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MATHEMATICS

Use of Expansions with a Negative Basis in the Arithmometer of a Digital Computer

by

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In designing a modern automatic digital computer it is important to obtain a possibly uniform organisation of the machine. This not only gives a clearer scheme but also helps to eliminate some of the defective actions,, improves the control and even simplifies the process of constructing such a machine. The present paper contains a suggestion of a more uniform conception of the arithmetical unit of a digital computer. Similar investigations, though for a different purpose, have been made by Shannon [1].

In the arithmetical units of the digital computers so far constructed, many complications, with regard both to organisation and to the technical aspect, are due to the necessity of performing the basic arithmetical operations (addition, subtraction, multiplication and division) on numbers with both signs. This is connected with the practice of representing negative numbers in arithmetical unit either as a system of digits with a sign (I B M-701, BARK) or by means of a complement (ENIAC, EDSAC). In effect, it is necessary to perform one algorithm on signs and another on numbers, or to perform complicated and non-uniform algorithms.

The method suggested here eliminates this inconvenience by introducing expansions of numbers with a negative basis. As in the case of a positive basis, an expansion of a real number with an integer basis $g<-1$ is what we call the series

$$a=\sum_{i=-\infty}^{m} c_i g^i,$$

m — being an integer and c_i a natural number satisfying the inequality

$$0\leqslant c_i\leqslant -g-1,\quad i- \text{ an integer},\quad -\infty<i\leqslant m.$$

It is easy to prove the following theorems on expansions with a negative basis.

THEOREM 1. *Every real number has an expansion with the integer basis* $g<-1$.

THEOREM 2. *If a number has a finite expansion, that expansion is unique.*

THEOREM 3. *Expansions of real numbers with the integer basis* $g<-1$ *are unique, with the exception of numbers* a *of the form*

$$a=E\frac{g^k}{(-g+1)}+C\cdot g^{k+1},$$

where $E=\pm 1$ *and* C *and* k *are any integers. These numbers have two distinct infinite expansions.*

On the basis of these theorems we can formulate the algorithms of the basic arithmetical operations on expansions of numbers with an integer negative basis. Since most arithmetical units of existing digital computers compute on expansions with the basis 2, we shall confine ourselves to giving the algorithms of operations on expansions with the basis -2, which, moreover, will facilitate to some extent the comparison of the method here presented with those hitherto applied. It should be observed, however, that it presents no difficulties to arrive at analogous algorithms for expansions with any integer basis $g<-1$.

Algorithms of basic operations

Let us take two numbers: $\alpha=\sum_{i=-n_1}^{m_1} a_i(-2)^i$, $\beta=\sum_{i=-n_2}^{m_2} b_i(-2)^i$. Let their sum be $\alpha+\beta=\gamma$, $\gamma=\sum_{i=-n_3}^{m_3} c_i(-2)^i$, where n_j, m_j are non-negative integers $(j=1,2,3)$ and $a_i, b_i, c_i=0,1$.

The carry p_j, which occurs in adding two digits, a_j and b_j, will be defined in the same way as in the case of a positive basis:

$$c_j+p_j(-2)=a_j+b_j+p_{j-1}.$$

The above iteration formula obviously gives for p_j the values 0, 1 and -1. This observation makes it possible to arrange the following addition table:

a_j	0	0	1	1	0	0	1	1	0	0	1	1
b_j	0	1	0	1	0	1	0	1	0	1	0	1
p_{j-1}	0	0	0	0	1	1	1	1	-1	-1	-1	-1
c_j	0	1	1	0	1	0	0	1	1	0	0	1
p_j	0	0	0	-1	0	-1	-1	-1	1	0	0	0

This algorithm is more complicated than the algorithm for the sum of two positive numbers but, on the whole, not more difficult than the normal one, in which we must perform the operations on signs separately. Its advantage is that it treats all places in a number in the same way.

The algorithm of subtraction is easily reduced to addition by the formula

$$a-b=a+b+(-2)b,$$

which is easy to carry out, for it is obvious that $(-2)\,b$ is a shifting of number b one place to the left. For a number opposite to a we can give a similar formula:

$$-a=a+(-2)a.$$

Of course, this method of carrying out subtraction is not always useful (e. g., in working with machines having fixed points). However, listing a subtraction table is as easy as in the case of addition with a negative basis.

We have already mentioned that multiplication, by the basis, of an expanded number with the basis -2 results in its shifting one place to the left. It follows that the algorithm of multiplying numbers with the negative basis -2 is identical with multiplying positive numbers by the basis 2, provided the addition is performed according to the algorithm given at the beginning.

We shall now give the algorithm of division, which presents a different idea from the normal ones. With regard to the use of algorithms with a negative basis we wish to simplify as much as possible the method of investigating the remainder in division.

We define α and β as above. Write

$$N(\alpha)=\max[a_{n_1}\cdot 2^{-n_1},\dots,a_i 2^i,\dots,a_{m_1} 2^{m_1}] \quad \text{and} \quad \alpha_k=(-2)^k\cdot\alpha.$$

The initial value of the subtracted divisor α_p is found from the relation $N(\beta)=N(\alpha_p)$. The algorithm of forming the quotient β: $\alpha=w=\sum\limits_{i=p} w_i$ is found by defining w_i in the following way:

1° Perform the subtraction

$$v_i-\alpha_k=v_{i+1},$$

where

$$v_p=\beta.$$

a) If $N(v_{i+1})>N(\alpha_k)$, then $w_k=0$; we must then replace k by $k-1$ and perform 1°.

b) If $N(v_{i+1})=N(\alpha_k)$, then $w_k=(-2)^k$; we must perform 1° replacing i by $i+1$.

c) If $N(v_{i+1})<N(v_i)$, then $w_k=+(-2)^k$; we must then replace k by $k-1$ and i by $i+1$, and perform 1°.

This algorithm, although fairly complicated, does not seem more difficult to realise than those normally used.

Final remarks

There are a few small points that should also be observed with regard to the negative basis. In the present systems of representing numbers in the machine, the appearance of two zeros has often been very troublesome (e. g., $+0$ and -0 or 1.111111... and 0.00000). In the system which we are suggesting there exists only one 0. Consequently the elimination of one of the zeros causes that, for the fixed numerical interval (e. g., $|x|<1$), the zero is not its centre but divides it in relation 2:1. However, the asymmetry with respect to zero does not seem to be a drawback of our principle.

It will be observed, moreover, that the greatest advantage of the proposed principle of expanding numbers in a digital computer is the avoidance of the necessity of performing some operations on digits and other operations on signs; consequently, we avoid in the arithmometer any information of the nature of incoming symbols, which is a great drawback of realisations in series. In effect, we can reduce the number of systems with different actions in the arithmetic unit.

The use of expansions with a negative basis makes it possible to record real numbers solely with the aid of signs which are interpreted as digits.

In the principle here proposed the algorithm of multiplication is distinctly better, other algorithms do not seem more difficult than those normally used. However, they are all built on a uniform idea, which implies a simplification of the logical scheme of the machine and makes for a reduced probability of an incorrect action. This involves better control — "manual" or automatic — of the performance of the arithmetic unit.

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REFERENCES

[1] C. E. Shannon, *A symmetrical notation of number*, Amer. Math. Month., **57** (1950), 90.