PRACE IPI PAN • ICS PAS REPORTS

Ewa Orłowska, Zdzisław Pawlak

Expressive power of knowledge representation systems

432

April 1981

WARSZAWA

INSTYTUT PODSTAW INFORMATYKI POLSKIEJ AKADEMII NAUK
INSTITUTE OF COMPUTER SCIENCE POLISH ACADEMY OF SCIENCES
00-901 WARSAW, P.O. Box 22, POLAND

Ewa Orłowska, Zdzisław Pawlak

EXPRESSIVE POWER OF KNOWLEDGE REPRESENTATION SYSTEMS

432

Warsaw, March 1981

Mailing address: Ewa Orłowska
Zdzisław Pawlak
Institute of Computer Science
Polish Academy of Sciences
00-901 Warsaw,
PKiN, P.O. Box 22
POLAND

P r i n t e d   a s   a   m a n u s c r i p t

N a   p r a w a c h   r ę k o p i s u

Nakład 700 egz. Ark. wyd. 1,40; ark. druk. 2,00. Papier offset. kl. [illegible], 70 g, 70 x 100. Oddano do druku w marcu 198[illegible] [illegible]D.N. Zam. nr 225/0/81 L-20

Abstract • Содержание • Streszczenie

Attribute based knowledge representation systems are studied in the paper from the point of view of their expressive power represented by the indiscernibility relation. The information logic is introduced, providing a formal method to compare systems with respect to the expressive power.

Экспрессионная мощность систем репрезентации знаний

В работе исследуются аттрибутные системы репрезентации знаний. Вводится логика, обеспечивающая возможность сравнения систем с точки зрения их способности определения информации.

Moc ekspresyjna systemów reprezentowania wiedzy

W pracy badane są atrybutowe systemy reprezentowania wiedzy. Wprowadzona jest logika pozwalająca na porównywanie systemów z punktu widzenia definiowalności przez nie informacji.

Introduction

In the paper knowledge representation systems introduced by Pawlak [5] are studied from the point of view of their expressive power. We treat knowledge representation system as an abstract model of a knowledge about a perceived part of reality. The basic component of the system is a set of objects, and an object is anything which we want to store information about. Objects are treated in the system as a whole, however they need not be atomic or undivisible. Knowledge about objects is provided by means of attributes and attribute values, given in the system. Usually the properties of objects expressed by means of attributes of the system are not sufficient to distinguish all objects, the same property could correspond to different objects. As a consequence, by using the knowledge given by the system, we can define precisely only some sets of objects. Another sets of objects can be defined only approximately. Hence the ability of a system to characterize objects is limited. The term "expressive power of a system" reflects this ability in a sense.

In part I of the paper we consider a family of systems based on the same set of objects, and we introduce formal tools enabling us to define and to investigate such notions as set of objects precisely defined in a system, set of objects approximately defined in a system etc. We also define relations between systems which provide a tool to compare systems with respect to their expressive power.

In part II we define the information logic IL intended to be a formal tool to formulate and to infer facts concerning expressive power of systems.

The material presented in the paper is based on ideas of Pawlak [6].

Another approaches to the problem of precise and approximate definability of informations are presented, among others, in Hajek [1] and Moore [4].

In [1] the distinction is made between the observational stage in defining information and the data definition stage. Informations given by means of observational statements lead to a hypothesis and data definition statements lead to a precise information.

In [4] modal logic S5 is used to formalize the process of medical decision making. The authors idea is based on the fact that usually it is not possible to make the precise and generally valid decision. We are only able to make decisions which are true with respect to a level of certainty.

Part I. Definability of knowledge

1. Knowledge representation system

The notion of a knowledge representation system was introduced in Pawlak [5]. Following this paper we present in this section the definition of the knowledge representation system. The system is intended to be a mathematical model of data bases, information systems or any devices which provide a way of representing knowledge about some objects.

The basic component of the system is a non-empty set of objects, for example human-beings, books etc. The knowledge contained in the system is expressed through assignment of some characteristic features to the objects e.g. human-beings can be characterized by means of sex and age, books by means of title and author's name etc. These features are represented by attributes and the values of attributes. Hence the set A of attributes and for each $a \in A$ the set V_a of values of attribute a are components of the system. The function assigning attribute values to objects is the least component of the system.

We now present the formal definition of a knowledge representation system.

The knowledge representation system is a system

$$S = (X, A, \{V_a\}_{a \in A}, \rho)$$

where X is a non-empty set, whose elements are called objects,

A is a non-empty set, whose elements are called attributes,

V_a is a non-empty set whose elements are called values of attribute a,

ρ is a total function from set $X \times A$ into set $V = \bigcup_{a \in A} V_a$ such that $\rho(x,a) \in V_a$ for each $x \in X$ and each $a \in A$.

Example 1.1

Consider the following system:

$X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$

$A = \{\text{sex, age}\}$.

$V_{sex} = \{\text{male, female}\}$

$V_{age} = \{\text{young, medium, old}\}$

The function ρ is defined by means of the following table:

	sex	age
x_1	male	young
x_2	male	medium
x_3	female	old
x_4	male	medium
x_5	female	old
x_6	female	young

2. Definable informations

Given system $S = (X, A, V, \rho)$, by an information in S we mean any subset of set X of objects. We define the family $\{\tilde{a}\}_{a \in A}$ of equivalence relations on set X in the following way

$$(x,y) \in \tilde{a} \quad \text{iff} \quad \rho(x,a) = \rho(y,a)$$

This means that two objects belong to relation $\tilde{a}$ just in case they cannot be distinguished by means of attribute a.

Given subset B of set A of attributes, the relation $\bigcap_{b \in B} \tilde{b}$ will be denoted by $\tilde{B}$ and relation $\tilde{A}$ determined by the whole set A of attributes will be denoted by $\tilde{S}$. Relation $\tilde{S}$ is called indiscernibility relation, determined by system S. According to the given definition two objects belong to relation $\tilde{C}$, for $C \subseteq A$, if they are undistinguishable with respect to all the attributes from set C.

Example 2.1

Consider system S given in example 1.1.

We have

$(x_1, x_4) \in \widetilde{sex}$

$(x_1, x_6) \in \widetilde{young}$

$(x_2, x_4) \in \tilde{S} = \widetilde{\{\text{sex, young}\}} = \widetilde{sex} \cap \widetilde{young}$

$(x_3, x_5) \in \tilde{S}$.

For any system S equivalence classes of relation $\tilde{S}$ will be called elementary informations in S. Hence relation $\tilde{S}$ provides the classification of objects according to the knowledge contained in the system. Objects belonging to the elementary informations are undistinguishable with respect to this knowledge.

Example 2.2

The elementary informations in the system from example 1.1 are the following:

$\{x_1\}$ $\{x_2, x_4\}$ $\{x_3, x_5\}$ $\{x_6\}$.

We will say that information $Y \subseteq X$ is definable in system S if Y is either the empty set or the union (finite or infinite) of some elementary informations.

Example 2.3

Information $\{x_1, x_2, x_4, x_6\}$ is definable in the system from example 1.1, and informations $\{x_1, x_3\}$, $\{x_2, x_5\}$ are not definable in this system.

Observe, that set X and the empty set are definable informations. Moreover, the family of definable informations is closed under union, intersection and complement. Hence we have the **following** theorem.

Fact 2.1. The family of all informations definable in system S is a Boolean algebra.

We will denote this algebra by $\mathcal{B}_S$. Clearly, this algebra is the subalgebra of the algebra $P(X)$ of all subsets of set X.

Fact 2.2. Elementary informations are atoms in the algebra $\mathcal{B}_S$.

A system S is said to be selective if all the elementary sets are **singletons.**

Fact 2.3. If system S is selective, then every information in S is definable.

Obviously, the converse theorem is also true.

3. Approximations of informations

In this section we define two operations on informations, namely upper approximation operation and lower approximation **operation.** These operations enable us to assign to any information the pair of informations, which are definable in a given system and which, roughly speaking, are close enough to the given information.

By an upper approximation $\overline{S}Y$ of an information Y in system S we mean the least information definable in S and containing set Y.

By a lower approximation $\underline{S}Y$ of an information Y in system S we mean the greatest information definable in S and contained in set Y.

The following facts follow immediately from the given definitions.

Fact 3.1. For any system $S = (X, A, V, \rho)$ and for any information $Y \subseteq X$ the following conditions are satisfied:

(a) $\overline{S}Y = \{x \in X:$ for every $y \in X$ if $(x,y) \in \tilde{S}$ then $y \in Y\}$

(b) $\underline{S}Y = \{x \in X:$ there is an $y \in X$ such that $(x,y) \in \tilde{S}$ and $y \in Y\}$.

Fact 3.2. If information Y is definable in system S then $Y = \overline{S}Y = \underline{S}Y$.

Example 3.1

Consider system S given in example 1.1 and informations $Y_1 = \{x_1, x_2\}$, $Y_2 = \{x_1, x_3, x_6\}$, $Y_3 = \{x_2, x_4, x_6\}$. We have

$\overline{S}Y_1 = \{x_1, x_2, x_4\}$ $\qquad \underline{S}Y_1 = \{x_1\}$

$\overline{S}Y_2 = \{x_1, x_3, x_5, x_6\}$ $\qquad \underline{S}Y_2 = \{x_2, x_6\}$

$\overline{S}Y_3 = \underline{S}Y_3 = Y_3 = \{x_2, x_4, x_6\}$.

In the following we list some properties of upper approximation operation and lower approximation operation.

For any system $S = (X, A, V, \rho)$ and for any informations $Y, Z \subseteq X$ the following conditions are satisfied:

Fact 3.3

(a) $\underline{S}(Y \cap Z) = \underline{S}Y \cap \underline{S}Z$,

(b) $\underline{S}Y \subseteq Y$,

(c) $\underline{S}\underline{S}Y = \underline{S}Y$,

(d) $\underline{S}X = X$.

Fact 3.4

(a) $\bar{S}(Y \cup Z) = \bar{S}Y \cup \bar{S}Z$,

(b) $Y \subseteq \bar{S}Y$,

(c) $\bar{S}\bar{S}Y = \bar{S}Y$,

(d) $\bar{S}\emptyset = \emptyset$.

It follows from these two facts that the algebra $P(X)$ with additional operations $\bar{S}$ and $\underline{S}$ is a topological field of sets, where $\bar{S}$ is a closure operation and $\underline{S}$ is an interior operation.

Fact 3.5

(a) $\bar{S}Y = -(\underline{S}(-Y))$,

(b) $\underline{S}Y = -(\bar{S}(-Y))$.

Fact 3.6

(a) if $Y \subseteq Z$ then $\bar{S}Y \subseteq \bar{S}Z$,

(b) if $Y \subseteq Z$ then $\underline{S}Y \subseteq \underline{S}Z$.

4. Comparing of knowledge representation systems

We consider a family $\mathcal{S} = \{S_i\}_{i \in I}$ of knowledge representation systems of the form $S_i = (X, A_i, V_i, \rho_i)$. In all the systems set X of objects is the same and we wish to compare the systems from the point of view of their ability to define informations from the family $P(X)$ of sets.

System S_1 is said to be more accurate than system S_2 $(S_1 \leq S_2)$ iff $\tilde{S}_1 \subseteq \tilde{S}_2$. This means that the indiscernibility relation of System S_1 determines the finer partition of set X than the indiscernibility relation of system S_2. It follows that approximations of any information in system S_1 are closer to this information than in system S_2. Namely, the following theorems hold.

Fact 4.1. For any systems $S_1, S_2 \in \mathcal{S}$ and for any set $Y \subseteq X$ the following conditions are equivalent:

(a) $S_1 \leq S_2$

(b) $\bar{S}_1 Y \subseteq \bar{S}_2 Y$.

Proof:

Let $[x]_i$, $i = 1,2$ denote the equivalence class with respect to relation $\tilde{S}_i$, determined by object $x \in X$. If $\tilde{S}_1 \subseteq \tilde{S}_2$ then for any $x \in X$ we have $[x]_1 \subseteq [x]_2$, and hence condition (b) holds.

Let us now suppose that for any set $Y \subseteq X$ we have $\bar{S}_1 Y \subseteq \bar{S}_2 Y$ and non $S_1 \leq S_2$. Hence, there is a pair (x,y) of objects such that $(x,y) \in \tilde{S}_1$ and $(x,y) \notin \tilde{S}_2$. Consider set $Z = \{x\}$. We have $y \in \bar{S}_1\{x\}$ and $y \notin \bar{S}_2\{x\}$, which contradicts condition (b).

Fact 4.2. For any systems $S_1, S_2 \in \mathcal{S}$ and for any set $Y \subseteq X$ the following conditions are equivalent:

(a) $S_1 \leq S_2$

(b) $\underline{S}_2 Y \subseteq \underline{S}_1 Y$.

The proof follows from Fact 3.5 and Fact 4.1.

The given theorems enable us to identify the expressive power of a system with the indiscernibility relation of the system. Relation $\leq$ can be considered as a formal tool to compare expressive power of systems.

In the following we present some simple facts concerning relation $\leq$.

Fact 4.3. For any systems $S_1 = (X, A_1, V_1, \rho_1)$ and $S_2 = (X, A_2, V_2, \rho_2)$ if $A_1 \subseteq A_2$ then $S_2 \leq S_1$.

Fact 4.4. If $S_1 \leq S_2$ then algebra $\mathcal{B}_{S_2}$ is a subalgebra of algebra $\mathcal{B}_{S_1}$.

Obviously relation $\leq$ is an order in the family $\mathcal{S}$. Selective systems play the special role in this order.

Fact 4.5. Selective systems are minimal elements in the family $\mathcal{S}$ ordered by **relation** $\leq$.

Example 4.1.

Consider the following systems

S_1		a	b
	x_1	p_1	q_1
	x_1	p_2	q_2
	x_3	p_1	q_1
	x_4	p_2	q_1
	x_5	p_2	q_1

S_2		a	c	d
	x_1	p_1	r_1	s_1
	x_2	p_2	r_2	s_2
	x_3	p_1	r_2	s_1
	x_4	p_2	r_2	s_1
	x_5	p_2	r_2	s_1

The indiscernibility relations of these systems generate the following elementary informations:

$\tilde{S}_1 : \{x_1, x_3\} \quad \{x_2\} \quad \{x_4, x_5\}$

$\tilde{S}_2 : \{x_1\} \quad \{x_2\} \quad \{x_3\} \quad \{x_4, x_5\}$.

We have $S_2 \leq S_1$.

Consider information $Y = \{x_1, x_4\}$. We have

$\bar{S}_1 Y = \{x_1, x_3, x_4, x_5\}$,

$\bar{S}_2 Y = \{x_1, x_4, x_5\}$,

$\underline{S}_1 Y = \emptyset$,

$\underline{S}_2 Y = \{x_1\}$.

Part II. Information logic IL

The logic considered in this part is intended to provide a formal method for comparing the expressive power of knowledge representation systems. According to the considerations of section 4 the expressive power of a system is represented by the indiscernibility relation of the system and a system S_1 is considered to be more expressive than a system S_2 if indiscernibility relation $\tilde{S}_1$ is contained in indiscernibility relation $\tilde{S}_2$.

We define formalized language which enables us to formulate facts concerning informations in various knowledge representation systems.
We also give the deductive structure to the language and hence we will be able to recognize valid facts or to infer facts from given facts.

In particular using logic IL we can axiomatize the class of selective systems. We also can define the formula, depending on a pair S_1, S_2 of systems such that it is valid for S_1 and S_2 if and only if $S_1 \leq S_2$ holds.

5. Formalized language of logic IL

Expressions of the logic will represent informations, i.e. sets of objects. These expressions are built up from atomic expressions, i.e. variables by means of operations corresponding to set-theoretical operations and the upper approximation operation determined by a system and the lower approximation operation determined by a system. To construct expressions of the logic we use the following symbols:

symbols $p, q, p_1, q_1, \ldots$ of information variables
symbols $R, R_1, R_2, \ldots$ of indiscernibility relations variables,
symbols $\neg$, $\overline{R}$, $\underline{R}$ of unary information operations,
symbols $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$ of binary information operations,
symbol $\cap$ of binary relation operation.

Let Var In and Var Rel denote the set of all information variables and the set of all relation variables respectively.

The set Rel of relation expressions is the least set containing set Var Rel and closed with respect to the operation $\cap$.

The set For of all expressions (formulas) of the logic is the least set satisfying the following conditions:

Var In $\subseteq$ For,
if $A, B \in$ For then $\neg A$, $A \vee B$, $A \wedge B$, $A \rightarrow B$, $A \leftrightarrow B \in$ For,
if $R \in$ Rel and $A \in$ For then $\overline{R}A$ and $\underline{R}A \in$ For.

The expressions of the form $\neg A$, $A \vee B$ and $A \wedge B$ are intended to represent the complement of an information represented by A, and the union and intersection of informations represented by A and B respectively. The expression $A \rightarrow B$ represent the union of the complement of A and B. The expression $A \leftrightarrow B$ represents the intersection of informations given by $A \rightarrow B$ and $B \rightarrow A$. The expression $\overline{R}A$ represents the upper approximation with respect to indiscernibility relation R of the information given by A. Similarly the expression $\underline{R}A$ represents the lower approximation with respect to indiscernibility relation R of the information given by A.

6. Semantics of the language of logic IL

We define the meaning of expressions of our logic by means of the notion of a model. By a model we mean a triple

$$M = (X, m, v)$$

where X is a non-empty set, treated as the universe of informations,

m is a meaning function which assigns to every relation variable an equivalence relation on set X, and satisfies the condition $m(R_1 \cap R_2) = m(R_1) \cap m(R_2)$,

v is a function from set Var In into set $P(X)$.

By induction with respect to the length of a formula we define the notion of a satisfiability of formulas in a model. We will say that a formula A is satisfied in model M by object $x \in X$ (M, x sat A) iff the following conditions hold :

M, x sat p iff $x \in v(p)$ for $p \in$ VarIn,
M, x sat $\neg A$ iff non M, x sat A,
M, x sat $A \vee B$ iff M, x sat A or M, x sat B,
M, x sat $A \wedge B$ iff M, x sat A and M, x sat B,

M,x sat $A \to B$ iff non M,x sat A or M,x sat B,

M,x sat $A \leftrightarrow B$ iff M,x sat $(A \to B)$ and M,x sat $(B \to A)$,

M,x sat $\bar{R}A$ iff there is an $y \in X$ such that

$(x,y) \in m(R)$ and M,y sat A,

M,x sat $\underline{R}A$ iff for every $y \in X$ if $(x,y) \in m(R)$

then M,y sat A.

We will say that set Γ of formulas is satisfied in model M by object x (M,x sat Γ) if for each formula $A \in \Gamma$ we have M,x sat A.

Given model M, let in_M be the function from set For into set P(X) such that

$$in_M(A) = \{x \in X : M,x \text{ sat } A\}.$$

We will call this set information represented by A in model M.

Fact 6.1. The following conditions are satisfied:

(a) $in_M(p) = v(p)$ for $p \in VarIn$

(b) $in_M(\neg A) = -in_M(A)$

(c) $in_M(A \vee B) = in_M(A) \cup in_M(B)$

(d) $in_M(A \wedge B) = in_M(A) \cap in_M(B)$

(e) $in_M(A \to B) = -in_M(A) \cup in_M(B)$

(f) $in_M(A \leftrightarrow B) = in_M(A \to B) \cap in_M(B \to A)$

(g) $in_M(\bar{R}A) = \{x \in X$: there is an $y \in X$ such that $(x,y) \in m(R)$ and $y \in in_M A\}$

(h) $in_M(\underline{R}A) = \{x \in X$: for every $y \in X$ if $(x,y) \in m(R)$ then $y \in in_M A\}$.

According to the given semantics, if we assume that relation m(R) for $R \in Rel$ represents an indiscernibility relation of a knowledge representation system with the set X of objects, then informations represented by formulas $\bar{R}A$ and $\underline{R}A$ correspond to the upper and the lower approximation of information represented by A respectively.

We will say that formula A is valid in model M ($\models_M A$) iff for every $x \in X$ we have M,x sat A.

Fact 6.2. The following conditions are equivalent

(a) $\models_M A$

(b) $in_M(A) = X$.

Example 6.1.

Consider the system given by the following table

	a	b
x_1	u_1	t_2
x_2	u_2	t_3
x_3	u_1	t_2
x_4	u_1	t_1

The elementary sets in this system are the folowing $\{x_1,x_3\}$ $\{x_2\}$ $\{x_4\}$.

Consider models $M_i = (M,m,v_i)$, i = 1,2,3 such that

$X = \{x_1,x_2,x_3,x_4\}$

$m(R) = \widetilde{\{a,b\}}$

$v_1(p) = \{x_1,x_2\}$

$v_2(p) = \{x_1\}$ $v_2(q) = \{x_4\}$

$v_3(p) = \{x_2\}$.

We have

$in_{M_1} \overline{R}p = \{x_1, x_2, x_3\}$

$\models_{M_2} \underline{R}p \to q$

$\models_{M_3} \overline{R}p \leftrightarrow \underline{R}p$

M_3, x_1 sat $\neg p$.

We will say that formula A is valid ($\models A$) if it is valid in all models. Many examples of valid formulas will be given in the following sections. A set Γ of formulas is said to be valid in model M ($\models_M \Gamma$) if every formula $A \in \Gamma$ is valid in M. A formula A is satisfiable if M,x sat A for some model $M = (X,m,v)$ and for some object $x \in X$. A set Γ of formulas is satisfiable if there is a model $M = (X,m,v)$ and an object $x \in X$ such that M,x sat A for every $A \in \Gamma$. A formula A is a semantical consequence of a set Γ of formulas ($\Gamma \models A$) if for every model $M = (X,m,v)$ and for every $x \in X$ we have M,x sat A whenever M,x sat Γ.

7. System properties expressible in logic IL

In this section we list some formulas which express properties of knowledge representation systems or properties of informations in a system.

Fact 7.1. If formula

$$A \to B$$

is valid in a model M then $in_M A \subseteq in_M B$.

Fact 7.2. If formula

$$A \leftrightarrow B$$

is valid in a model M then $in_M A = in_M B$.

These facts follow immediately from the definition of satisfiability.

Fact 7.3. The following conditions are equivalent:

(a) for each $A \in$ For formula

$$\overline{R}A \to \underline{R}A$$

is valid in a model $M = (X,m,v)$,

(b) any system with set X of objects and with indiscernibility relation $m(R)$ is selective.

The above formulas say that for any information its upper approximation is contained in its lower approximation. But by using Fact 3.3(b) and Fact 3.4(b) we have $\underline{R}A \to \overline{R}A$ is valid in any model.

Hence, by Fact 7.2 for any information its upper and lower approximations in M coincide.

Fact 7.4. If for any $A,B \in$ For formula

$$\overline{R}(A \wedge B) \wedge \overline{R}(A \wedge \neg B)$$

is valid in a model $M = (X,m,v)$ then for any object $x \in X$ there is at least one object $y \in X$ such that $x \neq y$ and x,y cannot be distinguished in the system with indiscernibility relation $m(R)$.

Observe, that if the formula is valid in model M then for any object $x \in X$ there are objects y_1 and y_2 such that $y_1 \in in_M A$, $y_1 \in in_M B$, $(x,y_1) \in m(R)$, $y_2 \in in_M A$, y_2 $in_M \neg B$ and $(x,y_2) \in m(R)$. Since $m(R)$ is an equivalence relation we.

possibly have $y_1 = x$ or $y_2 = x$ but y_1 and y_2 are separated by the information represented by B.

Fact 7.5. If for any $A,B \in For$ formula

$$\overline{R}(A \wedge B) \wedge \overline{R}(A \wedge \neg B) \to \underline{R}A$$

is valid in a model $M = (X,m,v)$ then for any object $x \in X$ there is exactly one object $y \in X$ which does not equal x and such that x and y cannot be distinguished in the system with indiscernibility relation $m(R)$.

The formula on the left hand side of operation $\to$ guarantees the existence of an object y satisfying the conditions mentioned in Fact 7.3. The formula on the right hand side of $\to$ assures that this object is the only one satisfying these conditions.

It is easy to see that in the similar way to that followed in Fact 7.4 and Fact 7.5 we can define formulas which assure, that in a system given by the indiscernibility relation any object there is at least or exactly n, $n \geq 1$ objects distinguishable from it.

Fact 7.6. The following conditions are equivalent:

(a) for each $A \in For$ formula
$\overline{R}_1 A \to \overline{R}_2 A$
is valid in a model $M = (X,m,v)$,

(b) any system with set X of objects and with indiscernibility relation $m(R_1)$ is more accurate in the sense of relation $\leq$ than the system with set X of objects and with indiscernibility relation $m(R_2)$.

Fact 7.7. The following conditions are equivalent:

(a) $\models_M \overline{R}_1 A \to \overline{R}_2 A$

(b) $\models_M \underline{R}_2 A \to \underline{R}_1 A$

The validity in model M of formulas $\overline{R}_1 A \to \overline{R}_2 A$ assures that for any information its upper approximation in the system with indiscernibility relation $m(R_1)$ is contained in its upper approximation in the system with indiscernibility relation $m(R_2)$. Hence, by Fact 4.1 it is equivalent to condition (b) in Fact 7.6.

Fact 7.7 follows from Fact 4.2.

Hence by using logic IL we can express facts concerning relation $\leq$, so the logic enables us to compare the expressive power of systems.

8. Deductive system of logic IL

We give a deductive structure to the language of the logic in the usual way, first specifying a recursive set of axioms and inference rules, and then defining a theorem of the logic to be any formula obtainable from the axioms by repeated application of the rules.

Observe, that the following fact holds.

Fact 8.1. For any formulas A,B and for any model M the following conditions are satisfied:

(a) $in_M(A \vee B) = in_M(\neg A \to B)$,

(B) $in_M(A \wedge B) = in_M \neg(A \to \neg B)$,

(c) $in_M(A \leftrightarrow B) = in_M((A \to B) \wedge (B \to A))$,

(d) $in_M \overline{R}A = in_M \neg(\underline{R} \neg A)$.

It follows that operations $\vee, \wedge, \leftrightarrow, \overline{R}$ can be defined by using operations $\neg, \rightarrow, \underline{R}$. Hence it is sufficient to define the deductive system for the language based on operations $\neg, \rightarrow, \underline{R}$.

Axioms of logic IL

A1. All formulas having the form of a tautology of classical propositional calculus.

A2. $\underline{R}(A \rightarrow B) \rightarrow (\underline{R}A \rightarrow \underline{R}B)$,

A3. $\underline{R}A \rightarrow A$,

A4. $A \rightarrow \underline{R} \neg \underline{R} \neg A$

A5. $\underline{R}A \rightarrow \underline{R}\underline{R}A$

A6. $\underline{(R_1 \cap R_2)}A \leftrightarrow \underline{R_1}A \vee \underline{R_2}A$.

Rules of inference of IL

R1 $\dfrac{A, A \rightarrow B}{B}$

R2 $\dfrac{A}{\underline{R}A}$

This axiomatization corresponds very closely to the axiomatization for modal logic S5 [2]. The new axiom A6 characterizes the intersection of relations.

A proof of a formula A from a set Γ of formulas is a finite sequence of formulas each of which is either an axiom or an element of set Γ or else is obtainable from earlier formulas by a rule of inference, and A is the last formula in the sequence. A formula A is derivable from a set Γ ($\Gamma \vdash A$) whenever there is a proof of A from Γ. A formula A is a theorem ($\vdash A$) if there is a proof of A from the empty set. A set Γ is consistent if the formula of the form $A \wedge \neg A$ is not derivable from Γ.

Fact 8.2. (Soundness theorem)

For any formula A and for any set Γ of formulas the following conditions are satisfied:

(a) $\vdash A$ implies $\models A$

(b) $\Gamma \vdash A$ implies $\Gamma \models A$

(c) Γ satisfiable implies Γ consistent.

The proof is easily obtained by checking that all the axioms are valid formulas and the rules preserve validity.

In the following we list some theorems and metatheorems of the logic.

Fact 8.3. For any formulas A,B, and for any relation R the following conditions are satisfied:

(a) $\vdash A \rightarrow \overline{R}A$

(b) $\vdash \neg \underline{R}A \rightarrow \underline{R} \neg \underline{R}A$

(c) $\vdash \neg \underline{R}A \leftrightarrow \overline{R} \neg A$

(d) $\vdash \neg \overline{R}A \leftrightarrow \underline{R} \neg A$

(e) $\vdash \underline{R}\overline{R}A \leftrightarrow \overline{R}A$

(f) $\vdash \overline{R}\underline{R}A \leftrightarrow \underline{R}A$

(g) $\vdash \overline{R}\overline{R}A \leftrightarrow \overline{R}A$

(h) $\vdash \underline{R}\underline{R}A \leftrightarrow \underline{R}A$

(i) $\vdash \underline{R}(A \wedge B) \leftrightarrow \underline{R}A \wedge \underline{R}B$

(j) $\vdash \overline{R}(A \vee B) \leftrightarrow \overline{R}A \vee \overline{R}B$

(k) $\vdash \underline{R}A \vee \underline{R}B \rightarrow \underline{R}(A \vee B)$

(l) $\vdash \overline{R}(A \wedge B) \rightarrow \overline{R}A \wedge \overline{R}B$

(m) $\vdash \underline{R}(A \rightarrow B) \rightarrow (\overline{R}A \rightarrow \overline{R}B)$

(n) $\vdash (\overline{R}A \rightarrow \overline{R}B) \rightarrow \overline{R}(A \rightarrow B)$

(o) $A \in \Gamma$ implies $\Gamma \vdash A$

(p) $\Gamma \vdash A$ and $\Gamma \subseteq \Delta$ imply $\Delta \vdash A$

(r) $\vdash A$ implies $\Gamma \vdash A$

(s) $\Gamma \vdash A$ iff $\Gamma \cup \{\neg A\}$ consistent.

9. Completeness theorem

We prove the completeness theorem for the logic by using the Rasiowa-Sikorski method [7] adapted for modal logics by Mi rkowska [3].

Let Γ be a consistent set of formulas. We define relation $\sim$ in the set For as follows:

$$A \sim B \text{ iff } \Gamma \vdash A \to B \text{ and } \Gamma \vdash B \to A.$$

Fact 9.1

(a) Relation $\sim$ is the equivalence relation,

(b) Relation $\sim$ is the congruence with respect to $\vee, \wedge, \neg$.

(c) If $A \sim B$ then $\underline{R}A \sim \underline{R}B$ for any $R \in Rel$.

Let $For/_{\sim}$ denote the set of all equivalence classes of relation $\sim$.

Fact 9.2

(a) Algebra $(For/_{\sim}, \cup, \cap, -, 1, 0)$, where

$[A] \cup [B] = [A \vee B]$

$[A] \cap [B] = [A \wedge B]$

$-[A] = [\neg A]$

$1 = [A \vee \neg A]$

$0 = [A \wedge \neg A]$

is the non-degenerate Boole'an algebra,

(b) $[A] \leq [B]$ iff $\Gamma \vdash A \to B$

(c) $\Gamma \vdash A$ iff $[A] = 1$

(d) $[\neg A] \neq 0$ iff non $\Gamma \vdash A$.

Proofs of these two facts are similar to that presented in [7].

Let $\mathcal{F}$ be the family of all maximal filters in algebra $For/_{\sim}$. Set $\mathcal{F}$ is non-empty since the algebra is non-degenerate. We define canonical model $M_o = (X_o, m_o, v_o)$ as follows:

$X_o = \mathcal{F}$,

$m_o(R) = \{(F_1, F_2) \in \mathcal{F}^2$: for any formula A if $[\underline{R}A] \in F_1$ then $[A] \in F_2\}$,

$v_o(p) = \{F \in \mathcal{F} : [p] \in F\}$.

Fact 9.3

(a) For any $R \in Rel$ $m_o(R)$ is the equivalence relation,

(b) for any $R_1, R_2 \in Rel$ $m_o(R_1 \cap R_2) = m_o(R_1) \cap m_o(R_2)$.

Proof:

By A2 and Fact 9.2(b) we have $[\underline{R}A] \leq [A]$. Hence if $[\underline{R}A] \in F$ then $[A] \in F$, so relation $m_o(R)$ is reflexive. Let us now assume that $(F_1, F_2) \in m_o(R)$, $[\underline{R}A] \in F_2$ and suppose that $[A] \notin F_1$. Hence by the maximality of F_1 $[\neg A] \in F_1$. By A4 $[\underline{R}\neg\underline{R}A] \in F_1$. Hence $[\neg\underline{R}A] \in F_2$, contradiction. Hence relation $m_o(R)$ is symetric. Let us now assume that $(F_1, F_2) \in m_o(R)$, $(F_2, F_3) \in m_o(R)$, $[\underline{R}A] \in F_1$ and suppose $[A] \notin F_3$. By A5 we have $[\underline{R}\underline{R}A] \in F_1$ and hence $[\underline{R}A] \in F_3$. It follows that $[A] \in F_3$, contradiction. Hence relation $m(R_o)$ is transitive.

We now prove condition (b). Assume $(F_1, F_2) \in m_o(R_1)$, $(F_1, F_2) \in m_o(R_2)$ and $[\underline{(R_1 \cap R_2)}A] \in F_1$. By A6 and since filter F_1 is a prime filter we have $[\underline{R_1}A] \in F_1$ or $[\underline{R_2}A] \in F_1$. Conversely, if $[\underline{R_1}A] \in F_1$ and $[\underline{R_2}A] \in F_1$ then by A6 $[\underline{(R_1 \cap R_2)}A] \in F_1$ and hence $[A] \in F_2$.

Fact9.4. For any formula A and for any filter $F \in \mathfrak{F}$

M_o, F sat A iff $[A] \in F$.

Proof:

The proof is by induction with respect to the length of a formula.

Case 1 A is $p \in VarIn$.

We have M_o, F sat p iff $F \in v_o(p)$ iff $[p] \in F$.

Case 2 A is $\neg B$.

Condition M_o, F sat $\neg B$ is equivalent to non M_o, F sat B. By the induction hypothesis we have $[B] \notin F$. Since F is a maximal filter we have $[\neg B] \in F$.

Case 3 A is $B \to C$.

Condition M_o, F sat $B \to C$ is equivalent to non M_o, F sat B or M_o, F sat C. By the induction hypothesis we have $[B] \notin F$ or $[C] \in F$. Since F is a maximal filter, we have $[\neg B] \in F$ or $[C] \in F$. Since F is a prime filter, we have $[\neg B] \cup [C] \in F$, and hence $[B \to C] \in F$.

Case 4 A is $\underline{R}B$.

Assume M_o, F sat $\underline{R}B$ and suppose $[\underline{R}B] \notin F$.

Consider set $Z_{FR} = \{[C] : [\underline{R}C] \in F\}$. We now prove four facts about this set.

(4a) Set Z_{FR} is non-empty.

It follows from the fact that $[\underline{R}(A \vee \neg A)] \in Z_{FR}$.

(4b) Set Z_{FR} is a filter.

$[B_1] \cap [B_2] \in Z_{FR}$ iff $[B_1 \wedge B_2] \in Z_{FR}$. By the definition of Z_{FR} we have $[\underline{R}(B_1 \wedge B_2)] \in F$. By Fact 8.3 (i) $[\underline{R}B_1 \wedge \underline{R}B_2] \in F$ and this is equivalent to $[\underline{R}B_1] \in F$ and $[\underline{R}B_2] \in F$. Hence $[B_1] \in Z_{FR}$ and $[B_2]$ Z_{FR}.

(4c) Filter Z_{FR} is a proper filter. Suppose $[A \wedge \neg A] \in Z_{FR}$. Then we have $[\underline{R}(A \wedge \neg A)] \in F$ and hence $1 = [\underline{R}(A \vee \neg A)] \notin F$, contradiction.

(4d) Filter F' generated by set $Z_{FR} \cup \{[\neg B]\}$ is a proper filter.

We show that for any $[A_1], \ldots, [A_n] \in Z_{FR}$, $n \geq 1$, we have $[A_1] \cap \ldots \cap [A_n] \cap [\neg B] \neq 0$. For suppose not, then we have

$$\Gamma \vdash A_1 \wedge \ldots \wedge A_n \wedge \neg B \to A \wedge \neg A, \text{ and hence}$$

$$\Gamma \vdash A_1 \wedge \ldots \wedge A_n \to B.$$

By using rule R2, axiom A2 and Fact 8.3(i) we obtain

$$\Gamma \vdash \underline{R}A_1 \wedge \ldots \wedge \underline{R}A_n \to \underline{R}B.$$

Since $[A_1], \ldots, [A_n] \in Z_{FR}$ we have $[\underline{R}A_1], \ldots, [\underline{R}A_n] \in F$ and hence $[\underline{R}A_1 \wedge \ldots \wedge \underline{R}A_n] \in F$. So $[\underline{R}B] \in F$ and this is in conflict with the supposition under case 4. Hence filter F' can be extended to a maximal filter F'' such that $[\neg B] \in F''$ and for any formula C if $[\underline{R}C] \in F$ then $[C] \in F''$. Hence $(F, F'') \in m_o(R)$. By case 2 M_o, F'' sat $\neg B$ and this contradicts the assumption under case 4.

Let us now assume that $[\underline{R}B] \in F$. Let F' be a maximal filter such that $(F, F') \in m_o(R)$. Then $[B] \in F'$ and by the induction hypothesis we have M_o, F' sat B. Hence M_o, F sat $\underline{R}B$.

Fact 9.5. (Completeness theorem)

For any formula A and for any set Γ of formulas the following conditions are satisfied:

(a) $\models A$ implies $\vdash A$

(b) $\Gamma \models A$ implies $\Gamma \vdash A$

(c) Γ consistent implies Γ satisfiable.

Proof:

Part (a) is a particular case of part (b), hence we prove (b). Assume $\Gamma \models A$ and suppose non $\Gamma \vdash A$. By Fact 9.2(d) we have $[\neg A] \neq 0$. By the Rasiowa-Sikorski lemma [7] there is a maximal filter $F_0 \in \mathcal{F}$ such that $[\neg A] \in F_0$. By Fact 9.4 we have M_0, F_0 sat $\neg A$ for the canonical model M_0. Moreover, for any formula $B \in \Gamma$ we have $\Gamma \vdash B$ by Fact 8.3 (0) and hence $[B] \in F_0$. By Fact 9.4 M_0, F_0 sat B for any formula $B \in \Gamma$, contradiction.

Condition (c) follows immediately from Fact 9.4.

As a corollary we obtain the following theorem

Fact 9.6 (Compactness theorem)

Set Γ of formulas is satisfiable iff every finite subset Γ' of Γ is satisfiable.

The given completeness theorem enables us to consider formalized theories based on the presented logic. For instance we can consider the theory of selective systems by adjoining to the logical axioms A1,...,A6 the formula given in Fact 7.3.

10. Concluding remarks

In the paper we present the method of dealing with informations which may not be precisely defined. We assume the definition of an information is given up to an equivalence relation, determined by an information system. We consider these equivalence relations as tools representing the expressive power of systems and we compare the expressive

power of systems by comparing these equivalences with respect to the set-theoretical inclusion. We develop the formal logic in which facts concerning informations given by information systems are expressible. We present the complete set of axioms and inference rules for the logic, which provides the method of reasoning about expressive power of information systems.

References

1. P. Hajek, I. Havel, M. Chytil, The GUHA Method of Automatmated Hypothesis Determination. Computing 1 (1966) 293-308.
2. G.E. Hughes, M.J. Cresswell: An Introduction to Modal Logic. London (1968).
3. G. Mirkowska: Multimodal Logic, Studia Logica, to appear.
4. G.W. Moore, M. Grover, M.D. Hutchins: A Hintikka Possible Worlds Model for Certainty Levels in Medical Decision Making, Metamed to appear.
5. Z. Pawlak: Mathematical Foundations of Informational Retrieval. CC PAS Report 101 (1973).
6. Z. Pawlak: Classification of Objects by Means of Attributes. An Approach to Inductive Inference. ICS PAS Report, to appear.
7. H. Rasiowa, R. Sikorski: Mathematics of Metamathematics. PWN Warsaw (1970).

Contents