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ON A PROBLEM CONCERNING DEPENDENCE SPACES

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Abstract. A set Y of attributes of an information system is said to be dependent on a set X of attributes if the classification of objects defined by X is finer or as fine as the classification defined by Y. An important problem reads as follows. If Y depends on X find a minimal $X' \subseteq X$ such that Y depends on X'. A set X' is said to be a reduct of X if X' is a minimal subset of X defining the same classification of objects as X. The paper is devoted to the study of relationship between reducts and dependence. Both dependence and reducts can be defined in the so called dependence spaces and the above mentioned problem can be transformed into the problem of constructing reducts in a suitable dependence space. We also present some algorithms providing reducts in a dependence space; in this way, we obtain an algorithmic solution of our problem.

Key words: information system, dependence space, reduct, dependence

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1. Introduction

An information system S is an ordered quadruple (U, A, V, f) where U, A, V are finite nonempty sets and f is a mapping of $U \times A$ into V. The elements of U are said to be *objects*, elements of A *attributes*, elements of V *values of attributes*; if $u \in U$, $a \in A$, $v \in V$, then $f(u, a) = v$ means that the attribute a has the value v for the object u (see [5,6,7] where further literature is quoted).

For any $X \subseteq A$, we put

$$EQ(S, X) = \{(u, u') \in U \times U;\ f(u, a) = f(u', a) \text{ for any } a \in X\}\ .$$

Then $EQ(S, X)$ is an equivalence on the set U.

Let $X \subseteq A$, $Y \subseteq A$ be sets. We put $X \to Y$ (S) if $EQ(S, X) \subseteq EQ(S, Y)$; in this case, the set Y is said to be *dependent on* X *with respect to* S (cf. [3]).

The equivalence $EQ(S,X)$ is a classification of objects of the information system S. Then the dependence $X \to Y$ (S) means that the classification $EQ(S,X)$ corresponding to X is finer or as fine as the classification $EQ(S,Y)$ corresponding to Y. In a little different terminology, a set $X \subseteq A$ may be considered to be a test of objects in U; this test divides the objects in U into blocks where objects belonging to the same block are indiscernible by the test X. Then $X \to Y$ (S) means that the test X is better or as good as the test Y, i.e., that the test Y is superfluous if X has been performed.

Another interpretation of $X \to Y$ (S) is as follows. The attributes of X may be considered to be conditions while the attributes of Y are interpreted to be decisions where $X \cap Y = \emptyset$ is supposed. The condition $X \to Y$ (S) means that any two objects with the same values of conditions have the same values of decisions (cf. Section 6.3 of [6]).

From this point of view the following problem is important.

Problem A. Let $S = (U, A, V, f)$ be an information system, suppose that $X \subseteq A$, $Y \subseteq A$ are sets of attributes such that $X \to Y$ (S). Find a set X' such that X' is minimal with respect to inclusion among all sets T with the properties $T \subseteq X$, $T \to Y$ (S).

This means that we are looking for a minimal subset of X that is better or as good as Y regarding X and Y as tests; if considering elements in X to be conditions and elements in Y to be decisions, we are looking for a minimal set of conditions that provides the same values of decisions as X.

In this paper, we present a solution of Problem A. This problem will be formulated in a more abstract way for the so called dependence spaces (Problem A$'$) where dependence and reducts of sets can be defined and constructed. We prove that the solution of Problem A$'$ for a given dependence space coincides with finding a reduct of a set in another dependence space. The results are reformulated for information systems in such a way that Problem A is solved.

Hence, we prove that finding reducts in dependence spaces solves not only problem of finding reducts in information systems but also Problem A, i.e., that one algorithm solves several problems.

A useful instrument will be following relation.

$$K(S) = \{(X,Y) \in B(A) \times B(A);\ EQ(S,X) = EQ(S,Y)\}$$

where $B(A)$ denotes the set of all subsets of A. We obtain (see 6.4 of [5])

1. Theorem. *If $S = (U, A, V, f)$ is an information system, then $K(S)$ is a congruence on the semilattice $(B(A), \cup)$ where $\cup$ means the binary operation of union.* □

2. Dependence spaces

These investigations lead us to the following definition. Let A be a finite nonempty set, K a congruence on the semilattice $(B(A), \cup)$. Then the ordered pair (A, K) is said to be a *dependence space* (see Section 2 of [5], [3]).

If (A, K) is a dependence space, then any block of A/K has a greatest element owing to the finiteness of A. For any $X \in B(A)$ we denote by $CK(X)$ the greatest element of the block of K that contains X. We obtain (see Lemma 2.2 of [4] and Lemma 2 of [3])

2. Theorem. *If (A, K) is a dependence space, then the following assertions hold.*
(i) If $X \subseteq Y \subseteq Z \subseteq A$ and $(X, Z) \in K$ hold, then $(Y, Z) \in K$.
(ii) CK is a closure operator on A. □

Let (A, K) be a dependence space, $X \subseteq A$, $Y \subseteq A$ sets. We put $X \to Y$ (A, K) if $CK(Y) \subseteq CK(X)$; in this case the set Y is said to be *dependent on X with respect to (A, K)*.

3. Example. Let us have $W = \{a, b, c\}$, $N = \emptyset$, $A = \{a\}$, $B = \{b\}$, $C = \{c\}$, $D = \{a, b\}$, $E = \{a, c\}$, $F = \{b, c\}$. Suppose that K has the following blocks: $\{N, A\}$, $\{B, D\}$, $\{C, E\}$, $\{F, W\}$. Clearly, K is a congruence on $(B(W), \cup)$ and (W, K) is a dependence space. Let us have $Y = A$, $X = F$. It follows that $CK(Y) = A$, $CK(X) = W$. Thus $CK(Y) \subseteq CK(X)$ holds and we have $X \to Y$ (W, K). □

The abstract form of Problem A reads as follows.

Problem A′. Let (A, K) be a dependence space, suppose that $X \subseteq A$, $Y \subseteq A$ are sets such that $X \to Y$ (A, K). Find a set X' such that X' is minimal with respect to inclusion among all sets T with the properties $T \subseteq X$, $T \to Y$ (A, K).

Let (A, K) be a dependence space and $X \subseteq A$ a set. A set $X' \subseteq A$ is said to be a *K-reduct* of X if X' is minimal with respect to inclusion among all sets T with the properties $T \subseteq X$, $(X, T) \in K$. We denote by $RED(K, X)$ the set of all K-reducts of the set X.

Construction of K-reducts solves some important problems concerning information systems and contexts in the sense of Wille (cf. [9] and Section 7 of [5]).

Starting with a dependence space we construct a new one.

4. Theorem. *Let (A, K) be a dependence space, $Y \in B(A)$ a set. For any $X \in B(A)$, $X' \in B(A)$ put $(X, X') \in D(K, Y)$ if and only if one of the following conditions (i), (ii) is satisfied.*
(i) $CK(Y) \subseteq CK(X)$, $CK(Y) \subseteq CK(X')$;
(ii) $CK(Y) \nsubseteq CK(X) = CK(X')$.
Then $D(K, Y)$ is a congruence on $(B(A), \cup)$.

P r o o f. (1) Clearly, $D(K, Y)$ is reflexive and symmetric. Suppose $(X, X') \in D(K, Y)$, $(X', X'') \in D(K, Y)$. Then the following cases may occur.

(α) $CK(Y) \subseteq CK(X)$. Then $(X, X') \in D(K, Y)$ implies $CK(Y) \subseteq CK(X')$ by *(i)*; since $(X', X'') \in D(K, Y)$, we obtain $CK(Y) \subseteq CK(X'')$ by *(i)* and, therefore, $(X, X'') \in D(K, Y)$ by *(i)*.

(β) $CK(Y) \nsubseteq CK(X)$. Then $(X, X') \in D(K, Y)$ implies $CK(X) = CK(X')$ by *(ii)* and, therefore, $CK(Y) \nsubseteq CK(X')$. Then $(X', X'') \in D(K, Y)$ entails $CK(X') = CK(X'')$ by *(ii)*. Thus $CK(Y) \nsubseteq CK(X) = CK(X'')$ and we obtain $(X, X'') \in D(K, Y)$ by *(ii)*.

We have proved that $D(K, Y)$ is transitive and, hence, an equivalence on $B(A)$.

(2) Suppose $(X, X') \in D(K, Y)$, $(Z, Z') \in D(K, Y)$. Then the following cases may occur.

(γ) $CK(Y) \subseteq CK(X)$. Then $(X, X') \in D(K, Y)$ implies $CK(Y) \subseteq CK(X')$ by *(i)*. Thus $CK(Y) \subseteq CK(X) \cup CK(Z) \subseteq CK(X \cup Z)$, $CK(Y) \subseteq CK(X') \cup CK(Z') \subseteq CK(X' \cup Z')$ by 2 which means $(X \cup Z, X' \cup Z') \in D(K, Y)$ by *(i)*.

(δ) $CK(Y) \subseteq CK(Z)$. Replacing X by Z, X' by Z', Z by X, and Z' by X' in (γ) we obtain $(X \cup Z, X' \cup Z') \in D(K, Y)$.

(ε) $CK(Y) \not\subseteq CK(X)$, $CK(Y) \not\subseteq CK(Z)$. Then $(X, X') \in D(K, Y)$, $(Z, Z') \in D(K, Y)$ imply $CK(X) = CK(X')$, $CK(Z) = CK(Z')$ by (ii) which means $(X, X') \in K$, $(Z, Z') \in K$ and, hence, $(X \cup Z, X' \cup Z') \in K$ which entails that $CK(X \cup Z) = CK(X' \cup Z')$. Thus, either $CK(Y) \subseteq CK(X \cup Z) = CK(X' \cup Z')$ which means $(X \cup Z, X' \cup Z') \in D(K, Y)$ by *(i)* or $CK(Y) \not\subseteq CK(X \cup Z) = CK(X' \cup Z')$ which means $(X \cup Z, X' \cup Z') \in D(K, Y)$ by *(ii)*.

We have proved that $D(K, Y)$ is a congruence on $(B(A), \cup)$. □

Thus, $(A, D(K, Y))$ is a new dependence space which enables the construction of $D(K, Y)$-reducts. They are solutions of Problem A$'$ as it follows from the next result.

5. Theorem. *Let (A, K) be a dependence space and $X \subseteq A$, $Y \subseteq A$ be sets such that $X \to Y$ (A, K). Then for any $X' \subseteq A$ the following conditions are equivalent.*

(i) $X' \subseteq X$ and X' is minimal with respect to inclusion among all sets T such that $T \subseteq X$, $T \to Y$ (A, K).

(ii) X' is a $D(K, Y)$-reduct of X.

P r o o f. By hypothesis $CK(Y) \subseteq CK(X)$ holds. We prove that for any $T \subseteq X$ the conditions $T \to Y$ (A, K) and $(T, X) \in D(K, Y)$ are equivalent.

Indeed, $T \to Y$ (A, K) means $CK(Y) \subseteq CK(T)$ which is equivalent to $(T, X) \in D(K, Y)$ by *(i)* of 4 regarding that $CK(Y) \subseteq CK(X)$ holds.

If $(T, X) \in D(K, Y)$, then $CK(Y) \subseteq CK(X)$ implies that $CK(Y) \subseteq CK(T)$ by *(i)* of 4 which means $T \to Y$ (A, K).

The assertion of our Theorem is an immediate consequence of the just proved equivalence. □

6. Example. Let (W, K) be the dependence space defined in 3. Clearly, $CK(A) \subseteq CK(Z)$ for any $Z \in B(W)$. It follows that $(Z, Z') \in D(K, A)$ for any Z, $Z' \in B(W)$ which means that $B(W)$ is one block of $D(K, A)$. Since N is its least element, it is also the unique $D(K, A)$-reduct of X for any $X \in B(W)$. □

3. Reducts

Construction of K-reducts solves some important problems. It covers finding reducts in information systems and in contexts in the sense of Wille (cf. [9] and Section 7 of [5]). Furthermore, the solution of Problem A$'$ may be reduced to finding reducts in a suitable dependence space as we have seen. Since Problem A$'$ is an abstract formulation of Problem A, also Problem A may be reduced to looking for reducts in a suitable dependence space. Problem A has an analogue in the theory of Wille's contexts; this will be investigated in a subsequent paper. Thus, also this analogous problem may be solved by finding reducts in a suitable dependence space. Hence, looking for reducts in a dependence space seems to have great importance for various parts of Artifical Intelligence. Therefore, the main aim of this paper is to describe how to transform Problem A′ to finding reducts and to present algorithms for finding reducts in dependence spaces. Since any information system defines a dependence space and any dependence space defines an information system (cf. Section 6 of [5]), any algorithm for finding reducts in dependence spaces may be transformed into an algorithm for finding reducts in information systems and vice versa; thus, both types of algorithms have the same effectivity.

We now study the problem of constructing the set $RED(K, X)$ where (A, K) is a dependence space and $X \in B(A)$. The construction following directly from the definition of a K-reduct of X means to test all subsets X' of X and to state which of them satisfy the condition $(X, X') \in K$; furthermore, it is necessary to compare the sets satisfying this condition with respect to inclusion and to find all minimal elements among them. But this is too laborious because the above mentioned test is superfluous for many subsets X' of X if we have some information about them. Thus, our leading idea is as follows. We start with the set X and cancel successively its elements in such a way that any set X' obtained by cancelling some elements satisfies the condition $(X, X') \in K$; we stop this procedure if X' is such that for any X'' obtained from X' by cancelling one element, we obtain $(X, X'') \notin K$. Clearly, it is superfluous to test further subsets of X'. This X' is, clearly, a K-reduct of X (it satisfies $(X, X') \in K$ and is minimal among sets with this property). As we have seen, the resulting set X' is dependent on the order in which the elements of X have been cancelled.

Some algorithms for finding reducts are known (see, e.g., [1], [2], [8], and Section 2 of [5]). First, we describe an algorithm for finding one K-reduct, then an algorithm for finding all K-reducts of a set where the former is a step of the latter. Our description will be informal. Some steps of the algorithms require properties that can be proved from the preceding steps; such proofs are separated by square brackets from the description of the algorithm.

Let (A, K) be a dependence space.

We suppose that the set A is given by the list of its elements and that an effective criterion is presented that permits to decide for any $X, Y \in B(A)$ whether $(X, Y) \in K$ holds or not.

Let $Z \in B(A)$ be a set, o a linear ordering on Z. We now define the successor $SC(K, o, Z)$ and the set of bad elements $BD(K, o, Z)$ of Z with respect to K and o as follows.

$$SC(K,o,Z) = \begin{cases} Z \text{ if } (Z, Z - \{z\}) \notin K \text{ for any } z \in Z, \\ Z - \{a\} \text{ if } a \text{ is the least element in } Z \text{ with respect} \\ \quad \text{to } o \text{ such that } (Z, Z - \{a\}) \in K; \end{cases}$$

$$BD(K,o,Z) = \begin{cases} \emptyset \text{ if } (Z, Z - \{z\}) \notin K \text{ for any } z \in Z, \\ \{Z - \{z\};\ z \in Z,\ (z,a) \in o, z \neq a\} \text{ if } a \\ \quad \text{is the least element in } Z \text{ with respect to } o \text{ such that} \\ \quad (Z, Z - \{a\}) \in K, \end{cases}$$

where $(z, a) \in o$ means that z precedes a in o.

7. Algorithm for finding one K-reduct. Let (A, K) be a dependence space, $Z \subseteq X \subseteq A$ sets such that $(Z, X) \in K$. Suppose that o is a linear ordering on Z. We put $SC^{(0)}(K, o, Z) = Z$, $BD^{(0)}(K, o, Z) = \emptyset$ and proceed by induction: $SC^{(i+1)}(K, o, Z) = SC(K, o, SC^{(i)}(K, o, Z))$, $BD^{(i+1)}(K, o, Z) = BD^{(i)}(K, o, Z) \cup BD(K, o, SC^{(i)}(K, o, Z))$ for any $i \geq 0$ where o denotes also the restrictions of o to subsets of Z.

[By an easy induction we obtain $SC^{(i)}(K, o, Z) \subseteq Z$, $(SC^{(i)}(K, o, Z), Z) \in K$ and, therefore, $(SC^{(i)}(K, o, Z), X) \in K$ for any $i \geq 0$. Furthermore, $Y \in BD(K, o, SC^{(i)}(K, o, Z))$ implies that $(SC^{(i)}(K, o, Z), Y) \notin K$ and, therefore, $(Y, Z) \notin K$, $(Y, X) \notin K$ for any $i \geq 0$.]

There exists a least integer $q \geq 0$ such that $SC^{(q+1)}(K, o, Z) = SC^{(q)}(K, o, Z)$. We put $R(K, o, Z) = SC^{(q)}(K, o, Z)$, $P(K, o, Z) = BD^{(q)}(K, o, Z)$. □

8. Theorem. *Let (A, K) be a dependence space, $Z \subseteq X \subseteq A$ sets such that $(Z, X) \in K$. Suppose that o is a linear ordering on Z. Then the following assertions hold.*

(a) $R(K, o, Z)$ is a K-reduct of X.

(b) $(X, Y) \notin K$ holds for any $Y \in P(K, o, Z)$.

Proof. We have $R(K, o, Z) = SC^{(q)}(K, o, Z)$ for some $q \geq 0$ and we have proved that $SC^{(q)}(K, o, Z) \subseteq Z \subseteq X$, $(SC^{(q)}(K, o, Z), X) \in K$. Furthermore, $SC^{(q+1)}(K, o, Z) = SC^{(q)}(K, o, Z)$ implies that $SC(K, o, SC^{(q)}(K, o, Z)) = SC^{(q)}(K, o, Z)$ which entails $(SC^{(q)}(K, o, Z), SC^{(q)}(K, o, Z) - \{z\}) \notin K$ for any $z \in SC^{(q)}(K, o, Z)$ and, therefore, $SC^{(q)}(K, o, Z)$ is minimal with respect to inclusion among all subsets Y of X with the property $(Y, X) \in K$. Thus *(a)* holds. Property *(b)* has been proved above. □

We denote by $LO(X)$ the set of all linear orderings on X. By 8, we obtain

9. Corollary. *Let (A, K) be a dependence space, $X \subseteq A$ a set. Then $RED(K, X) = \{R(K, o, X); o \in LO(X)\}$.*

Proof. By 8, any $R(K, o, X)$ with $o \in LO(X)$ is a member of $RED(K, X)$. Conversely, if $X' \in RED(K, X)$, we define a linear ordering o on X in such a way that elements of the set $X - X'$ precede elements of X' in o. It is easy to see that $X' = R(K, o, X))$. □

By 9, a K-reduct of X corresponds to a linear ordering on X and any linear ordering on X defines a K-reduct of X. But the construction of all K-reducts of X on the base of all linear orderings on X is difficult in practice and requires many elementary steps that can be excluded. The construction of all K-reducts of X may be simplified using the following

10. Theorem. *Let (A, K) be a dependence space, $X \subseteq A$ a set, X' its K-reduct. Then the following assertions hold.*

(i) If Z is such that $X' \subseteq Z \subseteq X$, $X' \neq Z$, then Z is not a K-reduct of X.

(ii) If Z is such that $Z \subseteq X'$, $X' \neq Z$, then Z is not a K-reduct of X.

(iii) If Y, Z are such that $Z \subseteq Y \subseteq X$, $(X,Y) \notin K$, then Z is not a K-reduct of X.

P r o o f. *(i)* and *(ii)* follow directly from the definition of a K-reduct, *(iii)* is a consequence of 2. □

This theorem characterizes some subsets of X that can be excluded from the constructions of K-reducts if some K-reducts have been constructed.

We now present an algorithm for finding all K-reducts. We construct objects $Z(i)$, $RD(i)$, $EX(i)$, $CN(i)$, $MA(i)$, $T(i)$, $NS(i)$, $NC(i)$, $EM(i)$ by induction using algorithm of 7 as a step of this construction. The objects $Z(i)$, $T(i)$ are elements in $B(X)$, and $RD(i)$, $EX(i)$, $CN(i)$, $MA(i)$, $NS(i)$, $NC(i)$, $EM(i)$ are subsets of $B(X)$: $RD(i)$ consists of K-*ReD*ucts, $EX(i)$ of elements that can be *EX*cluded if constructing new K-reducts, $CN(i)$ consists of elements that are *C*a*N*didates where constructions of new K-reducts can start, $MA(i)$ consists of *MA*ximal elements in $CN(i)$, $NS(i)$ of elements in $MA(i)$ that are *N*ot *S*uitable for further construction, $NC(i)$ of elements that are *N*ot *C*ongruent to X with respect to K, $EM(i)$ of elements that have been *E*xa*M*ined.

An important role is played by a linear ordering o on X; its restrictions to subsets of X are also linear orderings and will be denoted by the same symbol o. Furthermore, O is a linear ordering on $B(X)$; $(Y,Z) \in O$ means that Y precedes Z in O.

11. Algorithm for finding all K-reducts. Let (A,K) be a dependence space, $X \in B(A)$ a set, o a linear ordering on X, O a linear ordering on $B(X)$. We put
$Z(0) = X$;
$RD(0) = \{R(K,o,Z(0))\}$;
$EX(0) = \{Y \in B(X); Y \subseteq R(K,o,Z(0))\} \cup \{Y \in B(X);$
$R(K,o,Z(0)) \subseteq Y\} \cup \{Y \in B(X);$ there exists $W \in P(K,o,Z(0))$ with $Y \subseteq W\}$;
$CN(0) = B(X) - EX(0)$;
$MA(0) = \{Y \in CN(0);\ Y$ is maximal in $CN(0)$ with respect to inclusion$\}$;

$$T(0) = \begin{cases} X \text{ if } (Y,X) \notin K \text{ for any } Y \in MA(0), \\ T \text{ where } T \text{ is the least element in } MA(0) \text{ with respect} \\ \quad \text{to } O \text{ such that } (T,X) \in K; \end{cases}$$

$$NS(0) = \begin{cases} MA(0) \text{ if } (Y,X) \notin K \text{ for any } Y \in MA(0), \\ \{Z \in MA(0); (Z,T(0)) \in O, Z \neq T(0)\} \text{ if there exists at least} \\ \quad \text{one element } Y \in MA(0) \text{ such that } (Y,X) \in K; \end{cases}$$

$NC(0) = \{Y \in B(X);$ there exists $W \in NS(0)$ with $Y \subseteq W\}$;
$EM(0) = EX(0) \cup NC(0)$.
Let $i \geq 0$ and $Z(i)$, $RD(i)$, $EX(i)$, $CN(i)$, $MA(i)$, $T(i)$, $NS(i)$, $NC(i)$, $EM(i)$ have been constructed. Suppose that $EM(i) \neq B(X)$. Then put
$Z(i+1) = T(i)$;
$RD(i+1) = RD(i) \cup \{R(K,o,Z(i+1))\}$;

$EX(i+1) = EX(i) \cup \{Y \in B(X); Y \subseteq R(K,o,Z(i+1))\} \cup \{Y \in B(X);$
$R(K,o,Z(i+1)) \subseteq Y\} \cup \{Y \in B(X);$ there exists $W \in P(K,o,Z(i+1))$ with $Y \subseteq W\}$;
$CN(i+1) = B(X) - EX(i+1)$;
$MA(i+1) = \{Y \in CN(i+1);\ Y$ is maximal in $CN(i+1)$with respect to inclusion$\}$;

$$T(i+1) = \begin{cases} X & \text{if } (Y,X) \notin K \text{ for any } Y \in MA(i+1), \\ T & \text{where } T \text{ is the least element in } MA(i+1) \text{ with respect} \\ & \text{to } O \text{ such that } (T,X) \in K; \end{cases}$$

$$NS(i+1) = \begin{cases} MA(i+1) \text{ if } (Y,X) \notin K \text{ for any } Y \in MA(i+1), \\ \{Z \in MA(i+1);\ (Z,T(i+1)) \in O,\ Z \neq T(i+1)\} \text{ if there} \\ \text{exists at least one element } Y \in MA(i+1) \text{ with } (Y,X) \in K; \end{cases}$$

$NC(i+1) = \{Y \in B(X);$ there exists $W \in NS(i+1)$ with $Y \subseteq W\}$;
$EM(i+1) = EM(i) \cup EX(i+1) \cup NC(i+1)$.

[We now prove some properties of constructed objects.

(A) If $T(i) = X$, then $EM(i) = B(X)$ holds.

Indeed, if $Z \in B(X)$, then either $Z \in EX(i)$ or $Z \in CN(i)$. In the first case, we obtain $Z \in EM(i)$ because $EX(i) \subseteq EM(i)$. In the second case, there exists $Y \in MA(i)$ such that $Z \subseteq Y$. By $T(i) = X$, we obtain $NS(i) = MA(i)$ which implies that $Z \in NC(i) \subseteq EM(i)$.

(B) If $EM(i) \neq B(X)$, then $Z(i+1) \in EM(i+1) - EM(i)$.

Indeed, $Z(i+1) \supseteq R(K,o,Z(i+1))$ implies that $Z(i+1) \in EX(i+1) \subseteq EM(i+1)$. Furthermore, $Z(i+1) = T(i)$ and $T(i) \neq X$ holds by (A). We have $EM(i) = EX(0) \cup NC(0) \cup \cdots \cup EX(i) \cup NC(i) = EX(i) \cup NC(0) \cup \cdots \cup NC(i)$. Since $T(i) \in MA(i) \subseteq CN(i)$ and $(T(i),X) \in K$, we obtain $T(i) \notin EX(i)$ and $T(i) \notin NC(j)$ for $0 \leq j \leq i$ by 2 which means that $Z(i+1) \notin EM(i)$.]

There exists a least integer $r \geq 0$ such that $EM(r) = B(X)$. Then we put $RD(K,X) = RD(r)$.

12. Theorem. *Let (A,K) be a dependence space and $X \subseteq A$ a set. Then $RD(K,X) = RED(K,X)$.*

P r o o f. By definition, $(T(i),X) \in K$ for any i with $0 \leq i \leq r$ which implies that $R(K,o,T(i))$ is a K-reduct of X by 8. By definition of $RD(i)$, we have $RD(i) \subseteq RED(K,X)$ for any i with $0 \leq i \leq r$. Thus, $RD(K,X) = RD(r) \subseteq RED(K,X)$.

Suppose $Y \in RED(K,X)$. Thus, $Y \in B(X) = EM(r)$ for some $r \geq 0$. Let j be the least integer with $0 \leq j \leq r$, $Y \in EM(j)$. Then either $Y \in NC(j)$ or $Y \in EX(j)$.

If $Y \in NC(j)$, there exists $W \in NS(j)$ such that $Y \subseteq W$. By definition of $NS(j)$, we have $(W,X) \notin K$ and $(Y,X) \notin K$ by 2 which is a contradiction to the condition $Y \in RED(K,X)$. It follows that $Y \in EX(j)$. Let h be the least integer with $0 \leq h \leq j$, $Y \in EX(h)$.

The case $Y \subseteq W$ for some $W \in P(K, o, Z(h))$ is excluded because $(Z(h), X) \in K$ holds by definition of $Z(h)$, $(W, Z(h)) \notin K$ holds by 8 which implies that $(W, X) \notin K$ by transitivity of K. Since $Y \in RED(K, X)$, we have $(Y, X) \in K$ and $Y \subseteq W \subseteq X$ implies that $(W, X) \in K$ by 2 which is a contradiction.

Thus, we have either $Y \subseteq R(K, o, Z(h))$ or $R(K, o, Z(h)) \subseteq Y$ by definition of $EX(h)$. Since $Y \in RED(K, X)$, we obtain $Y = R(K, o, Z(h))$ by 10. Thus, $Y \in RD(h) \subseteq RD(r) = RD(K, X)$. We have $RED(K, X) \subseteq RD(K, X)$. □

13. Example. Let (W, K) be the dependence space defined in 3. We construct $RED(K, W)$ using 11. Let the linear ordering o of elements in W be as follows: a, b, c; suppose that the linear ordering O of elements in $B(W)$ is the following: N, C, B, F, A, E, D, W.

We put $Z(0) = W$, $RD(0) = \{R(K, o, Z(0))\} = \{R(K, o, W)\}$. The set $R(K, o, W)$ will be constructed by 7: $SC^{(0)}(K, o, W) = W$, $BD^{(0)}(K, o, W) = \emptyset$, $SC^{(1)}(K, o, W) = SC(K, o, W) = F$, $BD^{(1)}(K, o, W) = \emptyset$, $SC^{(2)}(K, o, W) = SC(K, o, F) = F$, $BD^{(2)}(K, o, W) = \emptyset$. Since $SC^{(1)}(K, o, W) = F = SC^{(2)}(K, o, W)$, we have $R(K, o, W) = F$, $P(K, o, W) = \emptyset$ and, therefore, $RD(0) = \{F\}$, $EX(0) = \{N, B, C, F, W\}$, $CN(0) = \{A, D, E\}$, $MA(0) = \{D, E\}$, $T(0) = W$, $NS(0) = \{D, E\}$, $NC(0) = \{D, E, A, B, C, N\}$, $EM(0) = B(W)$.

It follows that $RD(K, W) = RD(0) = \{F\}$. Thus, $RED(K, W) = \{F\}$. □

4. Problem A′.

By 5, Problem A′ is solved by means of $D(K, Y)$-reducts. We may either look for one solution of the problem or for all solutions. Algorithms 7 and 11 are suitable for finding the required reducts. For the use of these algorithms the construction of objects $SC(D(K, Y), o, Z)$ and $BD(D(K, Y), o, Z)$ is necessary. This is described by the following

14. Theorem. *Let (A, K) be a dependence space, $X \subseteq A$, $Y \subseteq A$ sets such that $X \to Y$ (A, K), o a linear ordering on X. Suppose that $Z \subseteq X$ and $Z \to Y$ (A, K) hold. Then*

$$SC(D(K,Y), o, Z) = \begin{cases} Z \text{ if } CK(Y) \not\subseteq CK(Z - \{z\}) \text{ for any } z \in Z, \\ Z - \{a\} \text{ where } a \text{ is the least element in } Z \text{ with} \\ \text{respect to } o \text{ such that } CK(Y) \subseteq CK(Z - \{a\}); \end{cases}$$

$$BD(D(K,Y), o, Z) = \begin{cases} \emptyset \text{ if } CK(Y) \not\subseteq CK(Z - \{z\}) \text{ for any } z \in Z, \\ \{Z - \{z\}; z \in Z, (z, a) \in o, z \neq a\} \text{ where } a \text{ is the} \\ \text{least element in } Z \text{ with respect to } o \text{ such that} \\ CK(Y) \subseteq CK(Z - \{a\}). \end{cases}$$

P r o o f. Since $Z \to Y$ (A, K) holds, we have $CK(Y) \subseteq CK(Z)$. Hence, $CK(Y) \subseteq CK(Z - \{z\})$ is equivalent to $(Z, Z - \{z\}) \in D(K, Y)$. Thus, if replacing conditions appearing in the definition of $SC(D(K,Y), o, Z)$ and $BD(D(K,Y), o, Z)$ by the equivalent ones, we obtain the theorem. □

From 14 it follows that the construction of all $D(K,Y)$-reducts of X requires an effective criterion that permits to decide for some sets $Z \subseteq X$ whether $CK(Y) \subseteq CK(Z)$ holds or not. We suppose that an effective criterion for the decision whether $(X, Y) \in K$ holds or not is given for arbitrary sets $X \subseteq A$, $Y \subseteq A$. Then the required criterion reads as follows.

15. Theorem. *Let (A, K) be a dependence space, $X \subseteq A$, $Y \subseteq A$ sets. Then the following two conditions are equivalent.*

(i) $CK(Y) \not\subseteq CK(X)$.

(ii) There exists $a \in A$ such that $(Y, Y \cup \{a\}) \in K$, $(X, X \cup \{a\}) \notin K$.

P r o o f. If *(i)* holds, there exists $a \in CK(Y) - CK(X)$. Then $Y \subseteq Y \cup \{a\} \subseteq CK(Y)$ and, hence, $(Y, Y \cup \{a\}) \in K$ by 2. On the other hand, $(X, X \cup \{a\}) \in K$ does not hold because it implies that $a \in X \cup \{a\} \subseteq CK(X)$ which is a contradiction. Thus $(X, X \cup \{a\}) \notin K$ and *(ii)* holds.

If *(ii)* is satisfied, then $a \in Y \cup \{a\} \subseteq CK(Y)$. The hypothesis $a \in CK(X)$ would imply that $X \subseteq X \cup \{a\} \subseteq CK(X)$ and, hence, $(X, X \cup \{a\}) \in K$ contrary to the hypothesis. Thus $a \in CK(Y) - CK(X)$ and *(i)* holds. □

16. Example. Let us consider the dependence space (W, K) defined in 3. We have stated that $F \to A$ (W, K) holds. By 5, the set of all solutions of Problem A′ coincides with the set $RED(D(K,A), F)$. We apply 11 where O is the same as in 13 and o is the linear ordering on $F = \{b, c\}$ such that c precedes b in o.

We have $Z(0) = F$, $RD(0) = \{R(D(K,A), o, Z(0))\} = \{R(D(K,A), o, F)\}$. The set $R(D(K,A), o, F)$ will be constructed by 7 and 14; since the sets $CK(Z)$ are known for any $Z \subseteq W$, we do not use 15. We obtain $SC^{(1)}(D(K,A), o, F) = SC(D(K,A), o, F) = B$, $BD^{(1)}(D(K,A), o, F) = \emptyset$, $SC^{(2)}(D(K,A), o, F) = SC(D(K,A), o, B) = N$, $BD^{(2)}(D(K,A), o, F) = \emptyset$. Clearly, $SC^{(3)}(D(K,A), o, F) = N = SC^{(2)}(D(K,A), o, F)$ and, therefore, $R(D(K,A), o, F) = N$, $P(D(K,A), o, F) = \emptyset$. It follows that $RD(0) = \{N\}$, $EX(0) = B(F)$, $CN(0) = \emptyset = MA(0)$, $T(0) = F$, $NS(0) = \emptyset = NC(0)$, $EM(0) = B(F)$. Thus, $RD(D(K,A), F) = \{N\} = RED(D(K,A), F)$ in accord with 6. □

5. Applications to information systems

The connection of Problem A to Problem A′ is given by the following

17. Theorem. *Let $S = (U, A, V, f)$ be an information system, $X \subseteq A$, $Y \subseteq A$ sets of attributes. Then the conditions $X \to Y$ (S) and $X \to Y$ $(A, K(S))$ are equivalent.*

For the proof see Theorem 14 of [3]. □

Basing on this result we may reformulate 5 as follows.

18. Theorem. *Let $S = (U, A, V, f)$ be an information system, $X \subseteq A$, $Y \subseteq A$ sets of attributes such that $X \to Y$ (S). Then for any $X' \subseteq A$ the following conditions are equivalent.*

(i) $X' \subseteq X$ and X' is minimal with respect to inclusion among all sets T with the properties $T \subseteq X$, $T \to Y$ (S).

(ii) X' is a $D(K(S), Y)$-reduct of X. □

Hence, an information system $S = (U, A, V, f)$ defines two dependence spaces: $(A, K(S))$ that is constructed by means of the discernibility relation $EQ(S, X)$ and $(A, D(K(S), Y))$ that may be used to solve Problem A.

The construction of $D(K(S), Y)$-reducts requires the knowledge of the objects $SC(D(K(S), Y), o, Z)$ and $BD(D(K(S), Y), o, Z)$. Basing on 14 and 17, we obtain

19. Theorem. *Let $S = (U, A, V, f)$ be an information system, $X \subseteq A$, $Y \subseteq A$ sets of attributes such that $X \to Y$ (S), and o a linear ordering on X. Suppose that $Z \subseteq X$ and $Z \to Y$ (S) hold. Then*

$$SC(D(K(S), Y), o, Z) = \begin{cases} Z \text{ if } EQ(S, Z - \{z\}) \not\subseteq EQ(S, Y) \text{ for any } z \in Z, \\ Z - \{a\} \text{ where } a \text{ is the least element in } Z \text{ with} \\ \text{respect to } o \text{ such that } EQ(S, Z - \{a\}) \subseteq EQ(S, Y); \end{cases}$$

$$BD(D(K(S), Y), o, Z) = \begin{cases} \emptyset \text{ if } EQ(S, Z - \{z\}) \not\subseteq EQ(S, Y) \text{ for any } z \in Z, \\ \{Z - \{z\}; z \in Z, (z, a) \in o, z \neq a\} \text{ where } a \text{ is the} \\ \text{least element in } Z \text{ with respect to } o \text{ such that} \\ EQ(S, Z - \{a\}) \subseteq EQ(S, Y). \end{cases}$$ □

We complete these investigations by presenting the explicit definition of the congruence $D(K(S), Y)$.

20. Theorem. *Let $S = (U, A, V, f)$ be an information system, $Y \subseteq A$ a set. For any $X \subseteq A$ and any $X' \subseteq A$ the condition $(X, X') \in D(K(S), Y)$ is satisfied if and only if one of the following conditions (i), (ii) holds.*

(i) $EQ(S, X) \subseteq EQ(S, Y)$, $EQ(S, X') \subseteq EQ(S, Y)$.

(ii) $EQ(S, X') = EQ(S, X) \not\subseteq EQ(S, Y)$.

This follows from 17 and 4. □

The construction of $D(K(S), Y)$-reducts requires an effective criterion that permits to decide, for any $X \subseteq A$, whether $EQ(S, X) \subseteq EQ(S, Y)$ holds or not. This is presented in the following theorem.

21. Theorem. *Let $S = (U, A, V, f)$ be an information system and $X \subseteq A$, $Y \subseteq A$ sets of attributes. Then the following conditions are equivalent.*

(i) $EQ(S,X) \not\subseteq EQ(S,Y)$.

(ii) There exist $u \in U$, $u' \in U$, $a_0 \in Y$ such that $f(u,a_0) \neq f(u',a_0)$ and $f(u,a) = f(u',a)$ for any $a \in X$.

P r o o f. If (i) holds, then there exist $u, u' \in U$ such that $(u,u') \in EQ(S,X)$, $(u,u') \notin EQ(S,Y)$. It follows that $f(u,a) = f(u',a)$ for any $a \in X$ while there exists $a_0 \in Y$ such that $f(u,a_0) \neq f(u',a_0)$. Thus, (ii) holds.

If (ii) holds, then $(u,u') \in EQ(S,X)$ but $(u,u') \notin EQ(S,Y)$ which is (i). □

We now illustrate Problem A.

22. Example. Let us have $S = (U, A, V, f)$ where $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$, $A = \{a, b, c, d, e\}$, $V = \{0, 1, 2\}$, and f is defined by the following table.

	a	b	c	d	e
u_1	1	0	0	1	1
u_2	1	0	0	0	1
u_3	0	0	0	0	0
u_4	1	1	0	1	0
u_5	1	1	0	2	2
u_6	2	1	0	2	2
u_7	2	2	2	2	2

We put $X = \{a,b,c,d\}$, $Y = \{e\}$. Clearly $EQ(S,X) = \mathrm{id}_U$ and $EQ(S,Y)$ has the blocks $\{u_1,u_2\}$, $\{u_3,u_4\}$, $\{u_5,u_6,u_7\}$. Hence $X \to Y$ (S) holds.

The solution of Problem A means finding minimal sets $X' \subseteq X$ with $X' \to Y$ (S). By 17, this is the same as the construction of minimal sets $X' \subseteq X$ such that $X' \to Y$ $(A, K(S))$. By 5, this means finding $D(K(S),Y)$-reducts of X. We construct all $D(K(S),Y)$-reducts of X using 11.

Let o be a linear ordering on X where the order of elements is as follows: a, b, c, d. Furthermore, let O denote the linear ordering on $B(X)$ such that the order of its elements is the following: $\emptyset$, $\{d\}$, $\{c\}$, $\{c,d\}$, $\{b\}$, $\{b,d\}$, $\{b,c\}$, $\{b,c,d\}$, $\{a\}$, $\{a,d\}$, $\{a,c\}$, $\{a,c,d\}$, $\{a,b\}$, $\{a,b,d\}$, $\{a,b,c\}$, $\{a,b,c,d\}$ (alphabetic order of characteristic functions of elements in $B(X)$ where X is ordered by o). We put $Z(0) = X$, $RD(0) = \{R(D(K(S),Y),o,X)\}$. We now use 7 for constructing $R(D(K(S),Y),o,X)$. We obtain $SC^{(0)}(D(K(S),Y),o,X) = X$, $BD^{(0)}(D(K(S),Y),o,X) = \emptyset$. Using 19, we obtain $SC^{(1)}(D(K(S),Y),o,X) = \{a,b,d\}$, $BD^{(1)}(D(K(S),Y),o,X) = \{\{b,c,d\}, \{a,c,d\}\}$. Furthermore, $SC^{(2)}(D(K(S),Y),o,X) = SC(D(K(S),Y),o,\{a,b,d\}) = \{a,b,d\}$, $BD^{(2)}(D(K(S),Y),o,X) = \{\{b,c,d\},\{a,c,d\},\{b,d\},\{a,d\},\{a,b\}\}$. Thus, $SC^{(2)}(D(K(S),Y),o,X) = \{a,b,d\} = SC^{(1)}(D(K(S),Y),o,X)$ and, therefore, $R(D(K(S),Y),o,X) = \{a,b,d\}$, $P(D(K(S),Y),o,X) = \{\{b,c,d\}, \{a,c,d\}, \{b,d\}, \{a,d\}, \{a,b\}\}$. It follows that $RD(0) = \{\{a,b,d\}\}$, $EX(0) = \{\emptyset,\{a\},\{b\},\{d\},\{a,b\}, \{a,d\},\{b,d\},\{a,b,d\}, \{a,b,c,d\},\{c\},\{b,c\},\{c,d\},\{b,c,d\},\{a,c\}, \{a,c,d\}\}$, $CN(0) = \{\{a,b,c\}\} = MA(0)$, $T(0) = X$, $NS(0) = \{\{a,b,c\}\}$, $NC(0) = \{\emptyset,\{a\},\{b\},\{c\}, \{a,b\},\{a,c\}, \{b,c\},\{a,b,c\}\}$, $EM(0) = B(X)$.

It follows that $RD(K,X) = \{\{a,b,d\}\} = RED(K,X)$.

We have proved that $\{a,b,d\}$ is the only solution of Problem A.

The information system S is an example of a consistent decision table where X is the set of conditions and Y the set of decisions. Our procedure leads to a reduction of the set of conditions where the reduced set suffices to provide correct values of decisions. Cf. Section 6.3 of [6] where further examples can be found. □

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