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Definability of informations

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Abstract • Содержание • Streszczenie

In the paper we discuss the definability of informations in Pawlak information systems. We present some inductive inference methods for finding the description of an information which is minimal with respect to the number of attributes.

Системы репрезентации знаний. Информационная определимость.

В работе исследуется вопрос определимости информации в информационных системах ПАВЛЯКА при использовании языка информационной системы. Приводятся методы минимального, с точки зрения количества используемых характерных признаков, описания информации, основанного на методах математической индукции.

Systemy reprezentowania wiedzy. Definiowalność informacji

W pracy badany jest problem definiowalności informacji w systemach informacyjnych Pawlaka, przy użyciu języka systemu informacyjnego. Podane są, oparte na systemach wnioskowania indukcyjnego, metody poszukiwania minialnego, ze względu na ilość użytych atrybutów, opisu informacji.

INTRODUCTION

In the paper we develop the method of representing informations and the method of searching for minimal descriptions of informations.

An information is a set of entities, called objects. An object is anything which can be spoken of in the subject position of a natural language sentence (e.g. book, company). Objects need not be atomic or undivisible. They can be composed or structured, but are treated as a whole. The knowledge about objects is given through assignment of properties. A property is denoted by a verb phrase in a natural language sentence (e.g. is red, is tall). To express properties we use the notions of an attribute (e.g. colour, height) and an attribute value (e.g. blue, white; low, tall). A set of attributes, a set of values of these attributes and a function which assigns attribute values to objects can be treated as a potential knowledge.
The system consisting of these sets and a set of objects will be called knowledge representation system. Given an object, we are able to know its properties expressed by the admitted attributes. However in general, we are not able to distinguish the all objects, the same properties from the admitted set of properties may be assigned to possibly many objects. Hence we have to distinguish definable and undefinable informations.

A definable information is a set of objects for which there is a subset of the set of admitted attributes which characterizes it uniquely. If such set of attributes does not exist, then the information can be characterized only in an approximate way. We introduce formal tools to present the notion of definable information and the notion of description of information. We also consider the problem how to find a description of the information minimal with respect to the needed attributes.

1. KNOWLEDGE REPRESENTATION SYSTEM

The notion of a knowledge representation system was introduced in Pawlak [8] . Following this paper we present in this section the basic definitions and facts which will be used in the paper.

A system is intended to represent a knowledge about some objects. Hence the basic component of the system is a finite, non-empty set of objects, e.g. human beings, books etc. The knowledge comtained in the system will be expressed through assignment of some characteristic features to the objects, e.g. human beings can be characterized by means of sex or age, books by means of title or author's name etc. These features will be represented by attributes and the values of attributes. Hence the finite, non-empty set A of attributes, and for each $a \in A$ the finite set V_a of values of attribute a are the components of the system. The assignment of attribute values to objects from the formal point of view can be considered as the function. This function is the last component of the system.

Now, we give the formal definition of a knowledge representation system.

The knowledge representation system is the system

$$S = (X, A, \{V_a\}_{a \in A}, \rho)$$

where X is a non-empty, finite set, whose elements are called objects,

A is a non-empty, finite set, whose elements are called attributes,

V_a is a non-empty, finite, set, whose elements are called values of attribute a,

ρ is a total function from set $X \times A$ into set $V = \bigcup_{a \in A} V_a$ such that $\rho(x,a) \in V_a$ for every $x \in X$.

Example 1.1

Let us consider very simple knowledge representation system S, defined as follows

$X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$

$A = \{\text{sex, age}\}$

$V_{sex} = \{\text{male, female}\}$

$V_{age} = \{\text{young, medium, old}\}$.

The function ρ is defined by means of the following table.

	sex	age
x_1	male	young
x_2	male	medium
x_3	female	old
x_4	male	medium
x_5	female	old
x_6	female	young

Observe, that according to the given definition only one attribute value can be associated with each object, and for each object the value of each attribute is determined. Due to the fact, that sets X, A and V are finite, we can identify a system with the finite table, defining function ρ. The columns of the table are labelled with attributes and consist of values of the corresponding attributes. The rows of the table are labelled with objects and we admit on occurrence of identical rows with different labels. The order of columns and rows in the table is arbitrary.

Given system $S = (X,A,V,\rho)$, by an information in S we mean any subset of set X, and by a descriptor in S we mean any pair (a, p) where $a \in A$ and $p \in V_a$. Let D_S denote the set of all descriptors in S. Descriptors are intended to describe informations. For instance in the system from example 1.1 descriptors (sex male) and (age young) describe the information $\{x_1\}$, descriptors (sex female) and (age old) or (sex female) and (age young) describe the information $\{x_3, x_6\}$. However, it is easy to see that there are infomations, which cannot be described. Consider the system from ex. 1.1 and set $\{x_1, x_2\}$. Object x_1 is described by descriptors (sex male) and (age young). But the pair of descriptors (sex male) and (age medium) describes both objects x_2 and x_4. Thus the knowledge contained in our system is not sufficient to distinguish objects x_2 and x_4 and we are not able to describe set $\{x_1, x_2\}$ in an exact way.

Given system $S = (X,A,V,\rho)$, we define the family $\{\tilde{a}\}_{a \in A}$ of equivalence relations on set X in the following way:

$$(x,y) \in \tilde{a} \quad \text{iff} \quad \rho(x,a) = \rho(y,a)$$

This means, that two objects belong to relation $\tilde{a}$ just in case they cannot be distinguished by means of attribute a.

Given proper subset B of set A of attributes, the relation $\bigcap_{b \in B} \tilde{b}$ will be denoted by $\tilde{B}$, and the relation $\bigcap_{a \in A} \tilde{a}$ will be denoted by $\tilde{S}$. Hence, for set $C \subseteq A$ two objects belong to relation $\tilde{C}$ if and only if they are undistinguishable with respect to all the attributes from set C.

Example 1.2

Consider system S, given in ex. 1.1. The following facts are true:

$(x_1, x_2) \in \widetilde{sex}$

$(x_1, x_6) \in \widetilde{young}$

$(x_2, x_4) \in \tilde{S} = \widetilde{\{sex, young\}} = \widetilde{sex} \cap \widetilde{young}$

$(x_3, x_5) \in \tilde{S}$.

For any system S the equivalence classes of relation $\tilde{S}$ will be called elementary sets of objects in S. Thus relation $\tilde{S}$ provides the classification of objects according to the knowledge contained in the system, and objects belonging to an elementary set are undistinguishable with respect to the knowledge represented by the system.

Example 1.3

The elementary sets for the system given in ex. 1.1 are the following:

$\{x_1\}$, $\{x_2,x_4\}$, $\{x_3,x_5\}$, $\{x_6\}$

Let us observe that according to our previous considerations, each information in system S can be described, by using descriptors, up to the equivalence $\tilde{S}$. In particular, all the informations which are proper subsets of elementary sets cannot be described.

System S will be called selective if all the equivalence classes of relation $\tilde{S}$ are one-element sets. It follows, that if system S is selective, then every information in S can be represented by means of descriptors.

More detailed considerations on comparing the expressive power of knowledge representation systems are given in Orłowska, Pawlak [7] .

2. KNOWLEDGE DESCRIPTION LANGUAGE

In this section we describe in the formal way the method of representation of informations by means of descriptors, namely we define the formalized language such that the expressions of this language will be interpreted as sets of objects of a knowledge representation system. Expressions of the language will provide a syntactical characterization of informations in the system, and informations will be considered as the semantical part of the knowledge contained in the system.

Assume, we are given system $S = (X,A,V,\rho)$. We define formalized language L_S as follows. Expressions of the language will be built up from the following symbols:

- symbols $a_1,a_2,\ldots,a_n$, $1 \le n < \omega$, denoting names of attributes from set A,
- symbols p,q,r, with indices if necessary, denoting names of values of attributes, i.e. elements of set V,
- symbol $-$ of unary operation,
- symbols $+$, $\cdot$, $\rightarrow$, $\leftrightarrow$ of binary operations,
- brackets (,).

For the sake of simplicity, in the following we will identify names of attributes and names of attribute values with attributes and values of attributes respectively.

Atomic expressions of language L_S are the descriptors. The set of all expressions of L_S is the least set containing the set D_S and closed under operations $-,+,\cdot,\rightarrow,\leftrightarrow$.

The language described above was defined in Marek, Pawlak [3] . Following this paper the expressions of language L_S will be called terms.

We now present the interpretation of terms, namely, we define inductively the function val_S from the set of all terms into the family of all subsets of set X, called the value of a term in system S:

$val_S(a,p) = \{x \in X : \rho(x,a) = p\}$,

$val_S(-t) = -\,val_S\, t$

$val_S(t_1+t_2) = val_S t_1 \cup val_2 t_2$

$val_S(t_1 \cdot t_2) = val_S t_1 \cap val_S t_2$

$val_S(t_1 \rightarrow t_2) = -val_S t_1 \cup val_S t_2$,

$val_S(t_1 \leftrightarrow t_2) = val_S(t_1 \rightarrow t_2) \cap val_S(t_2 \rightarrow t_1)$.

According to the given interpretation terms represent sets of objects and operations on terms correspond to the set-theoretical operations on sets of objects. In particular operations $-,+,\cdot$ correspond to complement, union and intersection, respectively.

We will say that term t is valid in system S if $val_S t = X$. We will write $\models_S t$ whenever t is valid in S.

Example 2.1

Consider the system S given by the following table

	a	b
x_1	p_1	q_1
x_2	p_2	q_3
x_3	p_1	q_2
x_4	p_1	q_1

and terms

$t_1 = (ap_1)\cdot(b\ q_2)$

$t_2 = (ap_2)\cdot(bq_2)$

$t_3 = (ap_1) + -(bq_2)$

$t_4 = (bq_1) \rightarrow (ap_1)$

$t_5 = (ap_2) \leftrightarrow (bq_3)$

We have

$val_S\ t_1 = \{x_1, x_3\}$

$val_S\ t_2 = \emptyset$

$val_S\ t_3 = val_S\ t_4 = val_S\ t_5 = X$

The presented language enables us to represent syntactically the knowledge given in the form of a knowledge representation system S. More exactly, each term t can be considered as the syntactical presentation or description of an information which is identified with the set $val_S t$.

The valid terms play a special role. For instance the following facts are true.

Fact 2.1. If term $t_1 \rightarrow t_2$ is valid in system S, then $val_S t_1 \subseteq val_S t_2$.

Hence if $\models_S t_1 \rightarrow t_2$ than the information $val_S t_1$ is stronger, than the information $val_S t_2$.

Fact 2.2. If term $t_1 \leftrightarrow t_2$ is valid in system S, then $val_S t_1 = val_S t_2$.

This means that if $\models_S t_1 \leftrightarrow t_2$ than the informations described by terms t_1 and t_2 are equal.

In the following we present some important examples of terms valid in a system and we show how the set of all the terms valid in a system can be characterized syntactically. For the purpose we distinguish the special class of terms, called elementary terms.

Term t is said to be elementary if it is of the form $(a_1\ p_1)\cdot\ldots\cdot(a_n p_n)$ where $a_1,\ldots,a_n$ are all the distinct attributes from set A and $p_1,\ldots,p_n$ are some attribute values from sets $Va_1,\ldots,Va_n$ respectively.

Fact 2.3. If t is an elementary term then $val_S t$ is an elementary set or the empty set.

If follows, that elementary terms provide descriptions of equivalence classes of relation $\tilde{S}$.

Example 2.2

Consider system S from example 2.1.

The following are examples of elementary terms in the language L_S:

$t_1 = (a\ p_2)\cdot(b\ q_3)$

$t_2 = (a\ p_1)\cdot(b\ q_2)$

$t_3 = (a\ p_1)\cdot(b\ q_1)$

$t_4 = (a\ p_1)\cdot(b\ q_3)$.

We have

$val_S t_1 = \{x_2\}$

$val_S t_2 = \{x_1, x_3\}$

$val_S t_3 = \{x_4\}$

$val_S\ t_4 = \emptyset$.

The equivalence classes of relation S are the following:

$\{x_1, x_3\}$ $\{x_2\}$ $\{x_4\}$.

Let $t_1, \ldots, t_m$, $m \geqslant 1$ be all the elementary terms which have non-empty values in a system S. Then terms $t_S = t_1 + \ldots + t_m$ will be called description of system S.

Fact 2.4. Description t_S of system S is valid in S.

Consider term of the form

$(a\ p) \leftrightarrow\ -(a\ p_1)\cdot \ldots \cdot - (a\ p_l)$

where $p, p_1, \ldots, p_l$ are all the distinct values of attribute a. We denote this term by $t_{(ap)}$.

Fact 2.5. For each descriptor $(a\ p) \in D_S$ term $t_{(a\ p)}$ is valid in system S.

Example 2.3

Consider the system from ex. 2.1

We have

$t_S = (a\ p_1)\cdot(b\ q_2) + (ap_2)\cdot(bq_3) + (ap_1)\cdot(bq_1)$

$val_S\ t_S = \{x_1, x_2, x_3, x_4\}$

$t(ap_1) = ((ap_1) \leftrightarrow -(ap_2))$

$t(bq_2) = ((bq_2) \leftrightarrow -(bq_1) \cdot -(bq_3))$

We now state the fact which says that the set of all terms valid in system S can be characterized syntactically.

Fact 2.6. The following conditions are equivalent:

(i) term t is valid in system S,

(ii) term t can be derived from set $t_S \cup \{ t_{(a\ p)} : (a\ p) \in D_S \}$ of terms by using axioms and inference rules of the classical propositional calculus.

Due to this fact terms t_S and $t_{(a\ p)}$ for $(a\ p) \in D_S$ can be considered as axioms of the system S. Axiom t_S provides the description of the set of objects and the elementary sets of objects, and axioms $t_{(a\ p)}$ provide description of the set of attributes and the set of values of attributes.

In the following we consider the problem of normal forms for terms.

By a literal we mean a term of the form $(a\ p)$ or $-(a\ p)$ for $(a\ p) \in D_S$. Term t is said to be in disjunctive normal form if it is of the form $t_1 + \ldots + t_m$, $m \geqslant 1$, and each t_i, $i = 1, \ldots, m$ is an intersection of finite number of literals.

Fact 2.7. For each term t of language L_S there is term t' of L_S in disjunctive normal form such that $val_S t = val_S t'$.

To obtain such term t' for a given term t we use the following equalities, which are counterparts of the well known laws of the classical propositional calculus.

$val_S(t_1 \to t_2) = val_S(-t_1 + t_2)$

$val_S(t_1 \leftrightarrow t_2) = val_S(t_1 \cdot t_2 + -t_1 \cdot -t_2)$

$val_S(t_1(t_2 + t_3)) = val_S(t_1 \cdot t_2 + t_1 \cdot t_3)$

$val_S(t_1 \cdot t_2) = val_S(t_2 \cdot t_1)$

$val_S(-(t_1 + t_2)) = val_S(-t_1 \cdot -t_2)$

$val_S(-(t_1 \cdot t_2)) = val_S(-t_1 + -t_2)$

$val_S(t \cdot t) = t = val_S(t + t)$

$val_S(t_1 \leftrightarrow t_2) = val_S(-t_1 \leftrightarrow -t_2)$

$val_S(- - t) = val_S t$

$val_S(t \cdot -t) = \emptyset$

$val_S(t + -t) = X$

A term is said to be in positive form if signs $-$, $\to$, $\leftrightarrow$ do not occur in it.

Fact 2.8. For every term t of language L_S there is term t' of L_S in the positive and disjunctive normal form such that $val_S t = val_S t'$.

To obtain such term t we use the previously listed equalities, and moreover the following one:

$val_S(-(ap)) = val_S((ap_1) + \dots + (ap_n))$

where $p, p_1, \dots, p_n$ are all the distinct values of attribute a.

A term is said to be complete if it is the union of some elementary terms.

It follows that any complete term is in the positive and disjunctive normal form.

Fact 2.9. For any term t of language L_S there is a complete term t' of L_S such that $val_S t = val_S t'$.

To obtain such term t' for a given term t we use all the previously given equalities and the following ones:

$val_S t = val_S(t \cdot t_a)$

where $t_a = (ap_1) + \dots + (ap_n)$, $a \in A$

and $p_1, \dots, p_n$ are all the distinct values of attribute a.

3. DEFINABLE INFORMATIONS

Given system $S = (X, A, V, \rho)$ and language L_S, we will say that set $Y \subseteq X$ is definable in system S if there is term t of L_S such that $Y = val_S t$.

Thus definable sets play the role of those informations contained in system S, which can be represented syntactically by means of expressions of language L_S.

Example 3.1

Consider system S from ex. 2.1.

It is easy to see that sets $\{x_1, x_3\}$ and $\{x_1, x_2, x_3\}$ are examples of definable sets. The respective terms are the following:

$t_1 = (ap_1) \cdot (bq_2)$ $\quad val_S\, t_1 = \{x_1, x_3\}$

$t_2 = (ap_1) \cdot (bq_2) + (ap_2) \cdot (bq_3)$ $\quad val_S\, t_2 = \{x_1, x_2, x_3\}$

Sets $\{x_1, x_2\}$ and $\{x_3, x_4\}$ are examples of sets which are not definable in our system. It follows from the following reasoning. Due to Fact 2.9 we can look for a description of set $\{x_1, x_2\}$ among complete terms. Since $x_1 \in val_S((ap_1)\cdot(bq_2))$ and $x_2 \in val_S((ap_2)\cdot(bq_3))$, we have to use at least elementary terms $(ap_1)\cdot(bq_2)$ and $(ap_2)\cdot(bq_3)$. But the value of the union of these terms is set $\{x_1, x_2, x_3\}$.

Observe, that set X and the empty set are definable in S by terms of the form $t + -t$ and $t \cdot -t$ respectively. Moreover, the family of sets definable in S is closed under union, intersection and complement. Hence the following fact is true.

Fact 3.1. The family of all sets definable in system S is a Boole'an algebra.

The question arises what are the conditions for definability of every subset of set X. The answer is the following.

Fact 3.2. If system $S = (X, A, V, \rho)$ is selective, then every subset of set X is definable in S.

It follows from the fact, that each elementary set of the selective system is the one-element set. Hence, we obtain the respective term by taking the union of all the elementary terms, whose values consist of elements of the considered set.

The presented notions of knowledge representation system and knowledge description language provide tools to formulate and to solve several problems from the field of artificial intelligence. In sections 4, 5, 6, 7 we present some elementary tasks, which are components of many artificial intelligence problems. We define these tasks and describe their solutions using the techniques introduced in sections 1, 2, 3.

4. CHARACTERISTIC DESCRIPTION OF A SET OF OBJECTS

In this section we consider the following problem. Given system $S = (X, A, V, \rho)$ and non-empty subset Y of set X of objects. We wish to define a term, which could be considered as a description of set Y. We distinguish two cases.

Case 4.1. Set Y is definable in system S. In this case we define, what is called the characteristic description of set Y. We take all the elementary terms $t_1, \ldots, t_m$, $m \geq 1$, such that their values are nonempty and $Y = val_S t_1 \cup \ldots \cup val_S t_m$. Then term $t_Y = t_1 + \ldots + t_m$ will be called characteristic description of set Y. It is easy to see that t_Y behave the property $val_S t_Y = Y$, so it is the term which defines set Y.

We now consider the membership problem. If we wish to know whether object y belongs to set Y, determined by its characteristic description, we simply consider the elementary term t' such that $y \in val_S t'$ and we check whether term $t' \to t_Y$ is valid in S. If it is the case, then $val_S t' \subseteq val_S t_Y$ and hence $y \in Y$. If it is not the case, then $y \notin Y$.

Example 4.1

We consider the system from ex. 2.1 and set $Y = \{x_1, x_2, x_3\}$. Term $t_Y = (ap_1)\cdot(bq_2) + (ap_2)\cdot(bq_3)$ is the characteristic description of set Y.

Case 4.2. Set Y is not definable in system S. In this case there is no term of language L_S which will define set Y and we are able to describe set Y using terms of L_S only in an approximate way. We define what is called upper description and lower description of set Y.

Case 4.2.1.

By upper description of set $Y = \{y_1,\ldots,y_m\}$ we mean term $\bar{t}_Y = t_1+\ldots+t_m$, $m \geqslant 1$, where $t_1,\ldots,t_m$ are elementary terms such that for each $i = 1,\ldots,m$ $y_i \in val_S t_i$. Observe, that $Y \subseteq val_S \bar{t}_Y$ and $val_S \bar{t}_Y = \{x \in X :$ there is an $y \in X$ such that $(x,y) \in \widetilde{S}$ and $y \in Y\}$.

Set $val_S \bar{t}_Y$ will be called upper approximation of set Y and will be denoted by $\bar{Y}$.

Given the upper description $\bar{t}_Y$ of set Y, the problem whether object y belongs to set Y can be answered in an approximate way. We consider elementary term t' such that $y \in val_S t'$ and we check whether term $t' \to \bar{t}_Y$ is valid in system S. If it is the case then we know that either $y \in Y$ or $y \in \bar{Y}-Y$. If term $t' \to \bar{t}_Y$ is not valid in S then we know that $y \notin Y$.

Example 4.2

We consider the system S from ex. 2.1 and set $Y = \{x_1,x_2\}$. Term $\bar{t}_Y = (ap_1)\cdot(bq_2) + (ap_2)\cdot(bq_3)$ is the upper description of Y and we have $val_S \bar{t}_Y = \{x_1,x_2,x_3\}$.

Case 4.2.2

By lower description of set $Y = \{y_1,\ldots,y_m\}$, $m \geqslant 1$, we mean term $\underline{t}_Y = t_1+\ldots+t_h$, $h \leq m$, where $t_1,\ldots,t_h$ are all the elementary terms such that $val_S t_i \subseteq Y$, $i = 1,\ldots h$.

Observe, that $val_S \underline{t}_Y \subseteq Y$ and $val_S \underline{t}_Y = \{x \in X$ for every $y \in X$ if $(x,y) \in \widetilde{S}$ then $y \in Y\}$.

Set $val_S \underline{t}_Y$ will be called lower approximation of set Y and will be denoted by $\underline{Y}$.

If we wish to know whether object y belongs to set Y, which is given by means of its lower description, then as previously we check the validity in system S of term $t' \to \underline{t}_Y$, where t' is the elementary term such that $y \in val_S t'$. If this term is valid in S, then we know that $val_S t' \subseteq val_S \underline{t}_Y$ and hence $y \in Y$. But if term $t' \to \underline{t}_Y$ is not valid in S, then we cannot answer the membership question, namely we cannot distinguish the case $y \notin Y$ and the case $y \in Y - \underline{Y}$.

Example 4.3

Consider the system S from ex. 2.1 and set $Y = \{x_1,x_2\}$. Term $\underline{t}_Y = (ap_2)\cdot(bq_3)$ is the lower description of Y, and we have $val_S \underline{t}_Y = \{x_2\}$.

In the following we will say that a term is a description of the set of objects whenever it is characteristic description, upper description or lower description of this set.

5. DESCRIPTION OF A CLASSIFICATION OF A SET OF OBJECTS

Assume, we are given system $S = (X,A,V,\rho)$ and partition $Y_1,\ldots,Y_m$, $m \geqslant 2$, of set X, i.e. sets $Y_1,\ldots,Y_m$ are non-empty, pairwise disjoint subsets of set X and $X = Y_1 \cup \ldots \cup Y_m$.

The problem is to find descriptions of sets $Y_1,\ldots,Y_m$. The set of these descriptions will be called the description of the classification.

In this section we describe only those aspects of the problem which follow from the considerations of the previous section. The problem of selection of attributes, which are necessary to describe the partition, will be presented in sections 8, 9, 10.

Case 5.1. Sets $Y_1,\dots,Y_m$ are definable in system S. In this case for each set Y_i, $i = 1,\dots,m$, we proceed in the way, described in section 4 under case 4.1 and we obtain the description of the given partition by means of characteristic descriptions $t_{Y_1},\dots,t_{Y_m}$. Sets $val_S t_{Y_1},\dots, val_S t_{Y_m}$ are non-empty, pairwise disjoint sets and $val_S t_{Y_1} \cup \dots \cup val_S t_{Y_m} = X$.

In this case the membership question can be answered precisely, as it was described in case 4.1.

Example 5.1

We consider the system S from ex. 2.1 and sets $Y_1 = \{x_1,x_3\}$ and $Y_2 = \{x_2,x_4\}$.
We have $t_{Y_1} = (ap_1)\cdot(bq_2)$ and $t_{Y_2} = (ap_2)\cdot(bq_3) + (ap_1)\cdot(bq_1)$.
The family $\{val_S t_{Y_1}, val_S t_{Y_2}\}$ is the partition of set $\{x_1,x_2,x_3,x_4\}$ and it coincides with the partition $\{Y_1,Y_2\}$.

Case 5.2. There is an $i \in \{1,\dots,m\}$ such that set Y_i is not definable in S.

In this case we use case 4.1 to obtain characteristic descriptions of those sets Y_h, $h \in \{1,\dots,m\}$ which are definable in S, and case 4.2 for sets Y_i $i \in \{1,\dots,m\}$ which are not definable in S. Since the construction described in case 4.2 provides the approximate descriptions of sets Y_i, we obtain the approximate description of the given partition. If we choose upper description of Y_i then sets $Y_1,\dots,\overline{Y}_i,\dots,Y_m$ may not be pairwise disjoint. If we choose the lower description of Y_i then set $Y_1 \dots \underline{Y}_i \dots Y_m$ may be properly contained in set X.

It follows that in general the membership question could not be answered precisely, as it was described in section 4 case 4.2.

Example 7.2.

Consider the system S from ex. 2.1 and sets $Y_1 = \{x_1,x_2\}$ and $Y_2 = \{x_3,x_4\}$. Sets Y_1,Y_2 are not definable in S. Consider terms $\overline{t}_{Y_1} = (ap_1)\cdot(bq_2)+(ap_2)\cdot(bq_3) + (ap_1)\cdot(bq_1)$, $\overline{t}_{Y_2} = (ap_1)\cdot(bq_2) + (ap_1)\cdot(bq_1)$.

We have

$$val_S \overline{t}_{Y_1} = \{x_1,x_2,x_3\},$$

$$val_S \overline{t}_{Y_2} = \{x_1,x_3,x_4\}.$$

Hence family $\{val_S \overline{t}_{Y_1}, val_S \overline{t}_{Y_2}\}$ is not the partition of set X.

If we choose terms

$\underline{t}_{Y_1} = (ap_2)\cdot(bq_3)$,

$\underline{t}_{Y_2} = (ap_1)\cdot(bq_1)$,

then $val_S \underline{t}_{Y_1} = \{x_2\}$ and $val_S \underline{t}_{Y_2} = \{x_4\}$, and in this case we also do not obtain the partition of set X.

However, it is easy to see, that both pairs of terms $\overline{t}_{Y_1}, \underline{t}_{Y_2}$ and $\underline{t}_{Y_1}, \overline{t}_{Y_2}$ provide the partitions of set X.

6. SELECTION OF REPRESENTATIVE EXAMPLE

In this section we deal with the problem of defining what is called a representative subset of a set of objects. Given system $S = (X, A, V, \rho)$ and set $Y \subseteq X$, roughly speaking, we wish to find set $Z \subseteq X$, such that if we know a description of set Z, then we know the description of set Y.

Case 6.1. Set Y is definable in system S. Let t be the term which defines set Y. By the representative subset of set $Y = \{y_1, \dots, y_m\}$, $m \geq 1$, we mean a minimal set $Z = \{y_{i_1}, \dots, y_{i_h}\}$ $h \leq m$, $i_1, \dots, i_h \in \{1, \dots, m\}$ such that if $t_1, \dots, t_h$ are elementary terms with the property $y_{i_j} \in val_S t_j$, for all $j = 1, \dots, h$, then term $t \Leftrightarrow (t_1 + \dots + t_h)$ is valid in S. This means that $val_S t = val_S(t_1 + \dots + t_h)$.

In other words the representative subset of a definable set is obtained by taking exactly one element from each elementary set contained in the given set. Hence if Z is the representative subset of definable set Y then $Z \subseteq Y$.

Example 6.1.
Consider the system S from ex. 2.1 and set $Y = \{x_1, x_3, x_4\}$. The representative subsets of Y are sets $\{x_1, x_4\}$ and $\{x_3, x_4\}$. We have $val_S t_Y = val_S t_{\{x_1, x_4\}} = val_S t_{\{x_3, x_4\}}$.

Case 6.2. Set Y is not definable in system S. In the case we have to decide whether we wish to obtain the representative subset of the upper or lower approximation of Y and proceed in the way described in case 6.1.

It follows, that if we construct the representative subset of the upper approximation of set Y then we take single element from each elementary set which has the nonempty intersection with set Y, and if we construct the representative subset of the lower approximation of set Y, then we take single element from each elementary set, which is contained in set Y.

Example 6.2.
Consider the system S from ex. 2.1 and set $Y = \{x_1, x_4\}$. We have $\overline{Y} = \{x_1, x_3, x_4\}$ and $\underline{Y} = \{x_4\}$. The representative subsets of $\overline{Y}$ are sets $\{x_1, x_4\}$ and $\{x_3, x_4\}$. The representative subset of set $\underline{Y}$ is set $\{x_4\}$. We have

$$val_S t_{\{x_4\}} \subseteq val_S t_Y \subseteq val_S t_{\{x_1, x_3, x_4\}}.$$

Let us observe, that if system S is selective, i.e. each elementary set contains exactly one element, then the only representative subset of a set of objects is this set itself.

Example 6.3
Consider the selective system, obtained from the system S of ex. 2.1 by removing object x_3, and set $Y = \{x_1, x_2\}$. The only representative subset of Y is set $\{x_1, x_2\}$. None of sets $\{x_1\}$ and $\{x_2\}$ can be its representative subset since we have

$$val_S t_{\{x_1\}} \subsetneq val_S t_Y \quad \text{and}$$
$$val_S t_{\{x_2\}} \subsetneq val_S t_Y.$$

7. CLASSIFICATION DETERMINED BY REPRESENTATIVE EXAMPLES

In this section we deal with the generalization of the problem discussed in section 6.

Assume, we are given system $S = (X,A,V,\rho)$ and the partition $\{Y_1,\dots,Y_m\}$, $m \geq 2$, of set X. The problem is to find representative subsets of sets $Y_1,\dots,Y_m$. In this section we do not consider those aspects of the problem, which concern the choice of attributes, which are necessary for proper characterization of representative subsets, it will be the subject of the next sections.

Case 7.1. Sets $Y_1,\dots,Y_m$ are definable in S. Consider the representative subsets of sets $Y_1,\dots,Y_m$, obtained as in section 6, case 6.1.
These subsets provide the characteristic description $t_{Y_1},\dots,t_{Y_m}$ of sets $Y_1,\dots,Y_m$ respectively and the partition $\{val_S t_{Y_1},\dots,val_S t_{Y_m}\}$ coincides with partition $\{Y_1,\dots,Y_m\}$.

Example 9.1

We consider the system S from ex. 2.1 and sets
$Y_1 = \{x_1,x_3\}$ and $Y_2 = \{x_2,x_4\}$.
Representative subsets of Y_1 are $\{x_1\}$ and $\{x_3\}$, representative subset of Y_2 is set $\{x_2,x_4\}$. We have $val_S t_{\{x_1\}} = val_S t_{\{x_3\}} = val_S t_{Y_1} = Y_1$ and $val_S t_{\{x_2,x_4\}} = Y_2$.

Case 9.2. Among sets $Y_1,\dots,Y_m$ there is set Y_i which is not definable in S.
If we construct upper approximation $\overline{Y}_i$ of set Y_i, then the resulting family of sets $\{Y_1,\dots,\overline{Y}_i,\dots,Y_m\}$ may not be a partition, since the sets may not be pairwise disjoint.

Example 9.2

Consider sets $Y_1 = \{x_1,x_2\}$, $Y_2 = \{x_3\}$, $Y_3 = \{x_4\}$ from the system S of ex. 2.1. Sets Y_1,Y_2 are not definable in S and set Y_3 is definable in S. Consider sets $\overline{Y}_1 = \{x_1,x_2,x_3\}$ and $\overline{Y}_2 = \{x_1,x_3\}$. As the representative subsets of $\overline{Y}_1$ and $\overline{Y}_2$ we can take sets $\{x_1,x_2\}$ and $\{x_1\}$ respectively. We have $val_S t_{\{x_1,x_2\}} \cap val_S t_{\{x_1,x_3\}} = \{x_1,x_3\}$ and hence terms $t_{\{x_1,x_2\}}$, $t_{\{x_1,x_3\}}$, $t_{\{x_4\}}$ do not describe a partition of set X.

Similarly, if we choose the lower approximation $\underline{Y}_i$ of set Y_i, then the family $\{Y_1,\dots,\underline{Y}_i,\dots,Y_m\}$ may not be a partition, because the union of the sets may not coincide with set X.

Example 7.3

Consider sets Y_1, Y_2, Y_3 from ex. 7.2 and sets $\underline{Y}_1 = \{x_2\}$ and $\underline{Y}_2 = \emptyset$. The representative subsets of $\underline{Y}_1$ and $\underline{Y}_2$ are these sets themselves. Hence we have $val_S t_{\{x_2\}} \cup val_S t_{\emptyset} \cup val_S t_{\{x_4\}} = \{x_2,x_4\} \neq X$.
It follows, that we do not obtain a partition.

8. DEPENDENT ATTRIBUTES

In the previous sections we discussed the method of defining descriptions of sets of objects from the point of view of selectivity or non-selectivity of the given knowledge representation system, and what follows, from the point of view of definability or non-definability of sets of objects. In the following three sections we will consider the problem what attributes in the system are not necessary to describe properly a set of objects. Namely, we show what attributes can be omitted in defining descriptions of given sets, without loosing the adequacy of the description.

In sections 8 and 9 we describe the problem from the semantical point of view and in section 10 we present syntactical methods of reducing of set of attributes.

Given system $S = (X, A, V, \rho)$,
attribute b is said to be dependent on attribute a iff $\widetilde{a} \subseteq \widetilde{b}$. We define dependency relation $\Rightarrow$ on set A as follows:

$$a \Rightarrow b \quad \text{iff} \quad \widetilde{a} \subseteq \widetilde{b}.$$

We extend relation $\Rightarrow$ on the family 2^A of all subsets of set A:

$$B \Rightarrow C \quad \text{iff} \quad \widetilde{B} \subseteq \widetilde{C}.$$

For the sake of simplicity we will write $a \Rightarrow b$, $a \Rightarrow B$ and $B \Rightarrow a$ instead of $\{a\} \Rightarrow \{b\}$, $\{a\} \Rightarrow B$ and $B \Rightarrow \{a\}$ respectively.

The meaning of the dependency relation is the following: holding of condition $B \Rightarrow C$ assures that if a pair of objects cannot be distinguished by means of attributes belonging to set B, then it cannot be distinguished by attributes from set C, in other words values of attributes from sets C are determined by values of attributes from set B.

Example 8.1

Consider the system given by the following table

	a	b	c
x_1	p_1	q_1	r_1
x_2	p_2	q_2	r_3
x_3	p_1	q_3	r_2
x_4	p_2	q_2	r_3
x_5	p_1	q_1	r_4

The partitions generated by relations $\widetilde{a}, \widetilde{b}, \widetilde{c}$ and $\widetilde{a} \cap \widetilde{b}$ are the following:

$\widetilde{a}$ $\{x_1, x_3, x_5\}$, $\{x_2, x_4\}$

$\widetilde{b}$ $\{x_1, x_5\}$, $\{x_3\}$, $\{x_2, x_4\}$

$\widetilde{c}$ $\{x_1\}$, $\{x_3\}$, $\{x_2, x_4\}$, $\{x_5\}$

$\widetilde{a} \cap \widetilde{b}$ $\{x_1, x_5\}$, $\{x_3\}$, $\{x_2, x_4\}$.

We have

$$b \Rightarrow a, \quad c \Rightarrow \{a, b\}, \quad c \Rightarrow a, \quad \{c, b\} \Rightarrow a.$$

A subset B of set A of attributes is said to be dependent in system S if there is set $C \subsetneq B$ such that $\widetilde{C} = \widetilde{B}$. Set B is said to be independent in S if it is not dependent in S.

Example 8.2

Consider system S such that

$$X = \{x_1, x_2, x_3, x_4, x_5\}$$

$$A = \{a, b, c, d\}$$

and assume that attributes generate the following partitions of set X:

$\tilde{a}$ $\{x_1, x_2, x_5\}$ $\{x_3, x_4\}$

$\tilde{b}$ $\{x_1\}$ $\{x_2, x_3, x_4, x_5\}$

$\tilde{c}$ $\{x_1\ x_2\ x_3\ x_4\}$ $\{x_5\}$

$\tilde{d}$ $\{x_1\}$ $\{x_3\ x_4\}$ $\{x_2\ x_5\}$.

Relation $\tilde{S} = \tilde{a} \cap \tilde{b} \cap \tilde{c} \cap \tilde{d}$ provides the following partition:

$\tilde{S}$ $\{x_1\}$ $\{x_2\}$ $\{x_3\ x_4\}$ $\{x_5\}$

Set A is dependent in S because set $B = \{a,b,c\}$ obeys the property $\tilde{B} = \tilde{A}$.

It is easy to check that for sets $C = \{a,c,d\}$ and $D = \{c,d\}$ we also have $\tilde{C} = \tilde{D} = \tilde{A}$.

Given system $S = (X,A,V,\rho)$, we will say that set $B \subseteq A$ is the reduct of set A if B is the minimal set such that $\tilde{B} = \tilde{A}$.

Example 8.3

In system S from ex. 8.2 set D is the reduct of set A.

The corresponding system, obtained from system S by removing all the attributes from set $A - B$ where B is the reduct of A will be called the reduced system for S and will be denoted by S_B. It follows that $S_B = (X, B, \{V_b\}_{b \in B}, \rho \restriction X \times B)$. We will denote the set $\bigcup_{b \in B} V_b$ by V_B.

Observe that since for any reduced system S_B for system S we have $\tilde{S}_B = \tilde{S}$, so system S_B has the same abilities to characterize objects as system S. More exactly, the elementary sets in S_B are the same as in S and hence sets of definable objects in S_B are the same as in S.

Hence, if we are given system S with set A of attributes and we manage to find the reduced system S_B for S such that $B \subsetneq A$, then using language L_{S_B} we can describe the same informations as using language L_S.

9. ATTRIBUTES WHICH ARE REDUNDANT FOR DESCRIBING CLASSIFICATION

Assume, we are given system $S = (X,A,V,\rho)$ and the partition $Z = \{Y_1,\dots,Y_m\}$, $m \geqslant 2$, of set X. The problem is to find the minimal subset of the set A of attributes which is necessary to describe partition Z.

In other words, we are given terms which are descriptions of sets $Y_1,\dots,Y_m$, and we wish to know for which attributes $a \in A$ all descriptors of the form $(a\ p)$, $p \in V_a$, occurring in these terms are redundant.

In this section we explain the solution of the problem from the semantical point of view, and in the next section we present syntactic methods of checking the redundancy of attributes.

To solve the problem, first we should find the reduct of set A of attributes, namely the minimal subset B of set A such that relations $\tilde{S}$ and $\tilde{B}$ coincide. Then, we consider the reduced system S_B and proceed in the following way.

Case 9.1. System S_B is selective.

In this case we try to find subset C of set B which is the minimal set such that each set Y_i, $i = 1,\dots,m$, is the union of some equivalence classes of relation $\tilde{C}$. If set C is a proper subset of set B then we consider system

$S' = (X, C, V_C, \rho \upharpoonright X \times C)$, obtained from system S_B by removing all the attributes from set $B-C$. Let $t'_1, \ldots, t'_m$ be characteristic descriptions of sets $Y_1, \ldots, Y_m$ in the language $L_{S'}$ and let $t_1, \ldots, t_m$ be the characteristic descriptions of these sets in language L_{S_B}. It is easy to see that for each $i = 1, \ldots, m$ term $t_i \leftrightarrow t'_i$ is valid in S_B. Hence in this case terms $t'_1, \ldots, t'_m$, with redundant attributes removed, provide the description of the partition Z.

Case 9.2. System S_B is not selective

In this case for each set Y_i, $i = 1, \ldots, m$, we decide whether we are interested in description of its upper or lower approximation and we consider family $Z' = \{Y'_1, \ldots, Y'_m\}$ of sets such that $Y'_i = \overline{Y}_i$ or $Y'_i = \underline{Y}_i$, $i = 1, \ldots, m$. Then we proceed as in case 8.1.

Example 9.1

Assume, we are given the following system S:

	a	b	c	d	
x_1	2	2	0	2	
x_2	3	1	2	2	
x_3	2	2	1	2	Y_1
x_4	1	1	1	2	
x_5	3	2	0	0	
x_6	3	2	1	2	
x_7	1	2	2	2	Y_2
x_8	0	2	1	2	

Consider relations $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$. Equivalence classes of these relations are the following

$\tilde{a}$ $\{x_1\ x_3\}$ $\{x_2\ x_5\ x_6\}$ $\{x_4\ x_7\}$ $\{x_8\}$

$\tilde{b}$ $\{x_1\ x_3\ x_5\ x_6\ x_7\ x_8\}$ $\{x_2\ x_4\}$

$\tilde{c}$ $\{x_1\ x_5\}$ $\{x_3 x_4 x_6 x_8\}$ $\{x_2\ x_7\}$

$\tilde{d}$ $\{x_1\ x_2\ x_3\ x_4\ x_6\ x_7\ x_8\}$ $\{x_5\}$

Assume, we are given partition $Z = \{Y_1, Y_2\}$ of set X, where $Y_1 = \{x_1, x_2, x_3, x_4\}$ and $Y_2 = \{x_5, x_6, x_7, x_8\}$. It is easy to see that equivalence classes of relation $\tilde{a} \cap \tilde{b}$ cover sets Y_1, Y_2, and none of relations $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ have this property:

$$\tilde{a} \cap \tilde{b} \quad \underbrace{\{x_1\ x_3\}\ \{x_2\}\ \{x_4\}}_{Y_1}\ \underbrace{\{x_5\ x_6\}\ \{x_7\}\ \{x_8\}}_{Y_2}$$

Consider term t of language L_S which is the characteristic description of set Y_2:

t $(a3)\cdot(b2)\cdot(c0)\cdot(d0)$ +

+ $(a3)\cdot(b2)\cdot(c1)\cdot(d2)$ +

+ $(a1)\cdot(b2)\cdot(c2)\cdot(d2)$ +

+ $(a0)\cdot(b2)\cdot(c1)\cdot(d2)$

and term t' of language $L_{S'}$, where

$S' = (\{x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8\}, \{a, b\}, \{0, 1, 2, 3\}, \rho')$

and ρ' is obtained from the table of our system S be removing the columns labelled by c and d:

t' $(a3)\cdot(b2) + (a1)\cdot(b2) + (a0)\cdot(b2)$.

Term $t \leftrightarrow t'$ is valid in system S

Moreover, the following terms are valid in system S:

$(a3)\cdot(b2) + (a1)\cdot(b2) + (a0)\cdot(b2) \leftrightarrow$

$\leftrightarrow ((a3) + (a1) + (a0)) \cdot (b2)$,

$(a3) + (a1) + (a0) \leftrightarrow -(a2)$.

Hence term $t \leftrightarrow -(a2)\cdot(b2)$ is valid in S. So term $-(a2)\cdot(b2)$ is the description of set Y_2. Since $Y_1 = -Y_2$, term $(a2) + -(b2)$ is the description of set Z_1.

10. INDUCTIVE INFERENCE METHODS OF REDUCING ATTRIBUTES

In this section we deal with the following problem. Given system $S = (X,A,V,\rho)$, language L_S and term t of L_S, we are going to find term t' such that term $t \leftrightarrow t'$ is valid in S and term t' contains less descriptors than term t. Without the loss of generality we can assume that term t is in the complete form, that is it has the form of a union of some elementary terms. We assume that each elementary term occurring in t has non-empty value in S. It follows from the considerations of the previous sections, that we cannot remove from set t none of elementary terms as a whole, because it leads to the change of its value. If we remove some descriptors, then we obtain term t' such that $val_S t \subseteq val_S t'$. This means that term $t \rightarrow t'$ is valid in S and moreover, this validity follows from the logical laws only. Hence, from the point of view of logical inference, our task is to find term t' which is inductively implied by term t and the axioms of system S.

There are many papers concerning systems of inductive inference: 1,2,4,5,6,9,10 .
In this section we present some inductive inference systems for language L_S. We use the ideas of inductive resolution given in Morgan [5].

In section 2 we defined special terms, called literals, namely by a literal we mean a descriptor or the complement of a descriptor. A term which is an intersection of a finite number of literals will be called a clause.

It follows that an elementary term is a clause and by Fact 2.7 each term of language L_S can be replaced by the term which is a union of clauses. Given term t and set Z of terms in the disjunctive normal form let $Cl(t)$ and $Cl(Z)$ denote the set of all clauses occurring in t and in all the terms from set Z, respectively.

10.1. Inductive inference system I1

System I1 is based on the following rule of inductive inference

$$R \quad \frac{t\cdot(a\ p)\ ,\ \ t'\cdot-(a\ p)}{t\cdot t}$$

where t and t' are clauses,

$(a\ p) \in D_S$.

For a set Z of clauses, let $R(Z)$ denote the set containing set Z and closed with respect to rule R,

The following fact states that rule R is a kind of a falshood preserving rule.

Fact 10.1 if $val_S(t\cdot(a\ p)) = \emptyset$ and $val_S(t'\cdot-(ap)) = \emptyset$ then $val_S(t\cdot t') = \emptyset$.

Let Ax_S be the set consisting of term t_S and all the terms $t_{(a\ p)}$ for $(ap) \in D_S$. The set of complements of all the terms from set Ax_S will be denoted by $CoAx_S$.

Fact 10.2. For any terms t, t' in disjunctive normal form, if $Cl(t) \subseteq R(Cl(CoAx_S \cup \{t\}))$ then term $t' \rightarrow t$ is valid in S.

Due to this fact if we are given the set of clauses taken from the description t of a given set of objects and from the complements of the axioms of system S, then we can use rule R to derive clauses, which are the possible hypotheses to account term t. Theorem 10.2 assures that each union of a finite number of the produced clauses is the term t' such that $val_S t' \subseteq val_S t$. Our aim is to find among these terms the term which is minimal with respect to the number of attributes and provides the description of set $val_S t$.

It follows from considerations of sections 3, 4, 5, 6, 7 that none of the elementary terms occurring in term t can be dropped as a whole. Hence term t' have to consist of clauses $t_1,\dots,t_n$, $n \geq 1$ such that for each elementary term t_e from term t, there is an $i \in \{1,\dots,n\}$ such that $t_e \to t_i$ is valid in S. It is advisable to choose clauses $t_1,\dots,t_n$ in such a way, that for a certain attribute $a \in A$ none of descriptors (ap) for $p \in V_a$ occur in them. If we manage to do so, then attribute a is redundant with respect to the description of set $val_S t$.

Example 10.1

Consider the system S

	a	b
x_1	1	0
x_2	1	2
x_3	0	1
x_4	2	1
x_5	0	2

and term t = (a1)·(b0) + (a1)·(b2) + (a0)·(b1).

We have $val_S t = \{x_1, x_2, x_3\}$.

The clauses corresponding to term t are the following

1 (a1)·(b0)

2 (a1)·(b2)

3 (a0)·(b1)

In the following we list some of the clauses obtained from terms from the set $CoAx_S$

4	(a0)·(a1)	$-t_{(a0)}$
5	(a1)·(a2)	$-t_{(a2)}$
6	(b1)·(b2)	$-t_{(b2)}$
7	(b0)·(b1)	$-t_{(b0)}$
8	(b0)·(b2)	
9	−(b0)·− (b1)·− (b2)	
10	−(a0)·− (a2)·− (b0)·− (b2)	$-t_S$
11	−(a0)·− (a1)·− (b1)	
12	−(a1)·− (b1)·− (b2)	

Using rule R we produce the following clauses

13	(a1)·− (b1)·− (b2)	1,9
14	−(b1)·− (b2)	12,13
15	(b0)·− (b2)	7,14
16	(b0)	8,15
17	(a1)·− (b0)·− (b1)	2,9
18	(a1)·− (b1)	1,17
19	−(a0)·− (b1)	11,18
20	−(a0)·(b2)	6,19
21	−(a0)·− (a2)·− (b0)	10,20
22	−(a0)·− (a2)	16,21
23	−(a0)·(a1)	5,22
24	(a1)	4,23

Using these clauses we form term

t' = (a1) + (a0)·(b1).

which is the minimal term satisfying the desired conditions:

$val_S t = val_S t'$,

$\models_S$ (a1)·(b0) → (a1),

$\models_S$ (a1)·(b2) → (a1),

$\models_S$ (a0)·(b1) → (a0)·(b1).

Hence we can simplify term t not changing its value, but in our system there are no redundant attributes with respect to the description of set $\{x_1, x_2, x_3\}$.

10.2. Inference system I2

Observe that if we are given two clauses t_1 and t_2 such that term $t_1 \leftrightarrow t_2$ is valid in S, then obviously $val_S t_1 = \emptyset$ iff $val_S t_2 = \emptyset$. Hence we can define the family of falshood preserving rules:

$$R^{(ap)} \quad \frac{t \cdot (ap)}{t \cdot -(ap_1) \cdot \ldots \cdot -(ap_n)}$$

$$R_{(ap)} \quad \frac{t \cdot -(ap_1) \cdot \ldots \cdot -(ap_n)}{t \cdot (ap)}$$

where t is a clause, $a \in A$ and $p, p_1, \ldots, p_n$ are all the different values of attribute a.

Let $R_1(Z)$ denote the set of clauses containing set Z of clauses and closed with respect to the rules R, $R^{(ap)}$, $R_{(ap)}$ for all $(a\ p) \in D_S$.

Fact 10.3. For any terms t and t' in disjunctive normal form if $Cl(t') \subseteq R_1(Cl(\{-t_S, t\}))$ then term $t' \to t$ is valid in S.

Example 10.2

Consider the system given in example 10.1 and term $t = (a0)\cdot(b1) + (a2)\cdot(b1) + (a0)\cdot(b2)$. We have $val_S t = \{x_3, x_4, x_5\}$. The clauses from term t and some of the clauses from term $-t_S$ are listed below:

1	(a0)·(b1)	t
2	(a2)·(b1)	t
3	(a0)·(b2)	t
4	−(a0)·−(a1)·−(b1)	$-t_S$
5	−(a1)·−(b1)·−(b2)	$-t_S$

Using the admitted rules we produce the following clauses:

6	−(a1)·−(a2)·(b1)	1 $R^{(a0)}$
7	−(a1)·−(a2)·(b2)	2 $R^{(a0)}$
8	−(a1)·−(a2)·−(b1)	5,7 R
9	−(a1)·− (a2)	6,8 R
10	(a0)	9 $R_{(a0)}$
11	−(a0)·−(a1)·(b1)	2 $R^{(a2)}$
12	−(a0)·−(a1)	4,11 R
13	(a2)	12 $R_{(a2)}$

It is easy to see that term

$t = (a0) + (a2)$

satisfies the desired conditions. Hence attribute b is redundant with respect to the description of set $\{x_3, x_4, x_5\}$.

10.3. Inductive inference system I3

In this system we are not restricted to clauses, we can start from an arbitrary set of terms. We will use rules having the following schemes:

$R_{\equiv}$ $\dfrac{t_1}{t_2}$ where t_1, t_2 are terms such that $\models_S t_1 \Longleftrightarrow t_2$

R_+ $\dfrac{t_1 + t_2}{t_1}$ $\dfrac{t_1 + t_2}{t_2}$

where t_1, t_2 are arbitrary terms

R $\dfrac{t \cdot (ap),\ t' \cdot -(ap)}{t \cdot t'}$

where t and t' are clauses and $(a\ p) \in D_S$.

The following rules are examples of scheme $R_{\equiv}$

$\dfrac{-(t_1 + t_2)}{-t_1 \cdot -t_2}$ de Morgan rule

$\dfrac{t_1 \cdot (t_2 + t_3)}{t_1 \cdot t_2 + t_1 \cdot t_3}$ distributivity rule

R_C $\dfrac{((ap_{i_1}) + \dots + (ap_{i_l})) \cdot t}{-(ap_{j_1}) \cdot \dots \cdot -(ap_{j_k}) \cdot t}$

where t is an arbitrary term,

$p_{i_1}, \dots, p_{i_l}$, $p_{j_1}, \dots, p_{j_k}$ are all the different values of attribute a,

$k, l = 1, \dots, \text{card } V_a - 1$,

$l + k = \text{card } V_a$.

Rules $R^{(ap)}$ and $R_{(ap)}$ are the special cases of rule R_C.
All the above rules obey the falshood preserving property, namely if premises of a rule have the value $\not\subset$ then the conclusion has the value $\not\subset$ too.

For a set Z of terms let $R_2(Z)$ denote the set of terms containing set Z and closed with respect to all the rules of the form $R_{\equiv}$, R_+, R.

Fact 10.3. For any terms t, t' of language L_S if $t' \in R_2(\{-t_S, t\})$ then term $t' \rightarrow t$ is valid in S.

Example 10.3

We consider the system S given in example 10.1 and term $t = (a1)\cdot(b0) + (a1)\cdot(b2) + (a0)\cdot(b2)$, which is the description of set $\{x_1, x_2, x_5\}$. We start with terms t and $-t_S$ and we produce the following terms:

1. $(a1)\cdot(b0) + (a1)\cdot(b2) + (a0)\cdot(b2)$ — t
2. $-(a0)\cdot -(a1)\cdot -(b1)$ — from $-t_S$ by the Morgan rule, distributivity rule and R_+
3. $-(a1)\cdot -(b1)\cdot -(b2)$
4. $(a1)\cdot(b0)$ — 1 R_+
5. $(a0)\cdot(b2)$ — 1 R_+
6. $((a1) + (a0))\cdot(b2)$ — 1 distributivity rule, R_+
7. $-(a2)\cdot(b2)$ — 6 R_C
8. $(a1)\cdot -(b1)\cdot -(b2)$ — 4 R_C
9. $-(a1)\cdot -(a2)\cdot -(b1)$ — 3,7 R
10. $(a1)\cdot -(a2)\cdot -(b1)$ — 7,8 R
11. $-(a2)\cdot -(b1)$ — 9,10 R
12. $-(a2)\cdot((b0) + (b2))$ — 11 R_C
13. $-(a0)\cdot -(a1)\cdot((b0) + (b2))$ — $2R_C$
14. $(a2)\cdot((b0) + (b2))$ — 13 R_C
15. $(b0) + (b2)$ — 12, 14 R

Hence term (b0) + (b2) provides the description of set $\{x_1, x_2, x_5\}$ and attribute a is redundant with respect to this description.

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