{R}B$ and suppose that $[\underline{R}B] \notin F$. We consider set $Z_{FR} = \{[C]: [\underline{R}C] \in F\}$. We now prove four properties of this set.

(4a) Set Z_{FR} is non-empty

It follows from the fact that $[\underline{R}(A \vee \neg A)] \in Z_{FR}$.

(4b) Set Z_{FR} is a filter

We have $[B_1] \cap [B_2] \in Z_{FR}$ iff $[B_1 \wedge B_2] \in Z_{FR}$. Hence $[\underline{R}(B_1 \wedge B_2)] \in F$. Since $\vdash \underline{R}(A \wedge B) \leftrightarrow \underline{R}A \wedge \underline{R}B$, we have $[\underline{R}B_1 \wedge \underline{R}B_2] \in F$. It is equivalent to $[\underline{R}B_1] \in F$ and $[\underline{R}B_2] \in F$. Hence $[B_1] \in Z_{FR}$ and $[B_2] \in Z_{FR}$.

(4c) Filter Z_{FR} is a proper filter

Let us suppose that $0 \in Z_{FR}$. Then we have $[\underline{R}(A \wedge \neg A)] \in F$ and hence $1 = [\underline{R}(A \vee \neg A)] \notin F$, a contradiction.

(4d) Filter G generated by set $Z_{FR} \cup \{[\neg B]\}$ is a proper filter

We show that for any $[A_1], \ldots, [A_n] \in Z_{FR}$, for $n \geq 1$, we have $[A_1] \cap \cdots \cap [A_n] \cap [\neg B] \neq 0$. For suppose not, then we have $T \vdash A_1 \cap \cdots \cap A_n \wedge \neg B \rightarrow A \wedge \neg A$, and hence $T \vdash A_1 \wedge \cdots \wedge A_n \rightarrow B$. By rule R2 and axiom A2 we obtain $T \vdash \underline{R}A_1 \wedge \cdots \wedge \underline{R}A_n \rightarrow \underline{R}B$. Since $[A_1], \ldots, [A_n] \in Z_{FR}$, we have $[\underline{R}A_1], \ldots, [\underline{R}A_n] \in F$, and hence $[\underline{R}A1 \wedge \cdots \wedge \underline{R}A_n] \in F$. So $[\underline{R}B] \in F$ and this is in conflict with the supposition under case 4.

It follows that filter G can be extended to a maximal filter H such that $[\neg B] \in H$ and for any formula C if $[\underline{R}C] \in F$ then $[C] \in H$. Hence $(F, H) \in m_0(R)$. By case 2 we have M_0, H sat$\neg$B, and this is a contraction with the assumption under case 4.

Let us now assume that $[\underline{R}B] \in F$ and consider set Z_{FR}. We have $[B] \in Z_{FR}$. By the Kuratowski–Zorn lemma there is a maximal filter G which includes set Z_{FR}, and hence $(F, G) \in m_0(R)$ and $[B] \in G$. But Z_{FR} is included in every filter G such that $(F, G) \in m_0(R)$, thus [B] belongs to every such filter. By the induction hypothesis we have M_0, G sat B for all G satisfying $(F, G) \in m_0(R)$. Hence M_0, F sat $\underline{R}$B.

FACT 11.5 (COMPLETENESS THEOREM)

(a) $\models A$ implies $\vdash A$,
(b) $T \models A$ implies $T \vdash A$ and
(c) T consistent implies T satisfiable.

Proof
Let us assume that $T \models A$ and suppose not $T \vdash A$. By 11.2(d) we have $[\neg A] \neq 0$ and hence there is a maximal filter $F_0 \in \mathscr{F}$ such that $[\neg A] \in F_0$. By 11.4 we have M_0, F_0 sat $\neg$A for canonical model M_0. Moreover, for any formula $B \in T$ we have $T \vdash B$, and hence $[B] \in F_0$. By 11.4 M_0, F_0 sat B for any $B \in T$, a contradiction. This proves (b) from which (a) and (c) follow immediately.

FACT 11.6 (COMPACTNESS THEOREM)

The following conditions are equivalent:

(a) A set T of formulae is satisfiable and
(b) every finite subset of T is satisfiable.

Completness theorem enables us to consider theories based on the given logic. We can consider, for example, the theory of selective systems by adjoining the scheme of formulae given in 9.2(a) to the logical axioms A1–A6.

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