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ROUGH MEMBERSHIP FUNCTIONS: A TOOL FOR REASONING WITH UNCERTAINTY

Z. PAWLAK

Institute of Computer Science, Warsaw University of Technology
Nowowiejska 15/19, 00-665 Warszawa, Poland

A. SKOWRON

Institute of Mathematics, University of Warsaw
Banacha 2, 00-913 Warszawa, Poland

Abstract. A variety of numerical approaches for reasoning with uncertainty have been investigated in the literature. We propose *rough membership functions*, rm-functions for short, as a basis for such reasoning. These functions have values in the interval $[0, 1]$ and are computable on the basis of the observable information about the objects rather than on the objects themselves. We investigate properties of the rm-functions. In particular, we show that our approach is intensional with respect to the class of all information systems [P91]. As a consequence we point out some differences between the rm-functions and the fuzzy membership functions [Z65], e.g. the rm-function values for $X \cup Y$ $(X \cap Y)$ cannot be computed in general by applying the operation max (min) to the rm-function values for X and Y.

1. Introduction. One of the fundamental problems studied in artificial intelligence is related to the object classification, that is, the problem of associating a particular object with one of many predefined sets. We study that problem. Our approach is based on the observation that the classification of objects is performed on the basis of the accessible information about them. Objects with the same accessible information will be considered as indiscernible [P91]. Therefore we are faced with the problem of determining whether or not an object belongs to a given set when only some properties (i.e. attribute values) of the object are accessible.

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We introduce rough membership functions (rm-functions for short) which allow us to measure the degree with which any object with given attribute values belongs to a given set X. The information about objects is stored in data tables called information systems [P91]. Any rm-function $\mu_X^{\mathbb{A}}$ is defined for a given information system $\mathbb{A}$ and a given set X of objects.

The paper is structured as follows.

Section 2 contains a brief discussion of information systems [P91], information functions [Sk91] and rough sets [P91]. In Section 3 we define a partition of boundary regions [S91] and we present some basic properties of this partition, which we apply later.

In Section 4 we define the rm-functions and we study their basic properties.

In Section 5 we present formulas for computing the rm-function values $\mu_{X\cup Y}^{\mathbb{A}}(x)$ and $\mu_{X\cap Y}^{\mathbb{A}}(x)$ from the values $\mu_X^{\mathbb{A}}(x)$ and $\mu_Y^{\mathbb{A}}(x)$ (when it is possible, i.e. when the classified objects are not in a particular boundary region) if information encoded in the information system $\mathbb{A}$ is accessible. In the construction of those formulas we apply the partition of boundary regions related to X and Y defined in Section 3. One can interpret that result as follows: the computation of the rm-function values $\mu_{X\cup Y}^{\mathbb{A}}(x)$ and $\mu_{X\cap Y}^{\mathbb{A}}(x)$ (if one excludes a particular boundary region!) is *extensional* under the condition that the information system is fixed.

We also show, in Section 5, that our approach is *intensional* with respect to the set of all information systems (with a universe including the sets X and Y), namely it is not possible, in general, to compute the rm-function values $\mu_{X\cup Y}^{\mathbb{A}}(x)$ and $\mu_{X\cap Y}^{\mathbb{A}}(x)$ from $\mu_X^{\mathbb{A}}(x)$ and $\mu_Y^{\mathbb{A}}(x)$ when information about $\mathbb{A}$ is not accessible (Theorem 3).

In Section 5 we specify the maximal classes of information systems such that the computation of rm-function values for union and intersection is extensional when related to those classes, and is defined by the operations min and max (as in the fuzzy set approach [Z65, DP80]), i.e. the values $\mu_{X\cup Y}^{\mathbb{A}}(x)$ and $\mu_{X\cap Y}^{\mathbb{A}}(x)$ are obtained by applying the operations min and max to $\mu_X^{\mathbb{A}}(x)$ and $\mu_Y^{\mathbb{A}}(x)$, respectively (if $\mathbb{A}$ belongs to those maximal classes).

2. Information systems and rough sets. Information systems (sometimes called data tables, attribute-value systems, condition-action tables etc.) are used for representing knowledge. The information system notion presented here is due to Pawlak and was investigated by several researchers (see the references in [P91]).

The rough sets have been introduced as a tool for dealing with inexact, uncertain or vague knowledge in artificial intelligence applications, like for example knowledge based systems in medicine, natural language processing, pattern recognition, decision systems, approximate reasoning. Since 1982 the rough sets have been intensively studied and by now many practical applications based on the theory of rough sets have been implemented.

In this section we present some basic notions related to information systems and rough sets which will be necessary for understanding our results.

An information system is a pair $\mathbb{A} = (U, A)$, where U is a non-empty, finite set called the *universe*, and A is a non-empty, finite set of *attributes*, i.e.

$$a : U \to V_a \quad \text{for } a \in A\,,$$

where V_a is called the *value* set of a.

With every subset of attributes $B \subseteq A$ we associate a binary relation $IND(B)$, called the *B-indiscernibility relation*, and defined as follows:

$$IND(B) = \{(x, y) \in U^2 : \text{for every } a \in B,\ a(x) = a(y)\}\,.$$

By $[x]_{IND(B)}$ or $[x]_B$ we denote the equivalence class of x with respect to the equivalence relation $IND(B)$, i.e. the set $\{y \in U : x IND(B)\, y\}$.

If $x IND(B)\, y$ then we say that the objects x and y are *indiscernible* with respect to attributes from B. In other words, we cannot distinguish x from y in terms of attributes in B.

Some subsets of objects in an information system cannot be expressed exactly in terms of the available attributes, they can only be roughly defined.

If $\mathbb{A} = (U, A)$ is an information system, $B \subseteq A$ and $X \subseteq U$ then the sets

$$\underline{B}X = \{x \in U : [x]_B \subseteq X\} \quad \text{and} \quad \overline{B}X = \{x \in X : [x]_B \cap X \neq \emptyset\}$$

are called the *B-lower* and the *B-upper approximation of* X in $\mathbb{A}$, respectively.

A set X is said to be *B-definable* if $\overline{B}X = \underline{B}X$. It is easy to observe that $\underline{B}X$ is the greatest B-definable set contained in X, whereas $\overline{B}X$ is the smallest B-definable set containing X. One can observe that a set is B-definable iff it is the union of some equivalence classes of the indiscernibility relation $IND(B)$.

By $\mathbb{P}(X)$ we denote the power set of X.

Every information system $\mathbb{A} = (U, A)$ determines an *information function*

$$Inf_{\mathbb{A}} : U \to \mathbb{P}\Big(A \times \bigcup_{a\in A} V_a\Big)$$

defined as follows:

$$Inf_{\mathbb{A}}(x) = \{(a, a(x)) : a \in A\}\,.$$

Hence $x IND(A) y$ iff $Inf_{\mathbb{A}}(x) = Inf_{\mathbb{A}}(y)$.

We restrict our considerations to the information functions related to information systems but our results can be extended to the case of more general information functions [Sk91]. One can consider as *information function* an arbitrary function f defined on the set of objects U with values in some computable set C.

For example, one may take as the set U of objects the set $Tot_{\mathbb{A}}$ of total elements in the Scott information system $\mathbb{A}$ [Sc82] and as C a computable (an accessible) subset of the set D of sentences in $\mathbb{A}$. The information function f related to C can be defined as follows: $f(x) = x \cap C$ for $x \in Tot_{\mathbb{A}}$.

Every such general information function f defines the indiscernibility relation $IND(f) \subseteq U \times U$ as follows:

$$xIND(f)y \quad \text{iff} \quad f(x) = f(y).$$

3. An approximation of classifications. In this section we introduce and study the notion of approximation of classification. It was preliminarily considered in [S91, SG91]. The main idea is based on the observation that it is possible to classify boundary regions corresponding to sets from a given classification, i.e. a partition of the object universe.

Let $\mathbb{A} = (U, A)$ be an information system and let $\mathbb{X}, \mathbb{Z}$ be families of subsets of U such that $\mathbb{Z} \subseteq \mathbb{X}$ and $|\mathbb{Z}| > 1$, where $|\mathbb{Z}|$ denotes the cardinality of $\mathbb{Z}$. The set

$$\bigcap_{X \in \mathbb{Z}} BN_A(X) \cap \bigcap_{X \in \mathbb{X} - \mathbb{Z}} (U - BN_A(X))$$

is said to be the *$\mathbb{Z}$-boundary region* defined by $\mathbb{X}$ and $\mathbb{A}$ and is denoted by $Bd_{\mathbb{A}}(\mathbb{Z}, \mathbb{X})$.

By $CLASS\_APPR_{\mathbb{A}}(\mathbb{X})$ we denote the set family

$$\{\underline{A}X : X \in \mathbb{X}\} \cup \{Bd_{\mathbb{A}}(\mathbb{Z}, \mathbb{X}) : \mathbb{Z} \subseteq \mathbb{X} \text{ and } |\mathbb{Z}| > 1\}.$$

From the above definitions we get the following proposition [S91]:

PROPOSITION 1. *Let $\mathbb{A} = (U, A)$ be an information system and let $\mathbb{X}$ be a family of pairwise disjoint subsets of U such that $\bigcup \mathbb{X} = U$. Let $\mathbb{Z} \subseteq \mathbb{X}$ and $|\mathbb{Z}| > 1$. Then*

(i) *The set $Bd_{\mathbb{A}}(\mathbb{Z}, \mathbb{X})$ is definable in $\mathbb{A}$;*
(ii) *$CLASS\_APPR_{\mathbb{A}}(\mathbb{X}) - \{\emptyset\}$ is a partition of U;*
(iii) *If $x \in Bd_{\mathbb{A}}(\mathbb{Z}, \mathbb{X})$ then $[x]_A \subseteq \bigcup \mathbb{Z}$;*
(iv) *If $x \in Bd_{\mathbb{A}}(\mathbb{Z}, \mathbb{X})$ then for every $X \in \mathbb{X}$*

$$[x]_A \cap X \neq \emptyset \quad \textit{iff} \quad X \in \mathbb{Z};$$

(v) *The following equality holds:*

$$\underline{A}\Big(\bigcup \mathbb{Y}\Big) = \bigcup_{X \in \mathbb{Y}} \underline{A}X \cup \bigcup_{\substack{|\mathbb{Z}|>1 \\ \mathbb{Z} \subseteq \mathbb{Y}}} Bd_{\mathbb{A}}(\mathbb{Z}, \mathbb{X}), \qquad \text{where } \mathbb{Y} \subseteq \mathbb{X}.$$

Proof. (i) If $x \in Bd_{\mathbb{A}}(\mathbb{Z}, \mathbb{X})$ then $x \in BN_{\mathbb{A}}(X)$ for any $X \in \mathbb{Z}$ and $x \in U - BN_{\mathbb{A}}(X)$ for any $X \in \mathbb{X} - \mathbb{Z}$. From the definability in $\mathbb{A}$ of the sets $BN_{\mathbb{A}}(X)$ and $U - BN_{\mathbb{A}}(X)$ for $X \subseteq U$ we have $[x]_A \subseteq BN_{\mathbb{A}}(X)$ for any $X \in \mathbb{Z}$ and $[x]_A \subseteq U - BN_{\mathbb{A}}(X)$ for any $X \in \mathbb{X} - \mathbb{Z}$. Hence $[x]_A \subseteq Bd_{\mathbb{A}}(\mathbb{Z}, \mathbb{X})$. We proved that $Bd_{\mathbb{A}}(\mathbb{Z}, \mathbb{X}) \subseteq \underline{A}(Bd_{\mathbb{A}}(\mathbb{Z}, \mathbb{X}))$. Since $Bd_{\mathbb{A}}(\mathbb{Z}, \mathbb{X}) \supseteq \underline{A}(Bd_{\mathbb{A}}(\mathbb{Z}, \mathbb{X}))$ we get $Bd_{\mathbb{A}}(\mathbb{Z}, \mathbb{X}) = \underline{A}(Bd_{\mathbb{A}}(\mathbb{Z}, \mathbb{X}))$.

(ii) It is easy to observe that $CLASS\_APPR_{\mathbb{A}}(\mathbb{X})$ is a family of pairwise disjoint sets. We prove that

$$\bigcup CLASS\_APPR_{\mathbb{A}}(\mathbb{X}) = U\,.$$

If $x \in U$ then $x \in X$ for some $X \in \mathbb{X}$. If $x \in \underline{A}X$ then $x \in CLASS\_APPR_{\mathbb{A}}(\mathbb{X})$, otherwise $x \in \bar{A}X - \underline{A}X$. In the latter case let

$$\mathbb{Z}_x = \{X \in \mathbb{X} : [x]_A \cap X \neq \emptyset\}\,.$$

Then we have $|\mathbb{Z}_x| > 1$ and $x \in Bd_{\mathbb{A}}(\mathbb{Z}_x, \mathbb{X})$.

(iii) Let $x \in Bd_{\mathbb{A}}(\mathbb{Z}, \mathbb{X})$. Suppose that $y \notin \bigcup \mathbb{Z}$ for some $y \in [x]_A$. Since $\bigcup \mathbb{X} = U$ we have $y \in \bigcup \mathbb{X} - \bigcup \mathbb{Z}$. Hence $y \in X_0$ for some $X_0 \in \mathbb{X} - \mathbb{Z}$. In consequence, $X_0 \cap [y]_A = X_0 \cap [x]_A \neq \emptyset$. If $x \in Bd_A(\mathbb{Z}, \mathbb{X})$ then $x \in U - BN_A(X)$ for $X \in \mathbb{X} - \mathbb{Z}$. Since $U - BN_A(X)$ is definable in $\mathbb{A}$ we obtain $[x]_A \subseteq U - BN_A(X) = (U - \bar{A}X) \cup \underline{A}X$. Hence $[x]_A \subseteq \underline{A}X_0$ or $[x]_A \subseteq U - \bar{A}X_0$. Since $X_0 \cap [x]_A \neq \emptyset$ we get

$$(*) \qquad [x]_A \subseteq \underline{A}X_0\,.$$

From the assumption $x \in Bd_{\mathbb{A}}(\mathbb{Z}, \mathbb{X})$ we also have $x \in BN_A(X)$ for any $X \in \mathbb{Z}$, so

$$(**) \qquad [x]_A \cap X \neq \emptyset \quad \text{for any } X \in \mathbb{Z}\,.$$

From $(*)$ and $(**)$ we would have $X \cap X_0 \neq \emptyset$ for any $X \in \mathbb{Z}$ but this contradicts the assumption that $\mathbb{X}$ is a family of pairwise disjoint sets.

(iv) Let $x \in Bd_{\mathbb{A}}(\mathbb{Z}, \mathbb{X})$ and $X \in \mathbb{X}$.

Suppose that $[x]_A \cap X \neq \emptyset$, i.e. $x \in BN_A(X)$. Hence from the definition of $Bd_{\mathbb{A}}(\mathbb{Z}, \mathbb{X})$ we have $X \in \mathbb{Z}$.

If $X \in \mathbb{Z}$ then we have $x \in BN_A(X)$. Hence $[x]_A \cap X \neq \emptyset$.

(v) ($\subseteq$) If $x \in Bd_{\mathbb{A}}(\mathbb{Z}, \mathbb{X})$ we deduce from (iii) that $x \in \underline{A} \bigcup \mathbb{Z} \subseteq \underline{A} \bigcup \mathbb{Y}$. We also have $\underline{A}X \subseteq \underline{A} \bigcup \mathbb{Y}$ for any $X \in \mathbb{Y}$.

($\supseteq$) Let $x \in \underline{A} \bigcup \mathbb{Y}$, i.e. $[x] \subseteq \bigcup \mathbb{Y}$. If $x \notin \underline{A}X$ for $X \in \mathbb{Y}$ then let $\mathbb{Z}_x = \{X \in \mathbb{Y} : [x]_{\mathbb{A}} \cap X \neq \emptyset\}$. Hence $|\mathbb{Z}_x| > 1$ and $[x]_A \subseteq \bigcup \mathbb{Z}_x$. Thus, we have $x \in Bd_{\mathbb{A}}(\mathbb{Z}_x, \mathbb{X})$. ∎

4. Rough membership functions—definition and basic properties. One of the fundamental notions of set theory is the membership relation, usually denoted by $\in$. When one considers subsets of a given universe it is possible to apply characteristic functions for expressing the fact whether or not a given element belongs to a given set. We discuss the case when only partial information about objects is accessible. In this section we show it is possible to extend the characteristic function notion to that case.

Let $\mathbb{A} = (U, A)$ be an information system and let $\emptyset \neq X \subseteq U$. The *rough $\mathbb{A}$-membership function of the set X* (or *rm-function*, for short), denoted by $\mu_X^{\mathbb{A}}$, is defined as follows:

$$\mu_X^{\mathbb{A}}(x) = \frac{|[x]_A \cap X|}{|[x]_A|} \quad \text{for } x \in U\,.$$

The above definition is illustrated in Fig. 1.

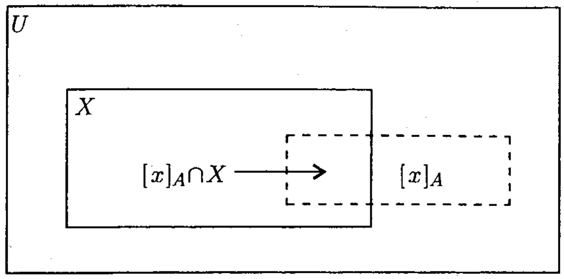


Fig. 1.

One can observe a similarity of the expression on the right hand side of the above definition with the one used to define conditional probability.

From the definition of $\mu_X^{\mathbb{A}}$ we have the following proposition characterizing some basic properties of rm-functions.

PROPOSITION 2. *Let* $\mathbb{A} = (U, A)$ *be an information system and let* $X, Y \subseteq U$. *The rm-function* $\mu_X^{\mathbb{A}}$ *has the following properties*:

(i) $\mu_X^{\mathbb{A}}(x) = 1$ *iff* $x \in \underline{A}X$;
(ii) $\mu_X^{\mathbb{A}}(x) = 0$ *iff* $x \in U - \overline{A}X$;
(iii) $0 < \mu_X^{\mathbb{A}}(x) < 1$ *iff* $x \in BN_{\mathbb{A}}(X)$;
(iv) *If* $IND(A) = \{(x, x) : x \in U\}$ *then* $\mu_X^{\mathbb{A}}$ *is the characteristic function of* X;
(v) *If* $xIND(A)y$ *then* $\mu_X^{\mathbb{A}}(x) = \mu_X^{\mathbb{A}}(y)$.
(vi) $\mu_{U-X}^{\mathbb{A}}(x) = 1 - \mu_X^{\mathbb{A}}(x)$ *for any* $x \in X$;
(vii) $\mu_{X \cup Y}^{\mathbb{A}}(x) \geq \max(\mu_X^{\mathbb{A}}(x), \mu_Y^{\mathbb{A}}(x))$ *for any* $x \in U$;
(viii) $\mu_{X \cap Y}^{\mathbb{A}}(x) \leq \min(\mu_X^{\mathbb{A}}(x), \mu_Y^{\mathbb{A}}(x))$ *for any* $x \in U$;
(ix) *If* $\mathbb{X}$ *is a family of pairwise disjoint subsets of* U *then*

$$\mu_{\bigcup \mathbb{X}}^{\mathbb{A}}(x) = \sum_{X \in \mathbb{X}} \mu_X^{\mathbb{A}}(x) \qquad \textit{for any } x \in U\,.$$

Proof. (i) We have $x \in \underline{A}X$ iff $[x]_A \subseteq X$ iff $\mu_X^{\mathbb{A}}(x) = 1$.

(ii) We have $x \in U - \overline{A}X$ iff $[x]_A \cap X = \emptyset$ iff $\mu_X^{\mathbb{A}}(x) = 0$.

(iii) We have

$$\begin{aligned} x \in BN_{\mathbb{A}}(X) \quad &\text{iff} \quad ([x]_A \cap X \neq \emptyset \text{ and } [x]_A \cap (U - X) \neq \emptyset) \\ &\text{iff} \quad (\mu_X^{\mathbb{A}}(x) > 0 \text{ and } \mu_X^{\mathbb{A}}(x) < 1)\,. \end{aligned}$$

(iv) If $IND(A) = \{(x, x) : x \in U\}$ then $|[x]_A| = 1$ for any $x \in X$. Moreover, $|[x]_A \cap X| = 1$ if $x \in X$ and $|[x]_A \cap X| = 0$ if $x \in U - X$.

(v) Since $[x]_A = [y]_A$ we have $\mu_X^{\mathbb{A}}(x) = \mu_X^{\mathbb{A}}(y)$.

(vi) We have

$$\mu_{U-X}^{\mathbb{A}}(x) = \frac{|[x]_A \cap (U - X)|}{|[x]_A|} = 1 - \frac{|[x]_A \cap X|}{|[x]_A|} = 1 - \mu_X^{\mathbb{A}}(x)\,.$$

(vii) We have

$$\mu^{\mathbb{A}}_{X\cup Y}(x) = \frac{|[x]_A \cap (X \cup Y)|}{|[x]_A|} \geq \frac{|[x]_A \cap X|}{|[x]_A|} = \mu^{\mathbb{A}}_X(x)\,.$$

In a similar way one can obtain $\mu^{\mathbb{A}}_{X\cup Y}(x) \geq \mu^{\mathbb{A}}_Y(x)$.

(viii) The proof runs as in the case (vi).

(ix) We have

$$\mu^{\mathbb{A}}_{\bigcup \mathbb{X}}(x) = \frac{|[x]_A \cap \bigcup \mathbb{X}|}{|[x]_A|} = \frac{|\bigcup\{[x]_A \cap X : X \in \mathbb{X}\}|}{|[x]_A|} = \sum_{X\in\mathbb{X}} \mu^{\mathbb{A}}_X(x)\,.$$

The last equality follows from the assumption that $\mathbb{X}$ is a family of pairwise disjoint sets. ■

The set $\{Inf_{\mathbb{A}}(x) : x \in U\}$ is called the $\mathbb{A}$-*information set* and is denoted by INF($\mathbb{A}$). For every $X \subseteq U$ we define the *rough* $\mathbb{A}$-*information function*, denoted by $\widehat{\mu}^{\mathbb{A}}_X$, as follows:

$$\widehat{\mu}^{\mathbb{A}}_X(u) = \mu^{\mathbb{A}}_X(x), \qquad \text{where } u \in \mathrm{INF}(\mathbb{A}) \text{ and } Inf_{\mathbb{A}}(x) = u\,.$$

The correctness of the above definition follows from (v) of Proposition 1.

If $\mathbb{A} = (U, A)$ is an information system then we define the *rough* $\mathbb{A}$-*inclusion* of subsets of U in the standard way, namely:

$$X \leq_{\mathbb{A}} Y \quad \text{iff} \quad \mu^{\mathbb{A}}_X(x) \leq \mu^{\mathbb{A}}_Y(x) \text{ for any } x \in U\,.$$

PROPOSITION 3. *If $X \leq_{\mathbb{A}} Y$ then $\underline{A}X \subseteq \underline{A}Y$ and $\bar{A}X \subseteq \bar{A}Y$.*

Proof. Follows from Proposition 2 (see (i) and (ii)). ■

The above definition of the rough $\mathbb{A}$-inclusion is not equivalent to the one of [P91]. Indeed, in [P91] the reverse implication to that formulated in Proposition 2 is not valid.

One can show that they are equivalent for any information system $\mathbb{A}$ only if $\bar{A}X \subseteq \underline{A}Y$. This is a consequence of our definition taking into account some additional information about objects from the boundary regions.

5. Rough membership functions for union and intersection. Now we present some results which are obtained as a consequence of our assumption that objects are observable by means of partial information about them represented by attribute values. In this section we prove that the inequalities in (vii) and (viii) of Proposition 2 cannot in general be replaced with equalities.

We also prove that for some boundary regions it is not possible to compute the values of the rm-functions for the union $X \cup Y$ and intersection $X \cap Y$ from the values of the rm-functions for X and Y only (if information about information systems is not accessible and there do not hold some special relations between the sets X and Y). These results show that the assumptions about properties of the fuzzy membership functions [DP80, p. 11] related to the union and intersection should be modified if one would like to take into account that objects are classified on the basis of partial information about them. We also present necessary and sufficient conditions for the following equalities (used in fuzzy set theory) to be true, for any $x \in U$:

$$\mu^{\mathbb{A}}_{X\cup Y}(x) = \max(\mu^{\mathbb{A}}_X(x), \mu^{\mathbb{A}}_Y(x)),$$
$$\mu^{\mathbb{A}}_{X\cap Y}(x) = \min(\mu^{\mathbb{A}}_X(x), \mu^{\mathbb{A}}_Y(x)).$$

These conditions are expressed by means of the boundary regions of a partition of U defined by X and Y or by means of some relationships which should hold for X and Y. In particular, we show that the above equalities are true for an arbitrary information system $\mathbb{A}$ iff $X \subseteq Y$ or $Y \subseteq X$.

First we prove the following two lemmas.

LEMMA 1. *Let $\mathbb{A} = (U, A)$ be an information system, $X, Y \subseteq U$ and $\mathbb{X} = \{X \cap Y, X \cap -Y, -X \cap Y, -X \cap -Y\}$. If $x \in U - Bd_{\mathbb{A}}(\mathbb{X}, \mathbb{X})$ then*

$\mu^{\mathbb{A}}_{X\cap Y}(x) =$

if $x \in Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y\}, \mathbb{X}) \cup Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}, \mathbb{X})$

then 0

else if $x \in Bd_{\mathbb{A}}(\{X \cap Y, X \cap -Y, -X \cap Y\}, \mathbb{X})$

then $\mu^{\mathbb{A}}_X(x) + \mu^{\mathbb{A}}_X(x) - 1$

else $\min(\mu^{\mathbb{A}}_X(x), \mu^{\mathbb{A}}_Y(x))$.

Proof. In the proof we apply Proposition 1(iii).

Let $x \in Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y\}, \mathbb{X}) \cup Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}, \mathbb{X})$. Hence $[x]_A \subseteq (X \cap -Y) \cup (-X \cap Y) \cup (-X \cap -Y)$, so $[x]_A \cap (X \cap Y) = \emptyset$ and $\mu^{\mathbb{A}}_{X\cap Y}(x) = 0$.

If $x \in Bd_{\mathbb{A}}(\{X \cap Y, X \cap -Y, -X \cap Y\}, \mathbb{X})$ then

$$[x]_A \subseteq X \cap Y \cup X \cap -Y \cup -X \cap Y.$$

Hence $[x]_A = [x]_A \cap (X \cap Y) \cup [x]_A \cap (X \cap -Y) \cup [x]_A \cap (-X \cap Y)$, so $[x]_A = [x]_A \cap X \cup [x]_A \cap Y$. We obtain $|[x]_A| = |[x]_A \cap X| + |[x]_A \cap Y| - |[x]_A \cap (X \cap Y)|$. Hence $\mu^{\mathbb{A}}_{X\cap Y}(x) = \mu^{\mathbb{A}}_X(x) + \mu^{\mathbb{A}}_Y(x) - 1$.

If $x \in \underline{A}(X \cap Y)$ then $[x]_A \subseteq X \cap Y$. Hence $\mu^{\mathbb{A}}_{X\cap Y}(x) = 1$. We also have $[x]_A \subseteq X$ and $[x]_A \subseteq Y$ because $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$. Hence $\mu^{\mathbb{A}}_X(x) = \mu^{\mathbb{A}}_Y(x) = 1$.

If $x \in \underline{A}(X \cap -Y)$ then $[x]_A \subseteq X \cap -Y$. Hence $[x]_A \cap (X \cap Y) = \emptyset$ and $[x]_A \cap Y \subseteq (X \cap -Y) \cap Y = \emptyset$, so

$$\mu^{\mathbb{A}}_{X\cap Y}(x) = \min(\mu^{\mathbb{A}}_X(x), \mu^{\mathbb{A}}_Y(x)).$$

If $x \in \underline{A}(-X \cap Y)$ the proof is analogous to the latter case.

If $x \in \underline{A}(-X \cap -Y)$ we obtain $\mu^{\mathbb{A}}_{X\cap Y}(x) = \mu^{\mathbb{A}}_X(x) = \mu^{\mathbb{A}}_Y(x) = 0$.

If $x \in Bd_{\mathbb{A}}(\{X \cap Y, X \cap -Y\}, \mathbb{X})$ we have $[x]_A = [x]_A \cap (X \cap Y) \cup [x]_A \cap (X \cap -Y)$. Hence $[x]_A \cap (X \cap Y) = [x]_A \cap Y$ and $[x]_A = [x]_A \cap X \subseteq X$. Hence $\mu^{\mathbb{A}}_{X\cap Y}(x) = \mu^{\mathbb{A}}_Y(x) \leq \mu^{\mathbb{A}}_X(x) = 1$.

If $x \in Bd_{\mathbb{A}}(\{X \cap Y, -X \cap Y\}, \mathbb{X})$ the proof is analogous to the latter case.

If $x \in Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap -Y\}, \mathbb{X})$ one can calculate that $\mu^{\mathbb{A}}_{X \cap Y}(x) = \mu^{\mathbb{A}}_{Y}(x) = 0 \leq \mu^{\mathbb{A}}_{X}(x)$. Similarly, in the case when $x \in Bd_{\mathbb{A}}(\{-X \cap Y, -X \cap -Y\}, \mathbb{X})$ one can calculate that $\mu^{\mathbb{A}}_{X \cap Y}(x) = \mu^{\mathbb{A}}_{X}(x) = 0 \leq \mu^{\mathbb{A}}_{Y}(x)$.

If $x \in Bd_{\mathbb{A}}(\{X \cap Y, -X \cap -Y\}, \mathbb{X})$ we have $\mu^{\mathbb{A}}_{X \cap Y}(x) = \mu^{\mathbb{A}}_{X}(x) = \mu^{\mathbb{A}}_{Y}(x)$. ∎

LEMMA 2. *Let* $\mathbb{A} = (U, A)$ *be an information system,* $X, Y \subseteq U$ *and* $\mathbb{X} = \{X \cap Y, X \cap -Y, -X \cap Y, -X \cap -Y\}$. *If* $x \in U - Bd_{\mathbb{A}}(\mathbb{X}, \mathbb{X})$ *then*

$\mu^{\mathbb{A}}_{X \cup Y}(x) =$

if $x \in Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y\}, \mathbb{X}) \cup Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}, \mathbb{X})$

then $\mu^{\mathbb{A}}_{X}(x) + \mu^{\mathbb{A}}_{Y}(x)$

else if $x \in Bd_{\mathbb{A}}(\{X \cap Y, X \cap -Y, -X \cap Y\}, \mathbb{X})$

then 1

else $\max(\mu^{\mathbb{A}}_{X}(x), \mu^{\mathbb{A}}_{Y}(x))$.

Proof. In the proof we apply Proposition 1(iii).

If $x \in Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y\}$ then

$$[x]_A = [x]_A \cap (X \cap -Y) \cup [x]_A \cap (-X \cap Y).$$

Hence $[x]_A \cap X = [x]_A \cap X \cap -Y$, $[x]_A \cap Y = [x]_A \cap -X \cap Y$. Since

$$[x]_A \cap (X \cup Y) = ([x]_A \cap X) \cup ([x]_A \cap Y) \quad \text{and}$$
$$([x]_A \cap X) \cap ([x]_A \cap Y) = [x]_A \cap X \cap -Y \cap -X \cap Y = \emptyset$$

we get $\mu^{\mathbb{A}}_{X \cup Y}(x) = \mu^{\mathbb{A}}_{X}(x) + \mu^{\mathbb{A}}_{Y}(x)$.

If $x \in Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}, \mathbb{X})$ then

$$[x]_A = [x]_A \cap (X \cap -Y) \cup [x]_A \cap (-X \cap Y) \cup [x]_A \cap (-X \cap -Y).$$

Since

$$[x]_A \cap (X \cup Y) = ([x]_A \cap X) \cup ([x]_A \cap Y) \quad \text{and}$$
$$([x]_A \cap X) \cap ([x]_A \cap Y) = [x]_A \cap X \cap -Y \cap -X \cap Y = \emptyset$$

we get $\mu^{\mathbb{A}}_{X \cup Y}(x) = \mu^{\mathbb{A}}_{X}(x) + \mu^{\mathbb{A}}_{Y}(x)$.

If $x \in Bd_{\mathbb{A}}(\{X \cap Y, X \cap -Y, -X \cap Y\}, \mathbb{X})$ then

$$[x]_A = [x]_A \cap (X \cap Y) \cup [x]_A \cap (X \cap -Y) \cup [x]_A \cap (-X \cap Y).$$

Hence $[x]_A \cap (X \cup Y) = [x]_A$, so $\mu^{\mathbb{A}}_{X \cup Y}(x) = 1$.

If $x \in \underline{A}(-X \cap -Y)$ then $[x]_A = [x]_A \cap (-X \cap -Y)$. Hence $[x]_A \cap (X \cup Y) = [x]_A \cap X = [x]_A \cap Y = \emptyset$.

If $x \in \underline{A}(X \cap Y)$ then $[x]_A = [x]_A \cap X \cap Y$. Hence $[x]_A \cap (X \cup Y) = [x]_A = [x]_A \cap X = [x]_A \cap Y$.

If $x \in \underline{A}(-X \cap Y)$ then $[x]_A = [x]_A \cap (-X \cap Y)$. Hence $[x]_A \cap (X \cup Y) = [x]_A \cap Y \neq \emptyset$ and $[x]_A \cap X = \emptyset$. If $x \in \underline{A}(-X \cap Y)$ then the proof is analogous.

If $x \in Bd_{\mathbb{A}}(\{X \cap Y, X \cap -Y\}, \mathbb{X})$ then

$$[x]_A = [x]_A \cap (X \cap Y) \cup [x]_A \cap (X \cap -Y).$$

Hence $[x]_A \cap (X \cup Y) = [x]_A \cap X \supseteq [x]_A \cap (X \cap Y) = [x]_A \cap Y$.

If $x \in Bd_{\mathbb{A}}(\{X \cap Y, -X \cap Y\}, \mathbb{X})$ then the proof is analogous.

If $x \in Bd_{\mathbb{A}}(\{X \cap Y, -X \cap -Y\}, \mathbb{X})$ then $\mu^{\mathbb{A}}_{X\cup Y}(x) = \mu^{\mathbb{A}}_X(x) = \mu^{\mathbb{A}}_Y(x)$.

If $x \in Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap -Y\}, \mathbb{X})$ then $\mu^{\mathbb{A}}_{X\cup Y}(x) = \mu^{\mathbb{A}}_X(x)$ and $\mu^{\mathbb{A}}_Y(x) = 0$.

If $x \in Bd_{\mathbb{A}}(\{-X \cap Y, -X \cap -Y\}, \mathbb{X})$ then $\mu^{\mathbb{A}}_{X\cup Y}(x) = \mu^{\mathbb{A}}_Y(x)$ and $\mu^{\mathbb{A}}_X(x) = 0$.

If $x \in Bd_{\mathbb{A}}(\{X \cap -Y, X \cap Y, -X \cap -Y\}, \mathbb{X})$ then $\mu^{\mathbb{A}}_{X\cup Y}(x) = \mu^{\mathbb{A}}_X(x) \geq \mu^{\mathbb{A}}_Y(x)$.

If $x \in Bd_{\mathbb{A}}(\{-X \cap Y, X \cap Y, -X \cap -Y\}, \mathbb{X})$ then $\mu^{\mathbb{A}}_{X\cup Y}(x) = \mu^{\mathbb{A}}_Y(x) \geq \mu^{\mathbb{A}}_X(x)$. ■

THEOREM 1. *Let $\mathcal{A}$ be a (non-empty) class of information systems with the universe including sets X and Y. The following conditions are equivalent:*

(i) $\mu^{\mathbb{A}}_{X\cap Y}(x) = \min(\mu^{\mathbb{A}}_X(x), \mu^{\mathbb{A}}_Y(x))$ *for any* $x \in U$ *and* $\mathbb{A} = (U, A) \in \mathcal{A}$;

(ii) $Bd_{\mathbb{A}}(\mathbb{Y}, \mathbb{X}) = \emptyset$ *for any* $\mathbb{Y} \supseteq \{X \cap -Y, -X \cap Y\}$ *and* $\mathbb{A} = (U, A) \in \mathcal{A}$, *where* $\mathbb{X} = \{X \cap Y, X \cap -Y, -X \cap Y, -X \cap -Y\}$.

Proof. (ii)$\rightarrow$(i). Follows from Lemma 1.

(i)$\rightarrow$(ii). Suppose that $Bd_{\mathbb{A}}(\mathbb{Y}, \mathbb{X}) \neq \emptyset$ for some $\mathbb{Y} \supseteq \{X \cap -Y, -X \cap Y\}$ and $\mathbb{A} \in \mathcal{A}$.

If $x \in Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y\}, \mathbb{X}) \neq \emptyset$ for some $\mathbb{A} \in \mathcal{A}$ then

$$[x]_A \cap (X \cap -Y) \neq \emptyset \quad \text{and} \quad [x]_A \cap (-X \cap Y) \neq \emptyset.$$

Hence $\mu^{\mathbb{A}}_X(x) > 0$ and $\mu^{\mathbb{A}}_Y(x) > 0$. Also, from Lemma 1, $\mu^{\mathbb{A}}_{X\cap Y}(x) = 0$. Thus we have $\mu^{\mathbb{A}}_{X\cap Y}(x) \neq \min(\mu^{\mathbb{A}}_X(x), \mu^{\mathbb{A}}_Y(x))$, contrary to (i).

If $x \in Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}, \mathbb{X})$ for some $\mathbb{A} \in \mathcal{A}$ and $x \in U$ then one gets a contradiction with (i) in the same manner as before.

If $x \in Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, X \cap Y\}, \mathbb{X}) \neq \emptyset$ for some $\mathbb{A} \in \mathcal{A}$ then $[x]_A = [x]_A \cap (X \cap -Y) \cup [x]_A \cap (-X \cap Y) \cup [x]_A \cap (X \cap Y)$. Hence

$$[x]_A \cap X = [x]_A \cap (X \cap -Y) \cup [x]_A \cap (X \cap Y) \quad \text{and}$$
$$[x]_A \cap Y = [x]_A \cap (-X \cap Y) \cup [x]_A \cap (X \cap Y).$$

Since $[x]_A \cap (X \cap -Y) \neq \emptyset$ and $[x]_A \cap (-X \cap Y) \neq \emptyset$ we would have $\mu^{\mathbb{A}}_X(x) > \mu^{\mathbb{A}}_{X\cap Y}(x)$ and $\mu^{\mathbb{A}}_Y(x) > \mu^{\mathbb{A}}_{X\cap Y}(x)$ but this contradicts (i).

If $x \in Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y, X \cap Y\}, \mathbb{X})$ for some $\mathbb{A} \in \mathcal{A}$ then

$$[x]_A = [x]_A \cap (X \cap -Y) \cup [x]_A \cap (-X \cap Y)$$
$$\cup [x]_A \cap (X \cap Y) \cup [x]_A \cap (-X \cap -Y).$$

Again we would have

$$[x]_A \cap X = [x]_A \cap (X \cap -Y) \cup [x]_A \cap (X \cap Y) \quad \text{and}$$
$$[x]_A \cap Y = [x]_A \cap (-X \cap Y) \cup [x]_A \cap (X \cap Y).$$

Since $[x]_A \cap (X \cap -Y) \neq \emptyset$ and $[x]_A \cap (-X \cap Y) \neq \emptyset$ we would have $\mu_X^{\mathbb{A}}(x) > \mu_{X \cap Y}^{\mathbb{A}}(x)$ and $\mu_Y^{\mathbb{A}}(x) > \mu_{X \cap Y}^{\mathbb{A}}(x)$, which contradicts (i).

This completes the proof of (i)$\rightarrow$(ii). ∎

THEOREM 2. *Let $\mathcal{A}$ be a (non-empty) class of information systems with the set of objects including sets X and Y. The following conditions are equivalent:*

(i) $\mu_{X \cup Y}^{\mathbb{A}}(x) = \max(\mu_X^{\mathbb{A}}(x), \mu_Y^{\mathbb{A}}(x))$ *for any* $x \in U$ *and* $\mathbb{A} = (U, A) \in \mathcal{A}$;

(ii) $Bd_{\mathbb{A}}(\mathbb{Y}, \mathbb{X}) = \emptyset$ *for any* $\mathbb{Y} \supseteq \{X \cap -Y, -X \cap Y\}$ *and* $\mathbb{A} = (U, A) \in \mathcal{A}$, *where* $\mathbb{X} = \{X \cap Y, -X \cap Y, X \cap -Y, -X \cap -Y\}$.

Proof. (ii)$\rightarrow$(i). Follows from Lemma 2.

(i)$\rightarrow$(ii). Suppose that $Bd_{\mathbb{A}}(\mathbb{Y}, \mathbb{X}) \neq \emptyset$ for some $\mathbb{Y} \supseteq \{X \cap -Y, -X \cap Y\}$ and $\mathbb{A} \in \mathcal{A}$.

If $x \in Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y\}, \mathbb{X}) \neq \emptyset$ for some $\mathbb{A} \in \mathcal{A}$ then

$$[x]_A \cap (X \cap -Y) \neq \emptyset \quad \text{and} \quad [x]_A \cap (-X \cap Y) \neq \emptyset .$$

Hence $\mu_X^{\mathbb{A}}(x) > 0$ and $\mu_Y^{\mathbb{A}}(x) > 0$. Also, from Lemma 2, $\mu_{X \cup Y}^{\mathbb{A}}(x) = \mu_X^{\mathbb{A}}(x) + \mu_Y^{\mathbb{A}}(x)$. This gives $\mu_{X \cup Y}^{\mathbb{A}}(x) > \mu_X^{\mathbb{A}}(x)$ and $\mu_{X \cup Y}^{\mathbb{A}}(x) > \mu_Y^{\mathbb{A}}(x)$, contrary to (i).

If $x \in Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}, \mathbb{X})$ for some $\mathbb{A} \in \mathcal{A}$ and $x \in U$ then one gets a contradiction with (i) as before.

If $x \in Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, X \cap Y\}, \mathbb{X}) \neq \emptyset$ for some $\mathbb{A} \in \mathcal{A}$ then $[x]_A = [x]_A \cap (X \cap -Y) \cup [x]_A \cap (-X \cap Y) \cup [x]_A \cap (X \cap Y)$ and $[x]_A \cap Z \neq \emptyset$ for $Z \in \{X \cap -Y, -X \cap Y, X \cap Y\}$. Hence

$$|[x]_A| > |[x]_A \cap X| \quad \text{and} \quad |[x]_A| > |[x]_A \cap Y| .$$

Thus $\mu_X^{\mathbb{A}}(x) < 1$ and $\mu_Y^{\mathbb{A}}(x) < 1$. However, $\mu_{X \cup Y}^{\mathbb{A}}(x) = 1$ from Lemma 2. This contradicts (i).

Now assume that $x \in Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y, X \cap Y\}, \mathbb{X})$ for some $\mathbb{A} \in \mathcal{A}$. Then

$$\begin{aligned}[x]_A = {} & [x]_A \cap (X \cap -Y) \cup [x]_A \cap (-X \cap Y) \\ & \cup [x]_A \cap (X \cap Y) \cup [x]_A \cap (-X \cap -Y)\end{aligned}$$

and

$$[x]_A \cap Z \neq \emptyset \quad \text{for } Z \in \{X \cap -Y, -X \cap Y, -X \cap -Y, X \cap Y\} .$$

Hence

$$[x]_A \cap (X \cup Y) = [x]_A \cap X \cup [x]_A \cap (-X \cap Y) ,$$
$$[x]_A \cap (X \cup Y) = [x]_A \cap Y \cup [x]_A \cap (X \cap -Y) .$$

Consequently, $\mu_{X \cup Y}^{\mathbb{A}}(x) > \mu_X^{\mathbb{A}}(x)$ and $\mu_{X \cup Y}^{\mathbb{A}}(x) > \mu_Y^{\mathbb{A}}(x)$. This contradicts (i).

The proof of (i)$\rightarrow$(ii) is complete. ∎

Now we characterize the conditions related to the boundary regions occurring in Theorems 1 and 2.

LEMMA 3. *Let $\mathcal{A}$ be a class of information systems with the set of objects including sets X and Y. The following conditions are equivalent for arbitrary* $\mathbb{A} = (U, A) \in \mathcal{A}$:

(i) $Bd_{\mathbb{A}}(\mathbb{Y}, \mathbb{X}) = \emptyset$ *for any* $\mathbb{Y} \supseteq \{X \cap -Y, -X \cap Y\}$, *where* $\mathbb{X} = \{X \cap Y, -X \cap Y, X \cap -Y, -X \cap -Y\}$;

(ii) $\alpha \vee \beta \vee \gamma \vee \delta \vee \varepsilon$ *holds, where*

$$\alpha := (X \subseteq Y \text{ or } Y \subseteq X);$$

$$\beta := (X - Y \neq \emptyset \text{ and } Y - X \neq \emptyset \text{ and } X \cup Y = U \text{ and } X \cap Y = \emptyset \\ \text{ and } Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y\}, \mathbb{X}) = \emptyset);$$

$$\gamma := (X - Y \neq \emptyset \text{ and } Y - X \neq \emptyset \text{ and } X \cup Y = U \text{ and } X \cap Y \neq \emptyset \\ \text{ and } Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y\}, \mathbb{X}) = \emptyset \\ \text{ and } Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, X \cap Y\}, \mathbb{X}) = \emptyset);$$

$$\delta := (X - Y \neq \emptyset \text{ and } Y - X \neq \emptyset \text{ and } X \cup Y \neq U \text{ and } X \cap Y = \emptyset \\ \text{ and } Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y\}, \mathbb{X}) = \emptyset \\ \text{ and } Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}, \mathbb{X}) = \emptyset);$$

$$\varepsilon := (X - Y \neq \emptyset \text{ and } Y - X \neq \emptyset \text{ and } X \cup Y \neq U \text{ and } X \cap Y \neq \emptyset \\ \text{ and } Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y\}, \mathbb{X}) = \emptyset \\ \text{ and } Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}, \mathbb{X}) = \emptyset \\ \text{ and } Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, X \cap Y\}, \mathbb{X}) = \emptyset \\ \text{ and } Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, X \cap Y, -X \cap -Y\}, \mathbb{X}) = \emptyset).$$

Proof. We have the following equivalences:

$$Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y\}, \mathbb{X}) = \emptyset \quad \text{iff} \\ X \subseteq Y \text{ or } Y \subseteq X \text{ or} \\ (X - Y \neq \emptyset \text{ and } Y - X \neq \emptyset \text{ and } Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y\}, \mathbb{X}) = \emptyset);$$

$$Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}, \mathbb{X}) = \emptyset \quad \text{iff} \\ X \subseteq Y \text{ or } Y \subseteq X \text{ or } X \cup Y = U \text{ or} \\ (X - Y \neq \emptyset \text{ and } Y - X \neq \emptyset \text{ and } X \cup Y \neq U \text{ and} \\ Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}, \mathbb{X}) = \emptyset);$$

$$Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, X \cap Y\}, \mathbb{X}) = \emptyset \quad \text{iff} \\ X \subseteq Y \text{ or } Y \subseteq X \text{ or } X \cap Y = \emptyset \text{ or} \\ (X - Y \neq \emptyset \text{ and } Y - X \neq \emptyset \text{ and } X \cap Y \neq \emptyset \text{ and} \\ Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, X \cap Y\}, \mathbb{X}) = \emptyset);$$

$$Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y, X \cap Y\}, \mathbb{X}) = \emptyset \quad \text{iff}$$
$$X \subseteq Y \text{ or } Y \subseteq X \text{ or } X \cap Y = \emptyset \text{ or } X \cup Y = U \text{ or}$$
$$(X - Y \neq \emptyset \text{ and } Y - X \neq \emptyset \text{ and } X \cap Y \neq \emptyset \text{ and } X \cup Y \neq U \text{ and}$$
$$Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y, X \cap Y\}, \mathbb{X}) = \emptyset).$$

Hence, taking the conjunction of the above equivalences, we obtain

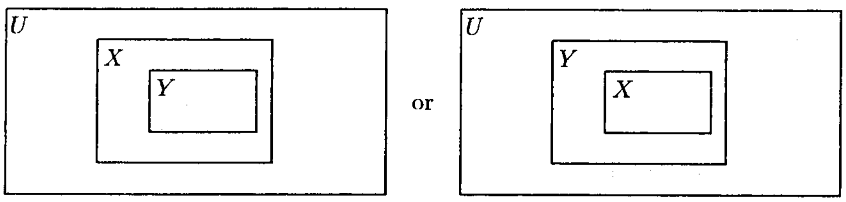
$$Bd_{\mathbb{A}}(\mathbb{Y}, \mathbb{X}) = \emptyset \quad \text{for any } \mathbb{Y} \supseteq \{X \cap -Y, -X \cap Y\}$$

iff one of the conditions $\alpha, \beta, \gamma, \delta, \varepsilon$ from (ii) is satisfied. ∎

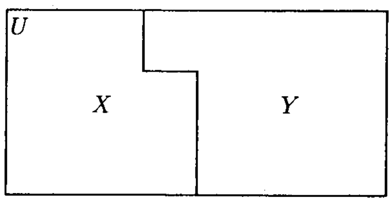
Let us remark that only when condition α holds, i.e. when $X \subseteq Y$ or $Y \subseteq X$, condition (ii) is independent of the properties of boundary regions in the information systems.

Below we illustrate the conditions formulated in (ii) of Lemma 3.

α:



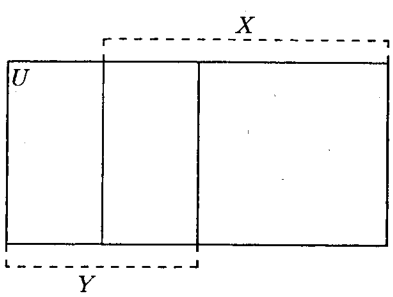
β:



X and Y form a partition of U. The condition for the boundary regions is

$$Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y\}, \mathbb{X}) = \emptyset.$$

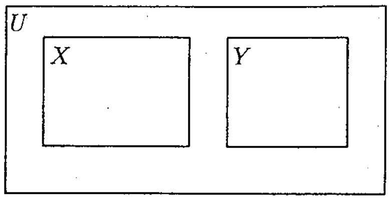
γ:



The conditions for the boundary regions are

$$Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y\}, \mathbb{X}) = \emptyset \quad \text{and}$$
$$Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, X \cap Y\}, \mathbb{X}) = \emptyset.$$

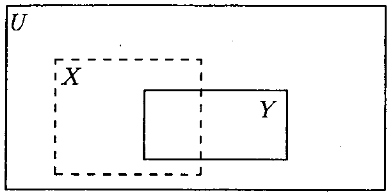
δ:



The conditions for the boundary regions are

$$Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y\}, \mathbb{X}) = \emptyset \quad \text{and}$$
$$Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}, \mathbb{X}) = \emptyset .$$

ε:



The conditions for the boundary regions are

$$Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y\}, \mathbb{X}) = \emptyset \quad \text{and}$$
$$Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}, \mathbb{X}) = \emptyset \quad \text{and}$$
$$Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, X \cap Y\}, \mathbb{X}) = \emptyset \quad \text{and}$$
$$Bd_{\mathbb{A}}(\{X \cap -Y, -X \cap Y, X \cap Y, -X \cap -Y\}, \mathbb{X}) = \emptyset .$$

Now we prove that the assumptions of Lemmas 1 and 2 related to the boundary region $Bd_{\mathbb{A}}(\mathbb{X}, \mathbb{X})$ cannot be removed because otherwise it will not be possible to compute the values of $\mu^{\mathbb{A}}_{X\cup Y}(x)$ and $\mu^{\mathbb{A}}_{X\cap Y}(x)$ from $\mu^{\mathbb{A}}_{X}(x)$ and $\mu^{\mathbb{A}}_{Y}(x)$ only.

THEOREM 3. *There is no function $F : [0,1] \times [0,1] \to [0,1]$ such that for any finite sets X and Y and any information system $\mathbb{A} = (U, A)$ such that $X, Y \subseteq U$ the following equality holds*:

$$\mu^{\mathbb{A}}_{X\cup Y}(x) = F(\mu^{\mathbb{A}}_{X}(x), \mu^{\mathbb{A}}_{Y}(x)) \quad \textit{for any } x \in U .$$

Proof. Take $X = \{1,2,3,5\}$ and $Y = \{1,2,3,4\}$. Let $U = \{1,\ldots,8\}$. It is easy to construct attribute sets A and A' such that $[1]_A = U$ and $[1]_{A'} = \{1,4,5,6\}$.

Thus we have

$$\mu^{\mathbb{A}}_{X}(1) = \mu^{\mathbb{A}}_{Y}(1) = 1/2 \quad \text{and} \quad \mu^{\mathbb{A}}_{X\cup Y}(1) = 5/8, \qquad \text{where } \mathbb{A} = (U, A)$$

and

$$\mu^{\mathbb{B}}_{X}(1) = \mu^{\mathbb{B}}_{Y}(1) = 1/2 \quad \text{and} \quad \mu^{\mathbb{B}}_{X\cup Y}(1) = 3/4, \qquad \text{where } \mathbb{B} = (U, A') . \blacksquare$$

Similarly one can prove

THEOREM 4. *There is no function $F : [0,1] \times [0,1] \to [0,1]$ such that for any finite sets X and Y and any information system $\mathbb{A} = (U, A)$ such that $X, Y \subseteq U$ the following equality holds*:

$$\mu^{\mathbb{A}}_{X \cap Y}(x) = F(\mu^{\mathbb{A}}_X(x), \mu^{\mathbb{A}}_Y(x)) \quad \textit{for any } x \in U\,. \ \blacksquare$$

Conclusions. We introduced the rough membership functions (rm-functions) as a new tool for reasoning with uncertainty. Their definition is based on the observation that objects are classified by means of partial information which is available. That definition allows us to overcome some problems which may be encountered if we use other approaches (like the ones mentioned in Section 5). We have investigated the properties of the rm-functions and, in particular, we have shown that the rm-functions are computable in an algorithmic way, so that their values can be derived without the help of an expert.

We would also like to point out one important topic for further research based on the results presented here. Our rm-functions are defined relative to information systems. We will look for a calculus with rules based on properties of rm-functions and also on belief and plausibility functions for information systems. One important problem to be studied is the definition of strategies which would allow reconstructing those rules when the information systems are modified by environment. In some sense we would like to embed rm-functions as well as the belief and plausibility functions [Sh76, S91, SG91] into a non-monotonic reasoning related to the information systems.

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