Choosing one of these classes, say (301, 149, 697), we can duplicate it by compounding (301, 149, 697) with (697, −149, 301).

This is done most simply by a method devised by the writer:

Let the classes be $(a, b, \cdots)$ and $(a', b', \cdots)$ where $a$ and $a'$ are prime to each other, and take $p$ and $q$ such that either $ap$ and $a'q$, or $a'p$ and $aq$ differ by unity. This can nearly always be done mentally, but when $a$ and $a'$ are not small, the values of $p$ and $q$ are more quickly found as the constituents of the penultimate convergent of the continued fraction representing $a/a'$ or $a'/a$. The required compound class is then given by $(aa', a'bp + ab'q, \cdots)$ or by $(aa', a'bq + ab'p, \cdots)$, care being taken that the signs of $p$ and $q$ are so chosen that the smaller of the two products, *e.g.*, $aq$ and $a'p$, say, shall be negative. Applying this to the case in hand, we get:

$$
\begin{array}{ll}
 & (697.301, 697.149(-19) + 301(-149)44, \cdots), \\
\text{or} & (209797, -1973207 -1973356, \cdots), \\
 & (209797, -3946563, \cdots), \\
 & (209797, +39580, 7468), \\
 & (7468, 2240, 697), \\
 & (697, 149, 301), \\
 & (301, -149, 697),
\end{array}
$$

which shows that (301, 149, 697) is a critical class, and each of the twelve other classes when similarly tested is found to be a critical class.

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1. G. B. Mathews, *Theory of Numbers*, Part I, 1892.
2. A. Scholz & O. Taussky, "Die Hauptideale der kubischen Klassenkörper imaginärquadratischer Zahlkörper: ihre rechnerische Bestimmung und ihr Einfluss auf den Klassenkörperturm," *Jahrbuch über die Fortschritte der Mathematik*, 60, 1934, p. 126, *J. reine angew. Math.* 171, 1934, p. 19–41.
3. T. Pepin, *Atti. Acad. Pont. Nuovi Lincei*, 33, 1881, p. 354–391.
4. C. F. Gauss, *Disquisitiones Arithmeticae*, Art. 306, VIII, 1801; in *Werke* I, 1863, p. 371; German transl. by H. Maser, 1889, p. 653–654.
5. The 11th member of the second series, *i.e.* −297675, has exponent 27, with 40 critical classes.

## TECHNICAL NOTES AND SHORT PAPERS

### Selected References on Use of High-Speed Computers for Scientific Computation

The author is often asked to recommend reading to orient mathematicians in the impact of high-speed computers on numerical analysis. The following list was prepared in answer to one such request, but does not pretend to be definitive. The author is indebted to C. B. Tompkins for several suggestions.

For a list of books not necessarily influenced by high-speed computers, but highly pertinent to their use, see G. E. Forsythe, "A numerical analyst's fifteen-foot shelf," *MTAC*, v. 7, 1953, p. 221–228.

## I. BOOKS

A. D. Booth & K. H. V. Booth, *Automatic Digital Calculators*, Butterworths Scientific Publications, London, 1953, 231 p.

B. V. Bowden, *Faster than Thought. A Symposium on Digital Computing Machines*, Isaac Pitman, London, 1953, 416 p.

Engineering Research Associates, inc., *High-Speed Computing Devices*, McGraw-Hill, New York, 1950, 451 p.

D. R. Hartree, *Numerical Analysis*, Clarendon Press, Oxford, 1952, 287 p.

Cecil Hastings, Jr., J. T. Hayward, & J. P. Wong, Jr., *Approximations for Digital Computers*, University Press, Princeton, 1955, 201 p.

A. S. Householder, *Principles of Numerical Analysis*, McGraw-Hill, New York, 1953 (extensive bibliography), 274 p.

National Bureau of Standards, *Monte Carlo Method*, Applied Mathematics Series 12, U. S. Gov. Printing Office, 1951, 42 p.

H. Rutishauser, *Automatische Rechenplanfertigung bei programmgesteuerten Rechenmaschinen*, Birkhäuser, Basel, 1952, 45 p.

M. V. Wilkes, D. J. Wheeler, & S. Gill, *The Preparation of Programs for an Electronic Digital Computer*, Addison-Wesley, Cambridge, Mass., 1951, 170 p.

## II. JOURNALS

Computers and Automation (New York)
Journal of the Association for Computing Machinery
Journal of Research of the U. S. National Bureau of Standards
Journal of the Society for Industrial and Applied Mathematics
Mathematical Reviews (Numerical and Graphical Methods section)
Mathematical Tables and Other Aids to Computation
Naval Research Logistics Quarterly
Proceedings of the Association for Computing Machinery (terminated)
Proceedings of the Cambridge Philosophical Society
Quarterly of Applied Mathematics
Quarterly Journal of Mechanics and Applied Mathematics
Vychislitel'naâ Matematika i Vychislitel'naâ Tekhnika (Moscow)
Zeitschrift für angewandte Mathematik und Mechanik
Zeitschrift für angewandte Mathematik und Physik

## III. SOME ARTICLES NOT IN ABOVE JOURNALS

American Mathematical Society, "Proceedings of symposia in applied mathematics," vol. 6, to appear.

V. Bargmann, D. Montgomery, & J. von Neumann, "Solution of linear systems of high order," Institute for Advanced Study, Princeton, 1946.

Wallace Givens, "Numerical computation of the characteristic values of a real symmetric matrix," ORNL 1574, Oak Ridge National Laboratory, 1954, 107 p.

H. H. Goldstine & J. von Neumann, "Planning and coding of problems for an electronic computing instrument," issued in three parts as part II, vols. 1–3, Institute for Advanced Study, Princeton, 1947–1948.

HARVARD UNIVERSITY, "Proceedings of a symposium on large-scale digital calculating machinery," Cambridge, Mass., 1948.

HARVARD UNIVERSITY, "Proceedings of a second symposium on large-scale digital calculating machinery," Cambridge, Mass., 1951.

M. A. HYMAN, "On the numerical solution of partial differential equations," thesis, Delft, 1953, 108 p.

N. METROPOLIS & S. ULAM, "The Monte Carlo method," Am. Stat. Assn., *Jn.*, v. 44, 1949, p. 335–341.

J. VON NEUMANN & H. H. GOLDSTINE, "Numerical inverting of matrices of high order," Am. Math. Soc., *Bull.*, v. 53, 1947, p. 1021–1099, and Am. Math. Soc., *Proc.*, v. 2, 1951, p. 188–202.

F. W. J. OLVER, "The evaluation of zeros of high-degree polynomials," R. Soc. London, *Phil. Trans.*, v. 244, 1952, p. 385–415.

E. STIEFEL, "Relaxationsmethoden bester Strategie zur Lösung linearer Gleichungssysteme," *Commentarii Mathematici Helvetici*, v. 29, 1955, p. 157–179.

JOHN TODD, "Motivation for working in numerical analysis," *Comm. on Pure and Applied Math.*, v. 8, 1955, p. 97–116.

DAVID YOUNG, "Iterative methods for solving partial difference equations of elliptic type," Am. Math. Soc., *Trans.*, v. 76, 1954, p. 92–111.

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## Modified Quotients of Cylinder Functions

The name in the title of this note is applied to the function, $\mathfrak{C}_\nu(z)$, defined by the following equation,

$$\mathfrak{C}_\nu(z) = \frac{zC_{\nu-1}(z)}{C_\nu(z)} \tag{1}$$

where $C_\nu(z)$ is a cylinder function [1] which satisfies the pair of recurrence formulae,

$$\frac{2\nu}{z} C_\nu(z) = C_{\nu-1}(z) + C_{\nu+1}(z) \tag{2}$$

$$2C_\nu'(z) = C_{\nu-1}(z) - C_{\nu+1}(z)$$

$\mathfrak{C}_\nu(z)$ has not repeated zeros and poles with possible exception of the origin and satisfies the following RICCATI's equation,

$$\frac{dy}{dz} + \frac{1}{z}(y^2 - 2\nu y) + z = 0. \tag{3}$$

Its derivative,

$$\frac{\partial \mathfrak{C}_\nu(z)}{\partial z} = \frac{z}{C_\nu^2(z)} \{C_{\nu-1}(z)C_{\nu+1}(z) - C_\nu^2(z)\} \tag{4}$$

has a close relation to the famous integral due to LOMMEL,

$$\int zC_\nu^2(z)dz = \left[\frac{z^2}{2}\{C_\nu^2(z) - C_{\nu-1}(z)C_{\nu+1}(z)\}\right]. \tag{5}$$

Introducing each of the three kinds of Bessel functions, $J_\nu(z)$, $Y_\nu(z)$, $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$, into the equation (1) in place of $C_\nu(z)$, we obtain corresponding kinds of modified quotients, $\mathfrak{J}_\nu(z)$, $\mathfrak{Y}_\nu(z)$, $\mathfrak{H}_\nu^{(1)}(z)$ and $\mathfrak{H}_\nu^{(2)}(z)$, respectively. In various boundary value problems of mathematical physics, we encounter quite often the Bessel functions in quotient forms [2]. It is obvious that the modified quotients defined here give a convenient approach to mathematical analysis and numerical estimates of these problems. Moreover there is a remarkable parallelism between the modified quotients and the trigonometric cotangent and tangent, as there is between the Bessel functions and the sine and cosine. Therefore the modified quotients should have the same *raison d'être* in cylinder functions as the cotangent and tangent in trigonometric functions.

From the above consideration it seems highly desirable to give permanent symbols to these modified quotients, to collect their formulae and to construct their tables. To this end an attempt was made [3], but further cooperation and criticism of interested workers are necessary.

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1. G. N. WATSON, *Theory of Bessel Functions*, Cambridge Univ. Press, 1922, p. 82 ff.
2. For example:
DENNISON BANCROFT, "The velocity of longitudinal waves in cylindrical bars," *Phys. Rev.*, 59, 1941, p. 588.
G. E. HUDSON, "Dispersion of elastic waves in solid circular cylinders," *Phys. Rev.*, 63, 1943, p. 46.
H. SUHL & L. R. WALKER, "Topics in guided-wave propagation through gyromagnetic media," *Bell System Tech. J.*, 33, 1954, p. 579.
3. M. ONOE, "Formulae and tables, modified quotients of cylinder functions," Report of the Institute of Industrial Science, University of Tokyo, No. 32, 1955.

## Flip-flop as Generator of Random Binary Digits

The aim of the present note is to show that a well known electronic element of digital computers, the flip-flop, may be used for generating a series of random binary digits with equal probabilities.

Let us consider a flip-flop as shown on fig. 1 and let A and B denote two possible stable states of the flip-flop. If we switch on the contact $S$, the flip-flop will be randomly set in one of its states A or B. We may obtain by the aid of the flip-flop a sequence of $2k$ random elements $X_1, X_2, \cdots, X_{2k}$, (abbreviated $\{X_{2k}\}$), where

$$X_j = \begin{cases} \text{A, if } j\text{-th switching on the contact } S\text{, set flip-flop in state A} \\ \text{B, if } j\text{-th switching on the contact } S\text{, set flip-flop in state B} \end{cases}$$

and $1 \leq j \leq 2k$.

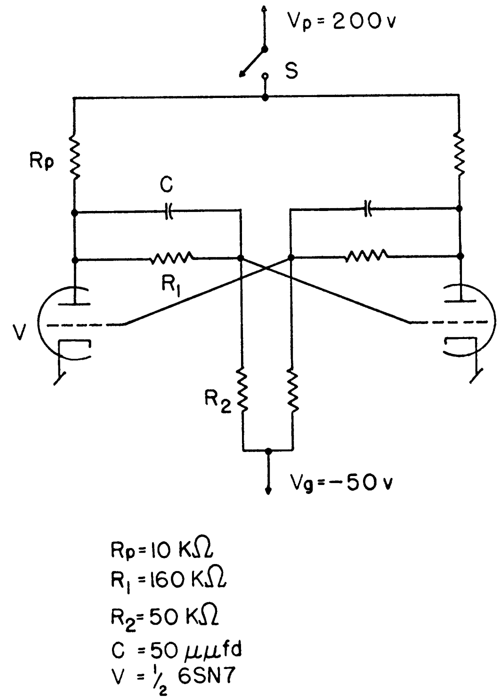


FIG. 1.

In this way we may obtain a finite random series of A and B which are statistically independent. One series produced by the aid of a flip-flop is given below:

AABAABBABBABBBABBAABABABBABAABABABBAA
BABABBBBBBBBABBBABAABBB.

Let $\{Y_k\}$ be the sequence of $k$ pairs of elements of $\{X_{2k}\}$ such that $Y_i = X_{2i-1}, X_{2i}$, where $1 \leq i \leq k$. Omitting in $\{Y_k\}$ all elements of the form AA and BB we obtain a third sequence whose elements are the pairs AB and BA only, denoted in the following by 0 and 1 respectively.

Let $p_j(\mathrm{A})$ and $p_j(\mathrm{B})$ denote probabilities that $j$-th switching on of contact $S$ set flip-flop in state A or B respectively and suppose that $p_j(\mathrm{A})$ and $p_j(\mathrm{B})$ are asymmetric, say $p_j(\mathrm{A}) > p_j(\mathrm{B})$. Supposing that the flip-flop does not change its properties during two successive switchings, we may write

(1) $$p_{2i-1}(\mathrm{A}) = p_{2i}(\mathrm{A})$$

(2) $$p_{2i-1}(\mathrm{B}) = p_{2i}(\mathrm{B}).$$

From 1 and 2 we have

(3) $$p_{2i-1}(\mathrm{A}) \cdot p_{2i}(\mathrm{B}) = p_{2i-1}(\mathrm{B}) \cdot p_{2i}(\mathrm{A}).$$

Because

(4) $$p_{2i-1}(\mathrm{A}) \cdot p_{2i}(\mathrm{B}) = p_i(0)$$

and

$$p_{2i-1}(\mathrm{B}) \cdot p_{2i}(\mathrm{A}) = p_i(1), \tag{5}$$

therefore

$$p_i(0) = p_i(1) \tag{6}$$

where $p_i(0)$ and $p_i(1)$ are probabilities of zeros and ones in the $i$-th place of the sequence $\{Y_k\}$.

The procedure above described may be used for production of binary random numbers by automatic digital computers. In this case the manual switch $S$ must be replaced by an electronic switch, of course.

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## Numerical Solution of the Schroedinger Equation for Central Fields

A fast program has been written for the ILLIAC to integrate the radial SCHROEDINGER equation

$$u''(r) - [l(l+1)/r^2 + 2V(r) - E]u = 0 \tag{1}$$

$$\text{boundary conditions:} \quad u(0) = 0 \quad u(r) \text{ bounded}$$

for any well-behaved potential $V(r)$. More generally the program can integrate any linear second-order differential equation which can be put in the form

$$u''(r) - g(r)u(r) = q(r) \tag{2}$$

with $q(r)$ vanishing at zero and infinity, $r^2g(r)$ bounded at zero, and $g(r)$ bounded for large $r$ [1].

A distinctive feature of the program is the use of the NOUMEROV [2], [3] method for the integration. This is faster than techniques (such as the RUNGE-KUTTA) which depend on making an estimate of one or more forward points and improving this by an iteration scheme. Here there is no iteration, but the error in the "estimate" of each forward point is of eighth order in the step-size $h$, so that the truncation error may still be kept small. The essence of the method is the elimination of all odd powers of $h$ from the TAYLOR expansion about any point by working with three points instead of two, followed by a change of dependent variable which removes the $h^4$-term. The calculation of a forward point to order $h^8$ thus requires the value of the dependent variable at the six preceding points.

For an equation in the form (2), the required new dependent variable is

$$y = u - (h^2/12)(gu + q). \tag{3}$$

The prescription for calculating forward points is

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \frac{g_n y_n + q_n}{1 - \frac{1}{12}h^2 g_n} - \frac{1}{240}\delta^6 y_n, \tag{4}$$

where the sixth central difference in the last term is given, to order $h^8$, by

(5) $$\delta^6 y_n = 3\delta^6 y_{n-2} - 2\delta^6 y_{n-3}.$$

This prescription can be used as soon as $y$ is known at the first six points.

To get this far, *i.e.*, to start the integration, an iteration method is used: The difference equation (4) without the last term is solved by successive approximations at the first six points, guessing a value of $y_0$ and improving this guess by applying the condition (seventh order approximation)

(6) $$\delta^6 y_2 = \delta^6 y_3.$$

For the inhomogeneous equation ($q \neq 0$), where the normalization is not arbitrary, the starting value $u_1 = u(h)$ is found by another iteration scheme: With an estimated $u_1$, the equation is integrated out to some large $r$, where the inhomogeneity $q$ has become negligible. Application of the condition that $u$ be small and monotonically decreasing leads to an improved estimate of $u_1$. This scheme, which seldom requires more than three iterations, is practical because of the speed of integration. (ILLIAC takes less than 40 milliseconds per $h$-step.)

In its present form, the program can—

—find eigenvalues $E$ (permitted negative energies)
—print out values of the wave functions $u = u_E^{(r)}$ at desired values of $r$, and find the nodes and extrema
—evaluate matrix elements $\int u_{E_i} u_{E_j}$ (with ILLIAC's electrostatic memory of $2^{10}$ words, at 40 bits per word, there is enough space to integrate several wave-functions simultaneously)
—solve self-consistent field problems (Schroedinger equation and Poisson equation solved simultaneously).

A "guide" to the program has been prepared, containing a detailed description of the mathematical method and instructions for use of the existing tapes, and is available in mimeographed form at the University of Illinois Digital Computer Laboratory.

The authors are indebted to Professors John Blatt and J. N. Snyder for helpful suggestions.

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1. If $q(r)$ is not identically zero, i.e., for the inhomogeneous equation, there is the additional restriction that $r^2 g(r)$ be bounded everywhere.

2. G. W. Pratt, jr., "Wave functions and energy levels for $Cu^+$ as found by the Slater approximations to the Hartree-Fock equations," *Phys. Rev.*, 88, 1952, p. 1217.
3. B. Noumerov, Monthly Notices *Roy. Astron. Soc.*, 84, 1924, p. 592.

## Note on Using the Reciprocal Function for a Linear Inverse Interpolation

Let it be required to find the argument, $x$, by linear interpolation from a table of values of $y(x)$. If there is also a table of values of the reciprocal function $\frac{1}{y(x)}$, then a better linear interpolation may result by using the reciprocal function. Frequently, in applications such as in triangle or compound interest problems, $y(x)$ is given as equal to a quotient $\frac{A}{B}$. Hence the solution of $\frac{1}{y(x)} = \frac{B}{A}$ for $x$ by linear interpolation can be made with equal facility. For example, if $\tan x = \frac{A}{B}$ is given for determination of $x$, we might just as easily determine $x$ from the relation $\cot x = \frac{B}{A}$, using the $\cot x$ table. It is the purpose of this note to determine the conditions for which interpolation will be improved by using the reciprocal function.

Suppose that $y(x)$ and $z(x)$ are two monotonic functions such that $yz = 1$. Since the slopes of $y$ and $z$ have the relation, $z'y + y'z = 0$, and hence are always opposite in sign, we choose $y$ as the increasing function.

Now the value, $x_a$, given by linear interpolation for the argument, $x$, in a table of values of $y(x)$ is given by

$$x_a = x_1 + \frac{y - y_1}{y_2 - y_1}(x_2 - x_1) \quad \text{where} \quad \begin{cases} x_1 < x_a < x_2 \\ y_1 < y < y_2. \end{cases}$$

(We assume that the function $y(x)$ is positive. Roundoff errors are not considered.) The error, $E_a$, due to the interpolation is given by $E_a = x_a - x$. See [1] for the maximum error inherent in determining a function by linear interpolation from tables. It is given by

$$|\text{error}| \leq \frac{(x_2 - x_1)}{8} M,$$

where $M$ denotes the maximum absolute value of $f''(x)$ in the interval $(x_1, x_2)$. However, the problem presented in this paper is one of inverse interpolation. Hence this error formula can be applied only after the function is inverted. For compound interest functions, see [2].

The corresponding interpolated value, $x_b$, and the error, $E_b$, may be determined for the reciprocal function, $z$. (See [3]. It should be noted that this paper is not considering reciprocal differences. We consider only the differences of the reciprocal function.) The condition that the interpolation is improved by using the reciprocal function is given by $|E_b| < |E_a|$. We note that $E_a < 0$ ($>0$) if $y'' > 0$ ($<0$) and $E_b > 0$ ($<0$) if $z'' > 0$ ($<0$) and thus we have the following cases:

*Case I.* $y'' < 0$ and $z'' > 0$, then $E_a$ and $E_b$ are both positive. In this case the result is poorer since always

$$(1)\quad E_b \equiv x_1 - x + \frac{z - z_1}{z_2 - z_1}(x_2 - x_1) > x_1 - x + \frac{y - y_1}{y_2 - y_1}(x_2 - x_1) \equiv E_a.$$

The difference in the errors is given by

$$(2)\qquad E_b - E_a = \frac{(y - y_1)(y_2 - y)(x_2 - x_1)}{(y_2 - y_1)y}.$$

*Case II.* $y'' > 0$ and $z'' < 0$, then $E_a$ and $E_b$ are both negative. Inequality (1) remains true, but now $|E_b| < |E_a|$ and hence the interpolation is always improved.

*Case III.* $y'' < 0$ and $z'' < 0$. This case is impossible. The reason for this is as follows: From $yz = 1$, one obtains $z'y + y'z = 0$ and $z''y + 2z'y' + zy'' = 0$. Hence $z'' + z^2y'' = 2(z')^2y$, and since $y > 0$, $y''$ and $z''$ cannot both be negative. In fact, if $y''$ (or $z''$) is negative, then $z''$ (or $y''$) is positive.

*Case IV.* $y'' > 0$ and $z'' > 0$, then $E_a$ is negative and $E_b$ is positive. The inequality is $E_b < |E_a|$, giving the condition,

$$(3)\qquad E_b - |E_a| = \frac{y - y_1}{y_2 - y_1}(x_2 - x_1)\left(1 + \frac{y_2}{y}\right) - 2(x - x_1) < 0$$

or

$$(4)\qquad E_b - |E_a| = \frac{z_1 - z}{z_1 - z_2}(x_2 - x_1)\left(1 + \frac{z_2}{z}\right) - 2(x - x_1) < 0.$$

Some examples follow:

| | | | |
|---|---|---|---|
| I | $\begin{cases} y = \sec x \left(0 \leqq x < \frac{\pi}{2}\right) \\ z = \cos x \end{cases}$ | $y'' > 0$<br>$z'' < 0$ | Case II<br>Result is always improved. |
| II | $\begin{cases} y = \sin x \left(0 < x \leqq \frac{\pi}{2}\right) \\ z = \csc x \end{cases}$ | $y'' < 0$<br>$z'' > 0$ | Case I<br>Result is always poorer. |
| III | $\begin{cases} y = \sqrt{x}\ (x > 0) \\ z = \left(\frac{1}{\sqrt{x}}\right) \end{cases}$ | $y'' < 0$<br>$z'' > 0$ | Case I<br>Result is always poorer. |
| IV | $\begin{cases} y = e^x\ (x > 0) \\ z = e^{-x} \end{cases}$ | $y'' > 0$<br>$z'' > 0$ | Case IV<br>$E_b - \lvert E_a\rvert = 0$ if $x = \frac{x_1 + x_2}{2}$<br>Result is improved if<br>$\frac{x_1 + x_2}{2} < x < x_2$. |

V
$$\begin{cases} y = (1+i)^n \text{ for } n & y'' > 0 \quad \text{Case IV} \\ & E_b - |E_a| = 0 \text{ if } n = \dfrac{n_1+n_2}{2} \\ z = (1+i)^{-n}\ (n > 0) & z'' > 0 \quad \text{Result is improved if} \\ & \dfrac{n_1+n_2}{2} < n < n_2. \end{cases}$$

VI
$$\begin{cases} y = s_{\overline{n}|}(i) \text{ for } n & \text{Case IV} \\ x = \dfrac{1}{s_{\overline{n}|}}(i)\ n > 0 & \text{Result is same as for } (1+i)^n. \end{cases}$$

VII
$$\begin{cases} y = a_{\overline{n}|}(i) \text{ for } n & y'' < 0 \quad \text{Case I} \\ z = \dfrac{1}{a_{\overline{n}|}}(i)\ (n > 0) & z'' > 0 \quad \text{Result is always poorer.} \end{cases}$$

VIII
$$\begin{cases} y = \tan\theta \text{ for } \left(0 < \theta < \dfrac{\pi}{2}\right) & y'' > 0 \quad \text{Case IV} \\ z = \cot\theta & z'' > 0. \end{cases}$$

The error in this example is given by

$$|E_a| = \left|\theta_1 + \frac{\tan\theta - \tan\theta_1}{\tan\theta_2 - \tan\theta_1}(\theta_2 - \theta_1) - \theta\right|.$$

In this case $|E_a|$ increases as $\theta$ increases from 0 to $\frac{\pi}{2}$ for constant values of $\theta_2 - \theta_1$ and $\theta - \theta_1$. That this is true is seen by the relation

$$\frac{\tan\theta - \tan\theta_1}{\tan\theta_2 - \tan\theta_1} = \frac{\sin(\theta - \theta_1)\cos\theta_2}{\sin(\theta_2 - \theta_1)\cos\theta}.$$

Now

$$|E_a(\theta, \theta_1, \theta_2)| = E_b\left(\frac{\pi}{2} - \theta, \frac{\pi}{2} - \theta_1, \frac{\pi}{2} - \theta_2\right).$$

Hence $E_b < |E_a|$ only if $\theta > \frac{\pi}{4}$, and in this case the interpolation is improved; but if $\theta < \frac{\pi}{4}$ the interpolation is poorer.

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1. J. B. Scarborough, *Numerical Mathematical Analysis*, Johns Hopkins Press, 1930, p. 107.
2. Hugh E. Stelson, "The accuracy of linear interpolation in tables of the mathematics of finance," *MTAC*, v. 3, 1949, p. 408–412.
3. L. M. Milne-Thomson, *The Calculus of Finite Differences*, Macmillan and Co., 1933, p. 104–123.
4. W. E. Milne, *Numerical Calculus*, Princeton Univ. Press, 1949, p. 219–236.