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Abstract • Содержание • Streszczenie

In this paper we define the notion of partial observability considered in statistical models in terms of "rough" sets. With each event we associate "inner" and "outer" probability, or an interval which end points are inner and outer probability respectively. The interval is called rough probability of the event. Some elementary properties of rough probabilities are given.

Приближенное множество и частичная наблюдаемость

В работе определено понятие частичной наблюдаемости, рассматриваемой в статистических моделях в терминах приближенных множеств. С каждым происшествием связаны "внутренняя" и "внешняя" вероятности или интервалы, которых окончаниями являются внутренние и внешние вероятности. Такой интервал называется приближенной вероятностью. Представлено элементарные свойства приближенных вероятностей.

Zbiory przybliżone i częściowa obserwowalność

W pracy zdefiniowano pojęcie częściowej obserwowalności rozważanej w modelach statystycznych w terminach zbiorów przybliżonych. Z każdym zdarzeniem związane jest "wewnętrzne" i "zewnętrzne" prawdopodobieństwo lub przedział, którego końcami są prawdopodobieństwo wewnętrzne i zewnętrzne. Przedział ten jest nazwany prawdopodobieństwem przybliżonym. Podano elementarne własności prawdopodobieństw przybliżonych.

1. Introduction

Pleszczyńska and Dąbrowska introduced partial observability in statistical models (see [1]).

We propose in this paper somewhat different formulation of this notion based on the concept of the rough set (see [2]) – which can be of interest in situations when the "exact" probability of some events is not known, but only the interval, to which this probability belongs.

As a departure point of our considerations we introduce the notion of a stochastic approximation space S and the inner and outer probability $\underline{P}_S(X)$, $\overline{P}_S(X)$ of the event X in the approximation space S is defined. In fact the inner and outer probability of an event X is the probability of the interior and closure of X respectively, in the topological space generated by the approximation space S. Some elementary properties of inner and outer probabilities are given and the notion of the rough (approximate) probability of an event X is defined, as $P^*_S(X) = \langle \underline{P}_S(X), \overline{P}_S(X) \rangle$, which is to understand as an interval to which the probability of X belongs. Elementary properties of rough probability are given.

We consider finite stochastic approximation spaces only, but the infinite case can be also treated in a similar way.

We assume that the reader is familiar with the basic notions of set theory and topology, and we use standard mathematical notation throughout the paper.

2. Approximation space, approximation of sets

In this section we recall after [1] basic notions concerning the concept of a rough set, used as a departure point of our considerations.

Let U be a certain set, and let R be an equivalence relation on U. The pair $A = (U,R)$ will be called an approximation space, and R will be referred to as an indiscernibility relation. If $x,y \in U$ and $R(x,y)$ we say that x and y are indistinguishable in A.

Equivalence classes of the relation R and the empty set ϕ will be called elementary sets (atoms) in A or in short elementary sets (atoms) if A is understood.

Every union of elementary sets in A will be called a composed set in A, or in short a composed set, if A is known.

The family of an composed sets in A is denoted by $Com(A)$.

Obviously $Com(A)$ is a Boolean algebra, i.e., the family of all composed sets is closed under intersection, union and complement of sets.

Let X be a certain subset of U. The least composed set in A containing X will be called the best upper approximation of X in A, in symbols $\bar{A}(X)$; the greatest composed set in A contained in X will be called the best lower approximation of X in A, and will be denoted by $\underline{A}(X)$.

3. Properties of approximations

One can easily check that the approximation space $A = (U,R)$ defines uniquely the topological space $T_A = (U, Com(A))$, and $Com(A)$ is the family of all open and closed sets in T_A, and U/R is a base for T_A.

From the definition of approximations follows that $\underline{A}(X)$ and $\bar{A}(X)$ are interior and closure of X in the topological space T_A respectively.

Thus for every $X,Y \subset U$ and every approximation space $A = (U,R)$ the following properties of approximations are valid:

(A1) $\underline{A}(X) \subset X \subset \bar{A}(X)$

(A2) $\underline{A}(U) = \bar{A}(U) = U$

(A3) $\underline{A}(\phi) = \bar{A}(\phi) = \phi$

(A4) $\bar{A}(\bar{A}(X)) = \underline{A}(\bar{A}(X)) = \bar{A}(X)$

(A5) $\underline{A}(\underline{A}(X)) = \bar{A}(\underline{A}(X)) = \underline{A}(X)$

(A6) $\bar{A}(X \cup Y) = \bar{A}(X) \cup \bar{A}(Y)$

(A7) $\underline{A}(X \cup Y) \supset \underline{A}(X) \cup \underline{A}(Y)$

(A8) $\bar{A}(X \cap Y) \subset \bar{A}(X) \cap \bar{A}(Y)$

(A9) $\underline{A}(X \cap Y) = \underline{A}(X) \cap \underline{A}(Y)$

(A10) $\bar{A}(-X) = -\underline{A}(X)$

(A11) $\underline{A}(-X) = -\bar{A}(X)$

Moreover we have the property

(A12) If $X \subset Y$, then $\bar{A}(X) \subset \bar{A}(Y)$ and $\underline{A}(X) \subset \underline{A}(Y)$

Let us also notice that $\underline{A}(X) = \bar{A}(X)$ if and only if X is a composed set in A.

Let $A = (U,R)$ and $A' = (U,R')$ be approximation spaces. If $R' \subset R$ we say that the space A' is finer than the space A or that the space A is coarser than the space A'.

If A' is finer than A, then the following is true:

(A13) $\underline{A}(X) \supset \underline{A}'(X)$

(A14) $\bar{A}(X) \subset \bar{A}'(X)$

for every $X \subset U$.

4. Observable and unobservable sets

Let $A = (U,R)$ be an approximation space.

If $X \subset U$ and $\overline{A}(X) = U$ we say that X is dense in A.

If $X \subset U$ and $\underline{A}(X) = \emptyset$ we say that X is co-dense in A.

If $X \subset U$ and X is both dense and co-dense in A we say that X is dispersed in A.

If we assume that we are able to observe only elementary sets and their unions, i.e., composed sets, than we can classify subsets of the approximation space $A = (U,R)$ in the following way:

1) If $\underline{A}(X) = \overline{A}(X)$ then X will be called observable in A, otherwise set X is unobservable in A.

Let $X \subset U$ and let X be unobservable in A. We introduce the following four categories of unobservable sets:

2) If $\underline{A}(X) \neq \emptyset$ and $\overline{A}(X) \neq U$, then we shall call X roughly observable in A.
3) If $\underline{A}(X) \neq \emptyset$ and $\overline{A}(X) = U$, set X will be called externally unobservable in A.
4) If $\underline{A}(X) = \emptyset$ and $\overline{A}(X) \neq U$, set X will be called internally unobservable in A.
5) If $\underline{A}(X) = \emptyset$ and $\overline{A}(X) = U$, then X will be called totally unobservable in A.

Let us now give an intuitive motivation for the above classification.

The notion of an observable set is obvious. As mentioned above if we are able to observe some elementary sets then the only observable sets are elementary sets and their unions.

If set X is roughly observable in A, that is to mean that we are able to observe set X with a certain approximation

only, i.e., to observe only lower and upper approximations of X.

If set X is externally unobservable we are able to observe only lower approximations, because $\overline{A}(X) = U$ means that we can not exclude any element $x \in U$ being possibly member of X;

If set X is internally unobservable in A it means that we are unable to say for sure that any $x \in U$ is a member of X.

Finally, if set X is totally unobservable in A it means that we cannot exclude any element $x \in U$ being possibly a member of X and we cannot also say for sure that any $x \in U$ is a member of X. Thus in fact we cannot observe set X in A.

5. Example

Let us consider an information system as an example of an approximation space.

By an information system S (see [3]) we mean

$$S = \langle U, A, V, \rho \rangle$$

where

U - is a finite set of objects

A - is a finite set of attributes

$V = \bigcup V_a$, $a \in A$, is a finite set of values of attributes

$\rho : U \times A \to V$ is an information function such that $\rho(x,a) \in V_a$ for every $x \in U$, $a \in A$.

Instead the function ρ we shall use the function ρ_x such that $\rho_x(a) = \rho(x,a)$ for every $x \in U$ and $a \in A$.

We say that objects $x, y \in U$ are indiscernible in S if $\rho_x = \rho_y$. The indiscernibility relation generated by the information system S will be denoted by $\tilde{S}$; obviously S is an equivalence relation, so each information system $S = \langle U, A, V, \rho \rangle$ generates an approximation space $A_S = (U, \tilde{S})$.

Let us consider an information system with objects $U = \{x_0,\dots,x_{11}\}$, three attributes $A = \{a,b,c\}$, the sets of values $V_a = \{u_1,u_2\}$, $V_b = \{v_1,v_2,v_3\}$ and $V_c = \{w_1,w_2\}$ and the information function defined by the table

U	a	b	c
x_0	u_1	v_1	w_2
x_1	u_1	v_1	w_2
x_2	u_1	v_2	w_1
x_3	u_2	v_3	w_1
x_4	u_2	v_1	w_2
x_5	u_2	v_3	w_1
x_6	u_1	v_2	w_1
x_7	u_1	v_3	w_2
x_8	u_2	v_1	w_2
x_9	u_1	v_2	w_1
x_{10}	u_1	v_3	w_2

Elementary sets (atoms) are in this system the following:

$E_1 = \{x_0,x_1\}$

$E_2 = \{x_2,x_6,x_9\}$

$E_3 = \{x_3,x_5\}$

$E_4 = \{x_4,x_8\}$

$E_5 = \{x_7,x_{10}\}$

Sets

$X_1 = \{x_0,x_1,x_4,x_8\}$

$Y_1 = \{x_3,x_5,x_7,x_8\}$

$Z_1 = \{x_2,x_3,x_5,x_6,x_9\}$

are observable in the information system.

Sets

$X_2 = \{x_0,x_3,x_4,x_5,x_8,x_{10}\}$

$Y_2 = \{x_1,x_7,x_8,x_{10}\}$

$Z_2 = \{x_2,x_3,x_4,x_8\}$

are roughly observable in the system and the following are the corresponding approximations

$\underline{A}(X_2) = \{x_3,x_4,x_5,x_8\} = E_3 \cup E_4$

$\bar{A}(X_2) = \{x_0,x_1,x_3,x_4,x_5,x_7,x_8,x_{10}\} = E_1 \cup E_3 \cup E_4 \cup E_5$

$\underline{A}(Y_2) = \{x_7,x_{10}\} = E_5$

$\bar{A}(Y_2) = \{x_0,x_1,x_4,x_7,x_8,x_{10}\} = E_1 \cup E_4 \cup E_5$

$\underline{A}(Z_2) = \{x_4,x_8\} = E_4$

$\bar{A}(Z_2) = \{x_2,x_3,x_4,x_5,x_6,x_8,x_9\} = E_2 \cup E_3 \cup E_4$

Sets

$X_3 = \{x_0,x_1,x_2,x_3,x_4,x_7\}$

$Y_3 = \{x_1,x_2,x_3,x_6,x_8,x_9,x_{10}\}$

$Z_3 = \{x_0,x_2,x_3,x_4,x_8,x_{10}\}$

are externally unobservable in the system.

Sets

$X_4 = \{x_0,x_2,x_3\}$

$Y_4 = \{x_1,x_2,x_4,x_7\}$

$Z_4 = \{x_2,x_3,x_4\}$

are internally unobservable in the system.

Sets

$X_5 = \{x_0,x_2,x_3,x_4,x_7\}$

$Y_5 = \{x_1,x_5,x_6,x_8,x_{10}\}$

$Z_5 = \{x_0,x_2,x_5,x_7,x_8\}$

are totally unobservable in the system.

In other words observable sets can be exactly described by means of attributes available in the information system. Roughly observable sets cannot be described exactly by means of the attributes of the information system. We can give only in this case the best lower and upper description of the set by the attributes. Extrenally unobservable sets have only nontrivial lower approximation, and internally unobservable sets have only nontrivial upper approximation (by trivial approximation we understand the empty set ϕ and the universe U). That is to mean that the set of attributes available in the information system in not powerful enough to exclude any object being a member of the considered set - in the first case and assert that some objects are for sure members of the set. Totally unobservable sets are impossible to describe by the attributes of the information system.

6. Stochastic approximation space, inner and outer probability

The purpose of this section is to give probabilistic interpretation of the notions given in previous sections.

Let $A = (U,R)$ be a finite approximation space (i.e., space where U is finite, any subset of U will be called an event in A. In particular one-element event is called primitive in A and elementary sets in A are called elementary (or atomic) events in A. Observable sets in A are called observable events in A.

By a stochastic approximation space we mean an ordered triple

$$S = (U,R,P)$$

where $A = (U,R)$ is an approximation space, called an underlying space, and P is a probability measure defined an observable sets in A.

Evidently $P(\phi) = 0$, $P(U) = 1$, and if $X = \bigcup_{i=1}^{n} X_i$ is an observable set in A and X_i are atomic sets in A, then $P(X) = \sum_{i=1}^{n} P(X_i)$.

Our aim is to evaluate probability of unobservable events. (We recall that we are not given probabilities of primitive events).

In order to investigate the problem we introduce inner and outer probability of an event in the stochastic approximation space $S = (U,R,P)$, denoted as $\underline{P}_S(X)$ and $\overline{P}_S(X)$ respectively and defined as follows:

$$\underline{P}_S(X) = P(\underline{A}(X))$$

$$\overline{P}_S(X) = P(\overline{A}(X)),$$

where $A = (U,R)$ is the underlying approximation space of $S = (U,R,P)$.

From the definition of the probability measure and properties of approximations we get the following properties of inner and outer probabilities:

(B1) If X is observable in A then $\underline{P}_S(X) = P(X) = \overline{P}_S(X)$

(B2) $\underline{P}_S(X) \leq P(X) \leq \overline{P}_S(X)$

(B3) $\underline{P}_S(\phi) = \overline{P}_S(\phi) = 0$

(B4) $\underline{P}_S(U) = \overline{P}_S(U) = 1$

(B5) $\underline{P}_S(-X) = (1 - \overline{P}_S(X))$

(B6) $\overline{P}_S(-X) = (1 - \underline{P}_S(X))$

(B7) $\underline{P}_S(X \cup Y) \geqslant \underline{P}_S(X) + \underline{P}_S(Y)$ provided $\underline{A}(X) \cap \underline{A}(Y) = \phi$

(B8) $\overline{P}_S(X \cup Y) = \overline{P}_S(X) + \overline{P}_S(Y)$ provided $\overline{A}(A) \cap \overline{A}(Y) = \phi$

(B9) $\underline{P}_S(X \cap Y) = \underline{P}_S(X) \cap \underline{P}_S(Y)$ provided $\underline{A}(X)$ and $\underline{A}(Y)$ are stochastically independent

(B10) $\overline{P}_S(X \cap Y) \leq \overline{P}_S(X) \cdot \overline{P}_S(Y)$ provided $\overline{A}(X)$ and $\overline{A}(Y)$ are stochastically independent.

In general case we have

(B11) $\underline{P}_S(X \cup Y) = \underline{P}_S(X) + \underline{P}_S(Y) - \underline{P}_S(X \cap Y)$

(B12) $\overline{P}_S(X \cup Y) = \overline{P}_S(X) + \overline{P}_S(Y) - \overline{P}_S(X \cap Y)$

Let $S = (U,R,P)$ and $S' = (U,R',P')$ be two stochastic approximation spaces; if $A' = (U,R')$ is finer than $A = (U,R)$, we shall say that also S' is finer than S.

Obviously the following is true:

(B13) $\underline{P}_{S'}(X) \geqslant \underline{P}_S(X)$

(B14) $\overline{P}_{S'}(X) \leq \overline{P}_S(X)$

7. Rough probability

With every event X in a stochastic approximation space $S = (U,R,P)$ we associate the interval $P_S^*(X) \subseteq \langle 0,1 \rangle$ defined as

$$P_S^*(X) = \langle \underline{P}_S(X), \overline{P}_S(X) \rangle ,$$

and $P_S^*(X)$ will be called rough probability of X in S.

Thus $P_S^*(X)$ is the interval to which belongs the probability of the unobservable event X.

Of course if X is an observable event in S then

$$\underline{P}_S(X) = \overline{P}_S(X) = P(X)$$

and

$$P_S^*(X) = \langle P(X), P(X) \rangle$$

or simple

$$P_S^*(X) = P(X),$$

i.e. $P_S^*(X)$ reduces to one point.

Certainly $P_S^*(X)$ has the following properties:

(C1) $P_S^*(\phi) = 0$

(C2) $P_S^*(U) = 1$

(C3) $P_S^*(-X) = \langle 1 - \overline{P}_S(X),\ 1 - \underline{P}_S(X) \rangle$

(C4) $P_S^*(X \cup Y) \subset \langle \underline{P}_S(X) + \underline{P}_S(Y), \overline{P}_S(X) + \overline{P}_S(Y) \rangle$

(C5) $P_S^*(X \cap Y) \subset \langle \underline{P}_S(X) \cdot \underline{P}_S(Y), \overline{P}_S(X) \cdot \overline{P}_S(Y) \rangle$

Obviously we have the following properties:

(a) If X is externally unobservable in A, then

$$P_S^*(X) = \langle \underline{P}_S(X), 1 \rangle ,$$

(b) If X is internally unobservable in A then

$$P_S^*(X) = \langle 0, \overline{P}_S(X) \rangle$$

(c) If X is totally unobservable in S, then

$$P_S^*(X) = \langle 0,1 \rangle .$$

In other words: if the event X is observable in A, then we can give exact probability $P(X)$: if X is roughly observable in A then we can give the interval $P_S^*(X)$ to which the probability of event X belongs; if X is externally unobservable in A then we can give only lower bound of the probability of X; if X is internally unobservable in A, then we can give only upper bound of the probability of X; if X is totally unobservable in A then we cannot give any bounds for the probability of X.

Moreover we have the following property:

If S' is finer than S, then

$$P^{\neq}_{S'}(X) \subset P^{\neq}_{S}(X)$$

for every $X \subset U$.

8. Uncertainty measure

In order to describe to what extent the probability of an event $X \subset U$ can be evaluated in the given stochastic approximation space S we introduce uncertainty measure $\eta_S(X)$ defines as below:

$$\eta_S(X) = \overline{P}_S(X) - \underline{P}_S(X),$$

which is simply the length of the interval $P^{\neq}_S(X)$.

By simple calculation one can show the following

(D1) $\eta_S(-X) = \eta_S(X)$

(D2) $\eta_S(X \cup Y) \leq \eta_S(X) + \eta_S(Y)$

(D3) $\eta_S(X \cap Y) \geqslant \eta_S(X) \cdot \eta_S(Y)$

(D4) If S' is finer than S, then

$$\eta_{S'}(X) \leq \eta_S(X).$$

9. Example

Consider a stochastic approximation space with the set of primitive events $U = \{x_0, \dots, x_{10}\}$ and the indiscernibility relation defined by the following equivalence classes:

$E_1 = \{x_0, x_1\}$

$E_2 = \{x_2, x_6, x_9\}$

$E_3 = \{x_3, x_5\}$

$E_4 = \{x_4, x_8\}$

$E_5 = \{x_7, x_{10}\}$.

We assume that we are given the probabilities of the atomic events:

$P(E_1) = 1/4$

$P(E_2) = 1/8$

$P(E_3) = 3/8$

$P(E_4) = 1/8$

$P(E_5) = 1/8$.

For observable events in the space

$X_1 = \{x_0, x_1, x_4, x_8\} = E_1 \cup E_4$

$Y_1 = \{x_3, x_5, x_7, x_8\} = E_3 \cup E_5$

$Z_1 = \{x_1, x_3, x_5, x_6, x_9\} = E_2 \cup E_3$

We have the following probabilities

$P(X_1) = 1/4 + 1/8 = 3/8$

$P(Y_1) = 3/8 + 1/8 = 1/2$

$P(Z_1) = 1/8 + 3/8 = 1/2$

For roughly observable events in the space

$X_2 = \{x_0, x_3, x_4, x_5, x_8, x_{10}\}$

$Y_2 = \{x_1, x_7, x_8, x_{10}\}$

$Z_2 = \{x_2, x_3, x_4, x_8\}$

we have the following lower and upper approximations

$\underline{A}(X_2) = E_3 \cup E_4$

$\overline{A}(X_2) = E_1 \cup E_3 \cup E_4 \cup E_5$

$\underline{A}(Y_2) = E_5$

$\overline{A}(Y_2) = E_1 \cup E_4 \cup E_5$

$\underline{A}(Z_2) = E_4$

$\overline{A}(Z_2) = E_2 \cup E_3 \cup E_4$

Inner and outer probabilities of these sets are the following:

$P(X_2) = 3/8 + 1/8 = 1/2$

$P(X_2) = 1/4 + 3/8 + 1/8 + 1/8 = 7/8$

$P(Y_2) = 1/8$

$P(Y_2) = 1/4 + 1/8 + 1/8 = 1/2$

$P(Z_2) = 1/8$

$P(Z_2) = 1/8 + 3/8 + 1/8 = 5/8$

and consequently the rough probabilities of these sets are

$P^*(X_2) = \langle 1/2, 7/8 \rangle$

$P^*(Y_2) = \langle 1/8, 1/2 \rangle$

$P^*(Z_2) = \langle 1/8, 5/8 \rangle$

For externally unobservable events in the space

$X_3 = \{x_0, x_1, x_2, x_3, x_4, x_7\}$

$Y_3 = \{x_1, x_2, x_3, x_6, x_8, x_9, x_{10}\}$

$Z_3 = \{x_0, x_2, x_3, x_4, x_8, x_{10}\}$

we have the following lower approximations.

$\underline{A}(X_3) = E_1$

$\underline{A}(Y_3) = E_2$

$\underline{A}(Z_3) = E_4$

and the corresponding rough probabilities are

$P^*(X_3) = \langle 1/4, 1 \rangle$

$P^*(Y_3) = \langle 1/8, 1 \rangle$

$P^*(Z_3) = \langle 1/8, 1 \rangle$

For internally unobservable events in the space

$X_4 = \{x_0, x_2, x_3\}$

$Y_4 = \{x_1, x_2, x_4, x_7\}$

$Z_4 = \{x_2, x_3, x_4\}$

we have the following upper approximations:

$\overline{A}(X_4) = E_1 \cup E_2 \cup E_3$

$\overline{A}(Y_4) = E_1 \cup E_2 \cup E_4 \cup E_5$

$\overline{A}(Z_4) = E_2 \cup E_3 \cup E_4$

which gives the following outer probabilities

$\overline{P}(X_4) = 1/4 + 1/8 + 3/8 = 3/4$

$\overline{P}(Y_4) = 1/4 + 1/8 + 1/8 + 1/8 = 5/8$

$\overline{P}(Z_4) = 1/8 + 3/8 + 1/8 = 5/8$

and consequently

$$P^*(X_4) = \langle 0,\ 3/4 \rangle$$

$$P^*(Y_4) = \langle 0,\ 5/8 \rangle$$

$$P^*(Z_4) = \langle 0,\ 5/8 \rangle$$

For totally unobservable sets

$$X_5 = \{x_0, x_2, x_3, x_4, x_7\}$$

$$Y_5 = \{x_1, x_5, x_6, x_8, x_{10}\}$$

$$Z_5 = \{x_0, x_2, x_5, x_7, x_8\}$$

we get

$$P^*(X_5) = \langle 0,1 \rangle$$

$$P^*(Y_5) = \langle 0,1 \rangle$$

$$P^*(Z_5) = \langle 0,1 \rangle$$

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