Rough Set Rudiments *

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1   Introduction

The problem of imperfect knowledge has been tackled for a long time by philosophers, logicians and mathematicians. Recently it became also a crucial issue in the area of artificial intelligence. There are many approaches to the problem of how to understand and manipulate imperfect knowledge. The most popular one is, no doubt, the fuzzy set theory proposed by Zadeh.

This paper presents still another attempt to this problem – proposed by the author in (Pawlak, 1982) and called rough set theory. The theory has attracted attention of many researchers and practitioners all over the world, who contributed essentially to its development and applications.

The rough set concept is a new mathematical approach to vagueness and uncertainty. The rough set philosophy is founded on the assumption that with every object of the universe of discourse we associate some information (data, knowledge). E.g., if objects are patients suffering from a certain disease, symptoms of the disease form information about patients. Objects characterized by the same information are indiscernible (similar) in view of the available information about them. The indiscernibility relation generated in this way is the mathematical basis of rough set theory.

Any set of all indiscernible (similar) objects is called an elementary set, and form a basic granule (atom) of knowledge about the universe. Any union of some elementary sets is referred to as crisp (precise) set – otherwise the set is rough (imprecise, vague).

Consequently each rough set has boundary-line cases, i.e., objects which cannot be with certainty classified as members of the set or of its complement. Obviously crisp sets have no boundary-line elements at all. That means that boundary-line cases cannot be properly classified by employing the available knowledge.

Thus, the assumption that objects can be "seen" only through the information available about them leads to the view that knowledge has granular structure. Due to the

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granularity of knowledge some objects of interest cannot be discerned and appear as the same (or similar). As, a consequence vague concepts, in contrast to precise concepts, cannot be characterized in terms of information about their elements. Therefore in the proposed approach we assume that any vague concept is replaced by a pair of precise concepts – called the lower and the upper approximation of the vague concept. The lower approximation consists of all objects which surely belong to the concept and the upper approximation contains all objects which possible belong to the concept. Obviously, the difference between the upper and the lower approximation constitute the boundary region of the vague concept. Approximations are two basic operations in the rough set theory.

Rough set theory overlaps to a certain degree many other mathematical theories. Particularly interesting is the relationship with fuzzy set theory and Dempster-Shafer theory of evidence. The concepts of rough set and fuzzy set are different since they refer to various aspects of imprecision (Pawlak and Skowron, 1994) whereas the connection with theory of evidence is more substantial (Skowron and Grzymal-Busse, 1994). Besides, rough set theory is related to discriminant analysis (Krusińska et al., 1992), Boolean reasoning methods (Skowron and Rauszer, 1992) and others. The relationship between rough set theory and decision analysis is presented in (Pawlak and Słowiński, 1994, Słowiński, 1993). More details concerning these relationships can be found in the references.

Despite of the relationships rough set theory can be viewed in its own rights, as an the independent discipline.

Rough set theory has found many interesting applications. The rough set approach seems to be of fundamental importance to AI and cognitive sciences, especially in the areas of machine learning, knowledge acquisition, decision analysis, knowledge discovery from databases, expert systems, inductive reasoning and pattern recognition. It seems of particular importance to decision support systems.

The main advantage of rough set theory is that it does not need any preliminary or additional information about data – like probability in statistics, or basic probability assignment in Dempster-Shafer theory and grade of membership or the value of possibility in fuzzy set theory.

The rough set theory has been successfully applied in many real-life problems in medicine, pharmacology, engineering, banking, financial and market analysis and others. Some exemplary applications are listed below.

There are many applications in medicine (Grzymala-Busse and Woolerly 1994, Słowiński K., et al., 1988, Słowiński K., 1992, Słowiński K., and Sharif 1993, Słowiński K., et al., 1995, Tanaka et al., 1992). In pharmacology the analysis of relationships between the chemical structure and the antimicrobial activity of drugs (Krysiński 1990, 1992, 1992, 1995) has been successfully investigated. Banking applications include evaluation of a bankruptcy risk (Słowiński R., and Zopounidis 1993, 1994) and market research (Golan and Edwards 1993, Ziarko and Katzberg 1989). Very interesting results have been also obtained in speaker independent speech recognition (Brindle 1994, Czyżewski 1995, Czyżewski and Kaczmarek 1993, 1995, 1995) and acoustics (Kostek 1995, 1995, 1995, 1995). The rough set approach seems also important for various engineering applications, like diagnosis of machines using vibroacoustics, symptoms (noise, vibrations) (Nowicki et al., 1992, 1992, 1992), material sciences (Jackson et al., 1994) and process control (Lin 1995, Mrozek 1992, Munakata 1995, Plonka and Mrózek 1995, Szladow and Ziarko 1992, Ziarko 1992, Ziarko and Katzberg 1989). Application in linguistics (Grzymala-Busse et al., 1995, Grzymala-Busse and Than 1993, Kobayashi and Yokomori 1995, Moradi et al., 1995)

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and environment (Gunn *et al.*, 1994), databses (Beaubouef and Petry 1995, Braubouef *et al.*, 1995, Cercone and Han 1993, Shenoi 1995, Ziarko 1991) are other important domains.

More about applications of the rough set theory can be found in (Grzymala-Busse 1995, Lin 1994, Słowiński R., 1992, Wang 1995, Ziarko 1993). Besides, many other fields of application, e.g., time series analysis, image processing and character recognition, are being extensively explored.

Application of rough sets requires a suitable software. Many software systems for workstations and personal computers based on rough set theory have been developed. The most known include LERS (Grzymala-Busse 1992), Rough DAS and Rough Class and DATALOGIC (Szladow 1993). Some of them are available commercially.

One of the most important and difficult problem in software implementation of the presented approach is optimal decision rule generation from data. Many various approaches to solve this task can be found in (Bazan *et al.*, 1995, 1994, Grzymala-Busse *et al.*, 1995, Skowron 1995, Skowron and Stepaniuk 1994, Tsumoto and Tanaka 1995, Wróblewski 1995). The relation to other methods of rule generation is dwelt in (Grzymala-Busse *et al.*, 1995).

The theory has many important advantages. Some of them are listed below.

- Provides efficient algorithms for finding hidden patterns in data.

- Finds minimal sets of data (data reduction).

- Evaluates significance of data.

- Generates sets of decision rules from data.

- It is easy to understand.

- Offers straightforward interpretation of obtained results.

- Most algorithms based on the rough set theory are particularly suited for parallel processing, but in order to exploit this feature fully, a new computer organization based on rough set theory is necessary.

Although rough set theory has many achievements to its credit, nevertheless several theoretical and practical problems require further attention.

Especially important is widly accessible efficient software development for rough set based data analysis, particularly for large collections of data analysis.

Despite of many valuable methods of efficient, optimal decision rule generation methods from data, developed in recent years based on rough set theory – more research here is needed, particularly, when quantitative attributes are involved. In this context also further discretization methods for quantitative attribute values are badly needed. Also an extensive study of a new approach to missing data is very important. Comparison to other similar methods still requires due attention, although important results have been obtained in this area. Particularly interesting seems to be a study of the relationship between neural network and rough set approach to feature extraction from data.

Last but not least, rough set computer is badly needed for more serious computations in decision support. Some research in this area is already in progress.

For basic ideas of rough set theory the reader is referred to (Grzymala-Busse 1995, Pawlak 1991, Pawlak *et al* 1995, Słowiński 1995, Szladow and Ziarko 1993).

2 Examples

For the sake of simplicity we first explain the above ideas intuitively, by means of examples.

Data are often presented as a table, columns of which are labeled by *attributes*, rows by *objects* of interest and entries of the table are *attribute values*. For example in a table containing information about patients suffering from a certain disease objects are *patients* (strictly specking their ID's), attributes can be for example *blood pressure, body temperature* etc., whereas the entry corresponding to object *Smiths* and the attribute *blood preasure* can be normal. Such tables are known as *information systems, attribute-value* tables or *information tables*. We will use here the term *information table*. Below some examples of information tables are presented.

Example 1

Suppose we are given data about 6 patients, as shown in Table 1.

Patient	Headache	Muscle-pain	Temperature	Flu
p1	no	yes	high	yes
p2	yes	no	high	yes
p3	yes	yes	very high	yes
p4	no	yes	normal	no
p5	yes	no	high	no
p6	no	yes	very high	yes

Table 1

Columns of the table are labelled by attributes (symptoms) and rows – by objects (patients), whereas entries of the table are attribute values. Thus each row of the table can be seen as information about specific patient. For example patient p2 is characterized in the table by the following attribute-value set

(Headache, yes), (Muscle-pain, no), (Temperature, high), (Flu, yes),

which form information about the patient.

In the table patients p2, p3 and p5 are indiscernible with respect to the attribute Headache, patients p3 and p6 are indiscernible with respect to attributes Muscle-pain and Flu, and patients p2 and p5 are indiscernible with respect to attributes Headache, Muscle-pain and Temperature. Hence, for example, the attribute Headache generates two elementary sets {p2, p3, p5} and {p1, p4, p6}, whereas the attributes Headache and Muscle-pain form the following elementary sets: {p1, p4, p6}, {p2, p5} and {p3}. Similarly one can define elementary sets generated by any subset of attributes.

Patient p2 has flu, whereas patient p5 does not, and they are indiscernible with respect to the attributes Headache, Muscle-pain and Temperature, hence flu cannot be characterized in terms of attributes Headache, Muscle-pain and Temperature. Hence p2 and p5 are the boundary-line cases, which cannot be properly classified in view of the available knowledge. The remaining patients p1, p3 and p6 display symptoms which enable us to classify them with certainty as having flu, patients p2 and p5 cannot be excluded as having flu and patient p4 for sure does not have flu, in view of the displayed symptoms. Thus the lower approximation of the set of patients having flu is the set {p1, p3, p6} and the

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upper approximation of this set is the set {p1, p2, p3, p5, p6}, whereas the boundary-line cases are patients p2 and p5. Similarly p4 does not have flu and p2, p5 cannot be excludes as having flu, thus the lower approximation of this concept is the set {p4} whereas – the upper approximation – is the set {p2, p4, p5} and the boundary region of the concept "not flu" is the set {p2, p5}, the same as in the previous case.

Example 2

Another simple information table is presented in Table 2.

Store	E	Q	L	P
1	high	good	no	profit
2	med.	good	no	loss
3	med.	good	no	profit
4	no	avg.	no	loss
5	med.	avg.	yes	loss
6	high	avg.	yes	profit

Table 2

In the table six stores are characterized by four attributes, shown below:

E – empowerment of sales personnel,

Q – perceived quality of merchandise,

L – high traffic location,

P – store profit or loss.

Let us observe that each store has different description in terms of attributes E, Q, L and P, thus all stores may be distinguished (discerned) employing information provided by all attributes. However, stores 2 and 3 are indiscernible in terms of attributes E, Q and L, since they have the same values of these attributes. Similarly, stores 1, 2 and 3 are indiscernible with respect to attributes Q and L, etc.

Each subset of attributes determines a partition (classification) of all objects into classes having the same description in terms of these attributes. For example, attributes Q and L aggregate all stores into the following classes {1,2,3},{4},{5,6}. Thus, each information table determines a family of classification patterns which are used as a basis of further considerations.

Suppose we are interested in the following problem: what are the characteristic features of stores having profit (or loss) in view of information available in Table 1. In other words, the question is whether we are able to describe set (concept) {1,3,6} (or {2,4,5}) in terms of attributes E, Q and L. It can be easily seen that this is impossible, since stores 2 and 3 display the same features in terms of attributes E, Q and L, but store 2 makes a profit, whereas store 3 has a loss. Thus information given in Table 1 is not sufficient to answer this question. However, we can give a partial answer to this question. Let us observe that if the attribute E has the value *high* for a certain store, then the store makes a profit, whereas if the value of the attribute E is *low*, then the store has a loss. Thus, in view of information contained in Table 1, we can say for sure that stores 1 and 6 make a profit, stores 4 and 5 have a loss, whereas stores 2 and 3 cannot be classified as making a profit or having a loss. Therefore we can give approximate answers only. Employing attributes

E, Q and L, we can say that stores 1 and 6 *surely* make a profit, i.e., *surely* belong to the set $\{1,3,6\}$, whereas stores 1,2,3 and 6 *possibly* make a profit, i.e., possibly belong to the set $\{1,3,6\}$. We will say that the set $\{1,6\}$ is the *lower approximation* of the set (concept) $\{1,3,6\}$, and the set $\{1,2,3,6\}$ – is the *upper approximation* of the set $\{1,3,6\}$. The set $\{2,3\}$, being the difference between the upper approximation and the lower approximation is refered to as the *boundary region* of the set $\{1,3,6\}$.

Example 3

Consider now another example of information table concerning analysis of conflict situtations. In a conflict at least two participants, called in what follows *agents*, are in dispute over some *issues*. The agents may be individuals, groups, states, parties etc. The relationship of each agent to a specific issue can be clearly depicted in a form of a table, as shown in an example of the Middle East conflict, which is taken with slight modifications from Casti (cf. Casti 1988).

Consider six agents

1 – Israel,
2 – Egypt,
3 – Palestinians,
4 – Jordan,
5 – Syria,
6 – Saudi Arabia,

and five issues

a – autonomous Palestinian state on the West Bank and Gaza,

b – Israeli military outpost along the Jordan River,

c – Israeli retains East Jerusalem,

d – Israeli military outposts on the Golan Heights,

e – Arab countries grant citizenship to Palestinians who choose to remain within their borders.

In the table below the attitude of six nations of the Middle East region to the above issues is presented: –1 means, that the agent is against, 1 – favorable and 0 neutral toward the issue. For the sake of simplicity we will write – and + instead of –1 and 1 respectively.

U	a	b	c	d	e
1	–	+	+	+	+
2	+	0	–	–	–
3	+	–	–	–	0
4	0	–	–	0	–
5	+	–	–	–	–
6	0	+	–	0	+

Table 3

In this example objects in the table are agents and attributes are issues. Observe, that there are not condition and decision attributes distinguished in the table. Each row of the table characterizes uniquely an agent, by his approach to the disputed issues. In conflict analysis the most important problems include finding a minimal set of issues that uniquely characterize agents, discovering dependencies between issues and evaluation of significance of disused issues to the debate.

Example 4

This example concerns study of discrete dynamic systems, called in what follows shortly discrete systems.

Suppose a finite set $a = \{a_1, a_2, \ldots, a_n\}$ of elements called *discrete systems (DS)* is given. With every discrete systems $a \in A$ a finite set of its internal *states* V_a is associated. The following two seemingly similar problems are interesting in discrete systems investigation.

1. Analysis (explanation). Suppose that discrete systems are changing their states according to some rules. The changes are watched by an observer who does not know the rules. The results of the observation can be presented in a form of a table as shown in the example below.

U	a	b	c	d	e
1	1	0	2	1	0
2	0	0	1	2	1
3	2	0	2	1	0
4	0	0	2	2	2
5	1	2	2	1	0

Table 4

In the table the set of attributes is $A = \{a, b, c, d, e\}$ represent discrete systems. Each row in the table contains record of observed states of discrete systems, and each record is labelled by an element from the set U, in this case by numbers 1,2,3,4 and 5. For example record 3 reveals that during this observation discrete systems a, b, c, d and e were in states 2,0,2,1 and 0, respectively.

The task of the observer is to find out, on the basis of his observations, the rules governing the behavior of the system. More specifically, his task is to find out whether the discrete systems are changing their states independently or the changes are interrelated functionally.

Two approaches here are possible, called *the Closed World Assumption (CWA)* and *Open World Assumption (OWA)*. In the first case we assume that the table contains all possible states combinations, the remaining ones being prevented by the intrinsic nature of the system. In other words the table contains the whole knowledge about the observed behavior of the system – whereas in the second case only a part of possible observations is contained in the in the table, i.e., it contains partial knowledge about the system behavior only.

2. Synthesis (specification). Tables as shown before can also treated as a *specification* of the system required behavior. In this case the problem is whether such specification

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defines *concurrent* or *sequential* system and what are the rules describing the system behavior.

Attributes can be *qualitative* or *quantitative*. If attribute values are *high*, *normal* etc. the attribute is qualitative, and if attribute values are numbers the attribute is quantitative. Obviously all attributes in Table 1 and Table 2 are qualitative. Notice that all attributes in Table 3 and Table 4 are also qualitative, despite as its values numbers are used, because numbers -1,0,1, 2 are consider as codes for expressions like *yes*, *no* etc.

3 Information Tables, Decision Tables and Decision Algorithms

Sometimes we distinguish in an information table two classes of attributes, called *condition* and *decision* (*action*) attributes. For example in Table 1 attributes Headache, Muscle-pain and Temperature can be considered as condition attributes, whereas the attribute Flu – as a decision attribute. In Table 2 attributes E, Q, L are condition attributes, whereas the attribute P, is a decision one. Such tables will be referred to as it decision tables. Let us observe that Table 3 and Table 4 are not decision tables.

Each row of a decision table deteremines a *decision rule*, which specifies *decisions* (*actions*) that should be taken when conditions pointed out by *condition* attributes are satisfied. For example in Table 1 the condition (Headache, no), (Muscle-pain, yes), (Temperature, high) determines uniquely the decision (Flu, yes). Objects in a decision table are used as labels of decision rules. Decision rules 2) and 5) in Table 1 have the same conditions by different decisions. Such rules are called *inconsistent* (*nondeterministic, conflicting*); otherwise the rules are referred to as *consistent* (*certain, deterministic, nonconflicting*). Sometimes consistent decision rules are called *sure* rules, and inconsistent rules are called *possible* rules. Decision tables containing inconsistent decision rules are called *inconsistent* (*nondeterministic, conflicting*); otherwise the table is *consistent* (*deterministic, non conflicting*).

The number of consistent rules to all rules in a decision table can be used as *consistency measure* of the decision table, and will be denoted by $\gamma(C, D)$, where C and D are condition and decision attributes respectively. Thus if $\gamma(C, D) = 1$ the decision table is consistent and if $\gamma(C, D) \neq 1$ the decision table is inconsistent. For example for Table 1 $\gamma(C, D) = 4/6$.

Decision rules are often presented as implications and are called "*if... then...*" rules. For example rule 1) in Table 1 can be presented as implication

if (Headache, no) and (Muscle-pain, yes) and (Temperature, high) then (Flu, yes).

A set of decision rules is called a *decision algorithm*. Thus with each decision table we can associate a decision algorithm consisting of all decision rules occurring in the decision tables. Hence Table 1 can be presented as a the following decision algorithm:

if (Headache, no) and (Muscle-pain, yes) and (Temperature, high) then (Flu, yes),
if (Headache, yes) and (Muscle-pain, no) and (Temperature, high) then (Flu, yes),
if (Headache, yes) and (Muscle-pain, yes) and (Temperature, very high) then (Flu, yes),

if (Headache, no) and (Muscle-pain, yes) and (Temperature, normal) then (Flu, no),
if (Headache, yes) and (Muscle-pain, no) and (Temperature, high) then (Flu, no),
if (Headache, no) and (Muscle-pain, yes) and (Temperature, very high) then (Flu, yes).

We must however, make distinction between decision tables and decision algorithms. A decision table is a collection of data, whereas a decision algorithm is a collection of implications, e.g., logical expressions. To deal with data we use various mathematical methods, e.g., statistics (see Krusińska *et al.*, 1992), but to analyze implications we must employ logical tools (see Pawlak 1991). Thus these two approaches are not equivalent, however for simplicity we will often present here decision rules in form of implications, without referring deeper to their logical nature, as it is often practiced in AI.

4 Rough Sets, Approximations and Vagueness

As mentioned in the introduction, the starting point of the rough set theory is the indiscernibility relation, generated by information about objects of interest. The indiscernibility relation is intended to express the fact that due to the lack of knowledge we are unable to discern some objects employing the available information. That means that, in general, we are unable to deal with single objects but we have to consider clusters of indiscernible objects, as fundamental concepts of our theory.

Now we present above considerations more formally.

Suppose we are given two finite, non-empty sets U and A, where U is the *universe*, and A – a set *attributes*. With every attribute $a \in A$ we associate a set V_a, of its *values*, called the *domain* of a. Any subset B of A determines a binary relation $I(B)$ on U, which will be called an *indiscernibility relation*, and is defined as follows:

$xI(B)y$ if and only if $a(x) = a(y)$ for every $a \in A$,
where $a(x)$ denotes the value of attribute a for element x.

Obviously $I(B)$ is an equivalence relation. The family of all equivalence classes of $I(B)$, i.e., partition determined by B, will be denoted by $U/I(B)$, or simple U/B; an equivalence class of $I(B)$, i.e., block of the partition U/B, containing x will be denoted by $B(x)$.

If (x,y) belongs to $I(B)$ we will say that x and y are *B-indiscernible*. Equivalence classes of the relation $I(B)$ (or blocks of the partition U/B) are refereed to as *B-elementary sets*. In the rough set approach the elementary sets are the basic building blocks (concepts) of our knowledge about reality.

The indiscernibility relation will be used next to define basic concepts of rough set theory. Let us define now the following two operations on sets

$$B_*(X) = \{x \in U : B(x) \subseteq X\},$$

$$B^*(X) = \{x \in U : B(x) \cap X \neq \emptyset\},$$

assigning to every subset X of the universe U two sets $B_*(X)$ and $B^*(X)$ called the *B-lower* and the *B-upper approximation* of X, respectively. The set

$$BN_B(X) = B^*(X) - B_*(X)$$

will be referred to as the *B-boundary region* of X.

If the boundary region of X is the empty set, i.e., $BN_B(X) = \emptyset$, then the set X is *crisp* (*exact*) with respect to B; in the opposite case, i.e., if $BN_B(X) \neq \emptyset$, the set X is to as *rough* (*inexact*) with respect to B.

One can easily show the following properties of approximations:

(1) $B_*(X) \subseteq X \subseteq B^*(X)$,

(2) $B_*(\emptyset) = B^*(\emptyset) = \emptyset, B_*(U) = B^*(U) = U$,

(3) $B^*(X \cup Y) = B^*(X) \cup B^*(Y)$,

(4) $B_*(X \cap Y) = B_*(X) \cap B_*(Y)$,

(5) $X \subseteq Y$ implies $B_*(X) \subseteq B_*(Y)$ and $B^*(X) \subseteq B^*(Y)$,

(6) $B_*(X \cup Y) \supseteq B_*(X) \cup B_*(Y)$,

(7) $B^*(X \cap Y) \subseteq B^*(X) \cap B^*_{\cdot}(Y)$,

(8) $B_*(-X) = -B^*(X)$,

(9) $B^*(-X) = -B_*(X)$,

(10) $B_*(B_*(X)) = B^*(B_*(X)) = B_*(X)$,

(11) $B^*(B^*(X)) = B_*(B^*(X)) = B^*(X)$,

where $-X$ denotes $U - X$.

It is easily seen that the lower and the upper approximation of a set are interior and closure operations in a topology generated by the indiscernibility relation.

One can define to the following four basic classes of rough sets, i.e., four categories of vagueness:

a) $B_*(X) \neq \emptyset$ and $B^*(X) \neq U$, iff X is *roughly B-definable*,

b) $B_*(X) = \emptyset$ and $B^*(X) \neq U$, iff X is *internally B-indefinable*,

c) $B_*(X) \neq \emptyset$ and $B_*(X) = U$, iff X is *externally B-definable*,

d) $B_*(X) = \emptyset$ and $B^*(X) = U$, iff X is *totally B-indefinable*.

The intuitive meaning of this classification is the following.

If X is roughly *B-definable*, this means that we are able to decide for some elements of U whether they belong to X or $-X$, using B.

If X is internally B-indefinable, this means that we are able to decide whether some elements of U belong to $-X$, but we are unable to decide for any element of U, whether it belongs to X or not, using B.

If X is externally B-indefinable, this means that we are able to decide for some elements of U whether they belong to X, but we are unable to decide, for any element of U whether it belongs to $-X$ or not, using B.

If X is totally B-indefinable, we are unable to decide for any element of U whether it belongs to X or $-X$, using B.

5   Numerical Characterization of Vagueness

Rough set can be also characterized numerically by the following coefficient

$$\alpha_B(X) = \frac{|B_*(X)|}{|B^*(X)|}$$

called *accuracy of approximation*, where $|X|$ denotes the cardinality of X. Obviously $0 \leq \alpha_B(X) \leq 1$. If $\alpha_B(X) = 1$, X is *crisp* with respect to B (X is *precise* with respect to B), and otherwise, if $\alpha_B(X) < 1$, X is *rough* with respect to B (X is *vague* with respect to B).

Let us depict above definitions by examples referring to Table 1. Consider the concept "flu", i.e., the set $X = \{p1, p2, p3, p6\}$ and the set of attributes $B = \{$Headache, Muscle-pain, Temperature$\}$. Concept "flu" is roughly B-definable, because $B_*(X) = \{p1, p3, p6\} \neq \emptyset$ and $B^*(X) = \{p1, p2, p3, p5, p6\} \neq U$. For this case we get $\alpha_B("flu") = 3/5$. It means that the concept "flu" can be characterized partially employing symptoms, Headache, Muscle-pain and Temperature. Taking only one symptom $B = \{$Headache$\}$ we get $B_*(X) = \emptyset$ and $B^*(X) = U$, which means that the concept "flu" is totally indefinable in terms of attribute Headache, i.e., this attribute is not characteristic for flu whatsoever. However, taking single attribute $B = \{$Temperature$\}$ we get $B_*(X) = \{p3, p6\}$ and $B^*(X) = \{p1, p2, p3, p5, p6\}$, thus the concept "flu" is again roughly definable, but in this case we obtain $\alpha_B(X) = 2/5$, which means that the single symptom Temperature is less characteristic for flu, than the whole set of symptoms, and patient p1 cannot be now classified as having flu in this case.

Beside accuracy of approximation one can also define two interesting coefficients, denoted $\gamma_*(X)$, $\gamma^*(X)$ and called a *quality* of *lower* (*upper*) *approximation* of X by B, respectively, and defined as follows:

$$\gamma_*(X) = \frac{|B_*(X)|}{|U|} \text{ and } \gamma^*(X) = \frac{|B^*(X)|}{|U|}.$$

Obviously the accuracy measure can be defined in terms of quality measures, as

$$\alpha_B(X) = \frac{\gamma_*(X)}{\gamma^*(X)}.$$

There is a very interesting relationship between the quality measures and of the basic concepts of evidence theory: the quality of lower approximation is a belief function and the quality of upper approximation is the plausibility function of the evidence theory. More about this relationship can be found in (Grzymala-Busse 1995, Skowron and Grzymala-Busse 1994).

The above defined coefficients can be used to describe rough sets numerically, in addition to the basic topological characterization of rough sets.

6 Rough Sets Membership Function and Uncertainty

Rough sets can be also defined using a *rough membership function*, defined as

$$\mu_X^B(x) = \frac{|X \cap B(x)|}{|B(x)|}.$$

Obviously

$$\mu_X^B(x) \in [0, 1].$$

Value of the membership function $\mu_X(x)$ is kind of conditional probability, and can be interpreted as a degree of *certainty* to which x belongs to X (or $1 - \mu_X(x)$, as a degree of *uncertainty*).

The rough membership function, can be used to define approximations and the boundary region of a set, as shown below:

$$B_*(X) = \{x \in U : \mu_X^B(x) = 1\},$$

$$B^*(X) = \{x \in U : \mu_X^B(x) > 0\},$$

$$BN_B(X) = \{x \in U : 0 < \mu_X^B(x) < 1\}.$$

It can be shown (Pawlak and Skowron 1994) that the rough membership function has the following properties:

a) $\mu_X^B(x) = 1$ iff $x \in B_*(X)$,

b) $\mu_X^B(x) = 0$ iff $x \in -B^*(X)$,

c) $0 < \mu_X^B(x) < 1$ iff $x \in BN_B(X)$,

d) If $I(B) = \{(x, x) : x \in U\}$, then $\mu_X^B(x)$ is the characteristic function of X,

e) If $xI(B)y$, then $\mu_X^B(x) = \mu_X^B(y)$ provided $I(B)$,

f) $\mu_{U-X}^B(x) = 1 - \mu_X^B(x)$ for any $x \in U$,

g) $\mu_{X \cup Y}^B(x) \geq \max(\mu_X^B(x), \mu_Y^B(x))$ for any $x \in U$,

h) $\mu_{X \cap Y}^B(x) \leq \min(\mu_X^B(x), \mu_Y^B(x))$ for any $x \in U$,

i) If $\mathbf{X}$ is a family of pair wise disjoint sets of U, then $\mu_{\cup \mathbf{X}}^B(x) = \sum_{X \in \mathbf{X}} \mu_X^B(x)$ for any $x \in U$,

The above properties show clearly the difference between fuzzy and rough memberships. In particular properties g) and h) show that the rough membership formally can be regarded as a generalization of fuzzy membership, for the max and the min operations for union and intersection of sets respectively for fuzzy sets are special cases of that for rough sets. But let us recall that the "rough membership", in contrast to the "fuzzy membership", has probabilistic flavor.

For example, the membership function for concept "flu" in Table 1 is given below.

$$\mu^B_{flu}(p1) = \frac{|\{p1,p2,p3,p6\} \cap \{p1\}|}{|\{p1\}|} = 1,$$

$$\mu^B_{flu}(p2) = \frac{|\{p1,p2,p3,p6\} \cap \{p2,p4\}|}{|\{p2,p4\}|} = 1/2,$$

$$\mu^B_{flu}(p3) = \frac{|\{p1,p2,p3,p6\} \cap \{p3\}|}{|\{p3\}|} = 1,$$

$$\mu^B_{flu}(p4) = \frac{|\{p1,p2,p3,p6\} \cap \{p4\}|}{|\{p4\}|} = 0,$$

$$\mu^B_{flu}(p5) = \frac{|\{p1,p2,p3,p6\} \cap \{p2,p5\}|}{|\{p2,p5\}|} = 1/2,$$

$$\mu^B_{flu}(p6) = \frac{|\{p1,p2,p3,p6\} \cap \{p6\}|}{|\{p6\}|} = 1.$$

The membership function can be also used as a basis to define the credibility of decision rules.

Let $\delta(x)$ denote the decision rule associated with object x. We will say also that x supports rule $\delta(x)$. Then the credibility factor of this rule can be defined as

$$C(\delta(x)) = \begin{cases} 1, & \text{if } \mu^B_X(x) = 0 \text{ or } 1, \\ \mu^B_X(x), & \text{if } 0 < \mu^B_X(x) < 1. \end{cases}$$

In this way any consistent rule will get one as a credibility factor value, while any inconsistent rule will get credibility factor, which is smaller then one (but not equal to zero). The closer is the credibility factor is to one the more credible is the rule. If a decision rule is supported by more then one object, then as the credibility factor we assume the credibility factor of the most supporting object. This coefficient can be seen as an alternative to Dempster-Shafer approach. The above ideas can be generalized in the same way as proposed by Ziarko in the variable precision rough model (Ziarko 1993). Let β be a real number such that $0 \le \beta < 0.5$. Approximations can be defined now as

$$B_{*\beta}(X) = \{x \in U : \mu^B_X(x) \ge 1 - \beta\},$$

$$B^*_\beta(X) = \{x \in U : \mu(x) > \beta\}.$$

Note that if $\beta = 0$, we get the previous case.

The boundary region $BN^\beta_B(X) = B^*_\beta(X) - B_{*\beta}(X) = \{x \in U : \beta < \mu^B_X(x) < 1 - \beta\}$. The following properties are obvious

a) if $x \in B_{*\beta}(X)$, then not necessarily $B(x) \subseteq X$,

b) if $x \in U - B^*_\beta(X)$, then not necessarily $X \cap B(x) = \emptyset$,

c) if $X \cap B(x) \ne \emptyset$, then not necessarily $x \in B^*_\beta(X)$,

d) if $B(x) \not\subseteq X$, then not necessarily $x \notin B^*_\beta(X)$.

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Besides, we have

$$B_*(X) \subseteq B_{*\beta}(X) \quad \text{and} \quad B^*(X) \supseteq B^*_\beta(X).$$

The idea of variable precision rough sets enables one to relax strict conditions superimposed on the definitions of approximations and can be viewed as a weaker form of the original definitions.

It can be easily seen that there exists a strict connection between vagueness and uncertainty. As we mentioned above vagueness is related to sets (concepts), whereas uncertainty is related to elements of sets. Rough set approach shows clear connection between these two concepts.

7 Dependency of Attributes

Another important issue in data analysis is discovering dependencies between attributes. Intuitively, a set of attributes D depends totally on a set of attributes C, denoted $C \Rightarrow D$, if all values of attributes from D are uniquely determined by values of attributes from C. In other words, D depends totally on C, if there exists a functional dependency between values of D and C. In Table 1 there are not total dependencies whatsoever. If in Table 1, the value of the attribute Temperature for patient $p5$ were "no" instead of "high", there would be a total dependency {Temperature} $\Rightarrow$ {Flu}, because to each value of the attribute Temperature there would correspond unique value of the attribute Flu.

Formally dependency can be defined in the following way. Let D and C be subsets of A. We say that B *depends totally* on C, if and only if $I(C) \subseteq I(D)$. That means that the partition generated by C is finer than the partition generated by D. Notice, that the concept of dependency discussed above corresponds to that considered in relational databases.

We would need also a more general concept of dependency of attributes, called a *partial dependency* of attributes. Let us first depict the idea by example, referring to Table 1. In this table, for example, the attribute Temperature determines uniquely only some values of the attribute Flu. That is, (Temperature, very high) implies (Flu, yes), similarly (Temperature, normal) implies (Flu, no), but (Temperature, high) does not imply always (Flu, yes). Thus the partial dependency means that only some values of D are determined by values of C.

Formally, the above idea can be formulated as follows. Let D and C be subsets of A. We say that D *depends in degree* $k, 0 \le k \le 1$, on C, denoted $C \Rightarrow_k D$, if

$$k = \frac{|POS_C(D)|}{|U|},$$

where

$$POS_C(D) = \bigcup_{X \in U/I(D)} C_*(X).$$

The expression $POS_C(D)$, called a *positive region* of the partition U/D with respect to C, is the set of all elements of U that can be uniquely classified to blocks of the partition U/D, by means of C. It can be easily seen that k can be interpreted as the consistency measure $\gamma(C, D)$.

Thus the coefficient k expresses the ratio of all elements of the universe, which can be properly classified to blocks of the partition U/D, employing attributes C. Notice that for $k = 1$ we get the previous definition of total dependency.

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Obviously, a decision table is consistent if and only if $k = 1$, otherwise, i.e., if $k \neq 1$, the decision table is inconsistent; if $k = 0$ we will say that the decision table is totally inconsistent.

For dependency {Headache, Muscle-pain, Temperature} $\Rightarrow$ {Flu} we get $k = 4/6 = 2/3$, because four out of six patients can be uniquely classified as having flu or not, employing attributes Headache, Muscle-pain and Temperature.

If we were interested in how exactly patients can be diagnosed using only the attribute Temperature, that is – in the degree of the dependence {Temperature} $\Rightarrow$ {Flu}, we would get $k = 3/6 = 1/2$, since in this case only three patients $p3, p4$ and $p6$ out of six can be uniquely classified as having flu. In contrast to the previous case patient $p4$ cannot be classified now as having flu or not. Hence the single attribute Temperature offers worse classification than the whole set of attributes Headache, Muscle-pain and Temperature. It is interesting to observe that neither Headache nor Muscle-pain can be used to recognize flu, because for both dependencies {Headache} $\Rightarrow$ {Flu} and {Muscle-pain} $\Rightarrow$ {Flu} we have $k = 0$.

Summing up: D is *totally* (*partially*) dependent on C, if *all* (*some*) elements of the universe U can be uniquely classified to blocks of the partition U/D, employing C.

8   Reduction of Attributes

We often face a question whether we can remove some data from a data-table preserving its basic properties, that is – whether a table contains some superfluous data. For example, it is easily seen that if we drop in Table 1 either the attribute Headache or Muscle-pain we get the data set which is equivalent to the original one, in regard to approximations and dependencies. That is we get in this case the same accuracy of approximation and degree of dependencies as in the original table, however using smaller set of attributes.

In order to express the above idea more precisely we need some auxiliary notions. Let B be a subset of A and let a belong to B.

- We say that a is *dispensable* in B if $I(B) = I(B - \{a\})$; otherwise a is *indispensable* in B.

- Set B is *independent* if all its attributes are indispensable.

- Subset B' of B is a *reduct* of B if B' is independent and $I(B') = I(B)$.

It means that a reduct is a minimal subset of attributes that enables the same classification of elements of the universe as the whole set of attributes. In other words, attributes that do not belong to a reduct are superfluous with regard to classification of elements of the universe.

For example in Table 3 we have two reducts $\{a, b, e\}$ and $\{b, d, e\}$.

Reducts have several important properties. In what follows we will present two of them.

First, we define a notion of a core of attributes. Let B be a subset of A. The *core* of B is the set off all indispensable attributes of B. The following is an important property, connecting the notion of the core and reducts

$$Core(B) = \bigcap Red(B),$$

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where $Red(B)$ is the set off all reducts of B.

Because the core is the intersection of all reducts, it is included in every reduct, i.e., each element of the core belongs to some reduct. Thus, in a sense, the core is the most important subset of attributes, for none of its elements can be removed without affecting of the classification power of attributes.

In Table 3 the core is the set $\{b, e\}$.

Further simplification of an information table can be obtained by elimination of some values of attribute from the table in such a way that we are still able to discern objects in the table as the original one. To this end we can apply similar procedure as to eliminate superfluous attributes, which is defined next.

- We will say that the value of attribute $a \in B$, is *dispensable* for x, if $B(x) = B_a(x)$; otherwise the value of attribute a is *indispensable* for x, where $B_a = B - \{a\}$.

- If for every attribute $a \in B$ the value of a is indispensable for x, then B will be called *independent* for x.

- Subset $B' \subseteq B$ is a *value reduct* of B for x, iff B' is independent for x and $B(x) = B'(x)$.

The set of all indispensable values of attributes in B for x will be called the *value core* of B for x, and will be denoted $CORE^x(B)$.

Also in this case we have

$$CORE^x(B) = \bigcap Red^x(B),$$

where $Red^x(B)$ is the family of all reducts of B for x.

It may happen that the set D depends not on the whole set C but on its subset C' and therefore we might be interested to find out this subset. In order to solve this problem we need the notion of a relative reduct, which will be defined and discussed next.

Let $C, D \subseteq A$, and let

$$POS_C(D) = \bigcup_{X \in U/I(D)} C_*(X)$$

- We will say that attribute $a \in C$ is *D-dispensable* in C, if $POS_C(D) = POS_{(C-\{a\})}(D)$; otherwise the attribute a is *D-indispensable* in C.

- If all attributes $a \in C$ are C-indispensable in C, then C will be called *D-independent.*

- Subset $C' \subseteq C$ is a *D-reduct* of C, iff C' is *D*-independent and $POS_C(D) = POS_{C'}(D)$.

The set of all *D*-indispensable attributes in C will be called *D-core* of C, and will be denoted by $CORE_D(C)$. In this case we have also the property

$$CORE_D(C) = \bigcap Red_D(C),$$

where $Red_D(C)$ is the family of all *D*-reducts of C.

If $D = C$ we will get the previous definitions.

For example in Table 1 there are two relative reducts with respect to Flu {Headache, Temperature} and {Muscle-pain, Temperature} of the set of condition attributes {Headache,

Muscle-pain, Temperature}. That means that either the attribute Headache or Muscle-pain can eliminated from the table and consequently instead of Table 1 we can use either Table 5

Patient	Headache	Temperature	Flu
p1	no	high	yes
p2	yes	high	yes
p3	yes	very high	yes
p4	no	normal	no
p5	yes	high	no
p6	no	very high	yes

Table 5

or Table 6

Patient	Muscle-pain	Temperature	Flu
p1	yes	high	yes
p2	no	high	yes
p3	yes	very high	yes
p4	yes	normal	no
p5	no	high	no
p6	yes	very high	yes

Table 6

For Table 1 the relative core of with respect to the set {Headache, Muscle-pain, Temperature} is the Temperature. This confirms our previous considerations showing that Temperature is the only symptom that enables, at least, partial diagnosis of patients.

We will need also a concept of a *value reduct* and *value core*. Suppose we are given a dependency $C \Rightarrow D$ where C is relative D-reduct of C. To further investigation of the dependency we might be interested to know exactly how values of attributes from D depends on values of attributes from C. To this end we need a procedure eliminating values of attributes form C which does not influence on values of attributes from D.

- We say that value of attribute $a \in C$, is D-*dispensable* for $x \in U$, if

$$C(x) \subseteq D(x) \text{ implies } C_a(x) \subseteq D(x);$$

otherwise the value of attribute a is D-*indispensable* for x.

- If for every attribute $a \in C$ value of a is D-indispensable for x, then C will be called D-*independent* for x.

- Subset $C' \subseteq C$ is a D-*reduct* of C for x (a value reduct), iff C' is D-independent for x and

$$C(x) \subseteq D(x) \text{ implies } C'(x) \subseteq D(x).$$

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The set of all D-indispensable for x values of attributes in C will be called the D-core of C for x (the vlaue core), and will be denoted $CORE_D^x(C)$.

We have also the following property

$$CORE_D^x(C) = \bigcap Red_D^x(C),$$

where $Red_D^x(C)$ is the family of all D-reducts of C for x.

Using the concept of a value reduct, Table 5 and Table 6 can be simplified as follow

Patient	Headache	Temperature	Flu
p1	no	high	yes
p2	yes	high	yes
p3	–	very high	yes
p4	–	normal	no
p5	yes	high	no
p6	–	very high	yes

Table 7

Patient	Muscle-pain	Temperature	Flu
p1	yes	high	yes
p2	no	high	yes
p3	–	very high	yes
p4	–	normal	no
p5	no	high	no
p6	–	very high	yes

Table 8

where "-" denotes "don't care" values.

We can also present the obtained results in a form of a decision algorithm.

For Table 7 we get

if (Headache, no) and (Temperature, high) then (Flu, yes),

if (Headache, yes) and (Temperature, high) then (Flu, yes),

if (Temperature, very high) then (Flu, yes),

if (Temperature, normal) then (Flu, no),

if (Headache, yes) and (Temperature, high) then (Flu, no),

if (Temperature, very high) then (Flu, yes).

and for Table 8 we have

if (Muscle-pain, yes) and (Temperature, high) then (Flu, yes),

if (Muscle-pain, no) and (Temperature, high) then (Flu, yes),

if (Temperature, very high) then (Flu, yes),

if (Temperature, normal) then (Flu, no),

if (Muscle-pain, no) and (Temperature, high) then (Flu, no),

if (Temperature, very high) then (Flu, yes).

The following important property

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a) $B' \Rightarrow B - B'$, where B' is a reduct of B,

connects reducts and dependency, and can be used for discovering of dependencies in data. Besides, we have the following properties

b) If $B \Rightarrow C$, then $B \Rightarrow C'$, for every $C' \subseteq C$,

in particular

c) If $B \Rightarrow C$, then $B \Rightarrow \{a\}$, for every $a \in C$.

Moreover, the following property is valid

d) If B' is a reduct of B, then neither $\{a\} \Rightarrow \{b\}$ nor $\{b\} \Rightarrow \{a\}$ holds, for every $a, b \in B'$, i.e., all attributes in a reduct are pairwise independent.

For example in Table 3 we have the following dependecies:

$$\{a, b, e\} \Rightarrow \{c, d\},$$
$$\{b, d, e\} \Rightarrow \{a, c\},$$

and consequently the first dependency yields

$$\{a, b, e\} \Rightarrow \{c\},$$
$$\{a, b, e\} \Rightarrow \{d\},$$

whereas the second, gives

$$\{b, d, e\} \Rightarrow \{a\},$$
$$\{b, d, e\} \Rightarrow \{c\}.$$

That means that issues discussed in the debate are related and should not be discussed simultinously.

9 Indiscernibility Matrices and Functions

To compute easily reducts and the core we will use discernibility matrix (Skowron *et al.*, 1991), which is defined next.

By an discernibility matrix of $B \subseteq A$ denoted $M(B)$ we will mean $n \times n$ matrix defined as:

$$(c_{ij}) = \{a \in B : a(x_i) \neq a(x_j)\} \text{ for } i, j = 1, 2, \dots, n.$$

Thus entry c_{ij} is the set of all attributes which discern objects x_i and x_j.

The discernibility matrix $M(B)$ assigns to each pair of objects x and y a subset of attributes $\delta(x, y) \subseteq B$, with the following properties:

i) $\delta(x, x) = \emptyset$,

ii) $\delta(x, y) = \delta(y, x)$,

iii) $\delta(x, z) \subseteq \delta(x, y) \cup \delta(y, z)$.

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These properties resemble properties of semi-distance, and therefore the function δ may be regarded as *qualitative semi-matric* and $\delta(x,y)$ – *qualitative semi-distance*. Thus the discernibility matrix can be seen as a *semi-distance (qualitative)* matrix.

Let us also note that for every $x, y, z \in U$ we have

iv) $|\delta(x,x)| = 0$,

v) $|\delta(x,y)| = |\delta(y,x)|$,

vi) $|\delta(x,z)| \leq |\delta(x,y)| + |\delta(y,z)|$.

It is easily seen that the core is the set of all single element entries of the discernibility matrix $M(B)$, i.e.,

$$CORE(B) = \{a \in B : c_{ij} = \{a\}, \text{ for some } i,j\}.$$

Obviously $B' \subseteq B$ is a reduct of B, if B' is the minimal (with respect to inclusion) subset of B such that

$$B' \cap c \neq \emptyset \text{ for any nonempty entery } c \ (c \neq \emptyset) \text{ in } M(B).$$

In other words reduct is the minimal subset of attributes that discerns all objects discernible by the whole set of attributes.

Every discernibility matrix $M(B)$ defines uniquely a *discernibility (boolean) function* $f(B)$ defined as follows.

Let us assign to each attribute $a \in B$ a binary boolean variable $\bar{a}$, and let $\Sigma\delta(x,y)$ denotes boolean sum of all boolean variables assigned to the set of attributes $\delta(x,y)$. Then the discernibility function can be defined by the formula

$$f(B) = \prod_{(x,y)\in U^2} \{\Sigma\delta(x,y) : (x,y) \in U^2 \text{ and } \delta(x,y) \neq \emptyset\}.$$

The following property establishes the relationship between disjunctive normal form of the function $f(B)$ and the set of all reducts of B. (Skowron *et al.*, 1991).

All constituents in the minimal disjunctive normal form of the function $f(B)$ are all reducts of B.

For example the indiscernibility matrix for Table 4 is the following:

	1	2	3	4	5
1					
2	a,c,d,e				
3	a	a,c,d,e			
4	a,d,e	c,e	a,d,e		
5	b	a,b,c,d,e	a,b	a,b,d,e	

Table 9

The core of the set of attributes $\{a,b,c,e,d\}$ is the set $\{a,b\}$. The discernibility function for this set is

$$(a+c+d+e)a(a+c+d+e)(a+d+e)(c+e)(a+d+e)b(a+b+c+d+e)(a+b)(a+b+d+e).$$

By employing the absorption law $((x+y)x = x)$ and by "multiplying" all the constituents we get the following disjunctive normal formula

$$ab(c + e) = abc + abe.$$

Thus the set of attributes has two reducts $\{a,b,c\}$ and $\{a,b,e\}$. That means that instead of Table 4 we can use either Table 10 or Table 11

U	a	b	c
1	1	0	2
2	0	0	1
3	2	0	2
4	0	0	2
5	1	2	2

Table 10

or Table 11

U	a	b	e
1	1	0	0
2	0	0	1
3	2	0	0
4	0	0	2
5	1	2	0

Table 11

since they provide the same partition of the universe $U = \{1,2,3,4,5\}$.

In order to compute the value core and value reducts for x we can also use the discernibility matrix as defined before and the discernibility function, which must be slightly modified:

$$f^x(B) = \prod_{y \in U} \{\Sigma \delta(x,y) : y \in U \text{ and } \delta(x,y) \neq \emptyset\}.$$

Now for Table 10 we can compute the discernibility matrix:

	1	2	3	4	5
1					
2	a,c				
3	a	a,c			
4	a	c	a		
5	b	a,b,c	a,b	a,b	

Table 12

and the discernibility functions:

$$f^1(A) = (a + c)ab = ab,$$
$$f^2(A) = (a + c)c(a + b + c) = c,$$
$$f^3(A) = a(a + c)(a + b) = a,$$
$$f^4(A) = ac(a + b) = ac,$$
$$f^5(A) = b(a + b + c)(a + b) = b.$$

This means that Table 10 and Table 11 can be simplified as shown below in Table 13 and Table 14 below:

U	a	b	c
1	1	0	–
2	–	–	1
3	2	–	–
4	0	–	2
5	–	2	–

Table 13

U	a	b	e
1	1	0	–
2	–	–	1
3	2	–	–
4	–	–	2
5	–	2	–

Table 14

Relative reducts and core can be computed also using discernibility matrix, which needs slight modification

$$c_{ij} = \{a \in C : a(x_i) \neq a(x_j) \text{ and } w(x_i, x_j)\},$$

where $w(x_i, x_j) \equiv x_i \in POS_C(D)$ and $x_j \notin POS_C(D)$ or
$\qquad x_i \notin POS_C(D)$ and $x_j \in POS_C(D)$ or
$\qquad x_i, x_j \in POS_C(D)$ and $(x_j, x_j) \notin I(D)$

for $i, j = 1, 2, \ldots, n$ (Skowron et al., 1991).

If the partition defined by D is definable by C then the condition $w(x_i, x_j)$ in the above definition can be reduced to $(x_i, x_j) \notin I(D)$.

Thus entry c_{ij} is the set of all attributes which discern objects x_i and x_j that do not belong to the same equivalence calss of the relation $I(D)$.

The remaining definitions need a little changes.

The D-core is the set of all single element entries of the discernibility matrix $M_D(C)$, i.e.,

$$CORE_D(C) = \{a \in C : c_{ij} = (a), \text{ for some } i, j\}.$$

Set $C' \subseteq C$ is the D-reduct of C, if C' is the minimal (with respect to inclusion) subset of C such that

$$C' \cap c \neq \emptyset \text{ for any nonempty entry } c(c \neq \emptyset) \text{ in } M_D(C).$$

Thus D-reduct is the minimal subset of attributes that discerns all equivalence classes of the relation $I(D)$ discernible by the whole set of attributes.

Every discernibility matrix $M_D(C)$ defines uniquely a *discernibility (boolean) function* $f_D(C)$ which is defined as before we have also the following property:

All constituents in the disjunctive normal form of the function $f_D(C)$ are all D-reducts of C.

For example, consider Table 4 with $C = \{a, b, c\}$ and $D = \{d, e\}$ as condition and decision attributes respectivelly. Discernibilty matrix for this table is given below:

	1	2	3	4	5
1	`				
2	a,c				
3	–	a,c			
4	a	c	a		
5	–	a,b,c	–	a,b	

Table 15

The discernibility function and its disjunctive normal form is the following:

$$ac(a + c)(a + b + c)(a + b) = ac.$$

Thus set $C = \{a, b, c\}$ has only one D-reeduct, which is $\{a, c\}$. The means the attribute b is dispensables, i.e., objects of the universe can be equally well classified to classes of the equivalence relation $I(\{d, e\})$ without attribute b, or what is the same Table 4 can be simplified as

U	a	c	d	e
1	1	2	1	0
2	0	1	2	1
3	2	2	1	0
4	0	2	2	2
5	1	2	1	0

Table 16

We can also eliminate in this case dispensable values of attributes using the discernibility matrices and functions.

For computing value reducts and the value core we use as a starting point the discernibility matrix $M_D(C)$ and discernibility function will have the form:

$$f_D^x(C) = \prod_{y \in U} \{ \Sigma \delta(x, y) : y \in U \text{ and } \delta(x, y) \neq \emptyset \}.$$

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For Table 16 we get the following discernibility functions and their disjunctive normal forms:

$$f_D^1(C) = (a+c)a = a,$$
$$f_D^2(C) = (a+c)c(a+c) = c,$$
$$f_D^3(C) = (a+c)a = a,$$
$$f_D^4(C) = ac,$$
$$f_D^5(C) = a.$$

That means that Table 16 can be presented in equivalent form as shown below:

U	a	c	d	e
1	1	–	1	0
2	–	1	2	1
3	2	–	1	0
4	0	2	2	2
5	1	–	1	0

Table 17

The above decision table can be also regarded as a set of decision rules of the form

$$a_1 \rightarrow d_1 e_0,$$
$$a_2 \rightarrow d_1 e_0,$$
$$c_1 \rightarrow d_2 e_1,$$
$$a_0 c_2 \rightarrow d_2 e_2,$$

or

$$a_1 \vee a_2 \rightarrow d_1 e_0,$$
$$c_1 \rightarrow d_2 e_1,$$
$$a_0 c_2 \rightarrow d_2 e_2,$$

where α_i means "attribute α has value i" and symbols "$\vee$" nad " $\rightarrow$ " denote propositional alternative and implication respectively.

10 Significance of Attributes and Approximate Reducts

As it follows from considerations concerning reduction of attributes, they can be not equally important, and some of them can be eliminated from an information table without loosing information contained in the table. The idea of attribute reduction can be generalized by introduction a concept of *significance of attributes*, which enable us evaluation of attributes not only by two-valued scale, *dispensable - indispensable*, but by assigning to an attribute a real number from the closed interval [0,1], expressing how important is an attribute in an information table.

Significance of a attribute can be evaluated by measuring effect of removing the attribute from an information table on classification defined by the table. Let us first start our consideration with decision tables.

Let C and D be sets of condition and decision attributes respectively and let a be a condition attribute, i.e., $a \in C$. As shown previously the number $\gamma(C, D)$ expresses a degree of consistency of the decision table, or the degree of dependency between attributes C and D, or accuracy of approximation of U/D by C. We can ask how the coefficient $\gamma(C, D)$ changes when removing an attribute a, i.e., what is the difference between $\gamma(C, D)$ and $\gamma((C - \{a\}, D)$. We can normalize the difference and define the significance of an attribute a as

$$\sigma_{(C,D)}(a) = \frac{(\gamma(C, D) - \gamma(C - \{a\}, D))}{\gamma(C, D)} = 1 - \frac{\gamma(C - \{a\}, D)}{\gamma(C, D)},$$

and denoted simple by $\sigma(a)$, when C and D are understood.

Obviously $0 \le \sigma(a) \le 1$. The more important is the attribute a the greater is the number $\sigma(a)$. For example for condition attributes in Table 1 we have the following results:

$\sigma(\text{Headache}) = 0,$
$\sigma(\text{Muscle-pain}) = 0,$
$\sigma(\text{Temperature}) = 0.75.$

Because the significance of the attribute Temperature or Muscle-pain is zero, removing either of the attribute from condition attributes does not effect the set of consistent decision rules, whatsoever. Hence the attribute Temperature is the most significant one in the table. That means that by removing the attribute Temperature, 75% (three out of four) of consistent decision rules will disappear from the table, thus lack of the attribute essentially effects the "decisive power " of the decision table.

For a reduct of condition attributes, e.g., Headache, Temperature, we get

$\sigma(\text{Headache}) = 0.25,$
$\sigma(\text{Temperature}) = 1.00.$

In this case, removing the attribute Headache from the reduct, i.e., using only the attribute Temperature, 25% (one out of four) consistent decision rule will be lost, and dropping the attribute Temperature, i.e., using only the attribute Headache 100% (all) consistent decision rules will be lost. That means that in this case making decisions is impossible at all, whereas by employing only the attribute Temperature some decision can be made.

Thus the coefficient $\sigma(a)$ can be understood as an error which occurs when attribute a is dropped. The significance coefficient can be extended to set of attributes as follows:

$$\sigma_{(C,D)}(B) = \frac{(\gamma(C, D) - \gamma(C - B, D))}{\gamma(C, D)} = 1 - \frac{\gamma(C - B, D)}{(\gamma C, D)},$$

denoted by $\varepsilon(B)$, if C and D are understood, where B is a subset of C.

If B is a reduct of C, then $\varepsilon(B) = 1$, i.e., removing any reduct from a set of decision rules unable to make sure decisions, whatsoever.

Any subset B of C will be called an *approximate reduct* of C, and the number

$$\varepsilon_{(C,D)}(B) = \frac{(\gamma(C,D) - \gamma(B,D))}{\gamma(C,D)} = 1 - \frac{\gamma(B,D)}{(\gamma C, D)},$$

denoted simple as $\varepsilon(B)$, will be called an *error of reduct approximation*. It expresses how exactly the set of attributes B approximates the set of condition attributes C. Obviously $\varepsilon(B) = 1 - \sigma(B)$ and $\varepsilon(B) = 1 - \varepsilon(C - B)$. For any subset B of C we have $\varepsilon(B) \le \varepsilon(C)$. If B is a reduct of C, then $\varepsilon(B) = 0$.

For example, either of attributes Headache and Temperature can be considered as approximate reducts of {Headache, Temperature}, and

$\varepsilon(\text{Headache}) = 1$,
$\varepsilon(\text{Temperature}) = 0.25$.

But for the whole set of condition attributes {Headache, Muscle-pain, Temperature} we have also the following approximate reduct

$\varepsilon(\text{Headache, Muscle-pain}) = 0.75$.

The concept of an approximate reduct is a generalization of the concept of a reduct considered previously. A minimal subset B of condition attributes C, such that $\gamma(C, D) = \gamma(B, D)$, or $\varepsilon_{(C,D)}(B) = 0$ is a reduct in the previous sense. The idea of an approximate reduct can be useful in cases when a smaller number of condition attributes is preferred over accuracy of classification.

11   Conflict Analysis

Exmaples 1 and 2 considered in the Introduction display certain kind of problems we are interested in, in connection with data analysis. But there are many others questions, regarding data analysis, not necessarily referring to approximate description of set, dependencies in data or data reduction. Example 3 and Example 4, given in the introduction, illustrate this problem.

In this section we will consider a little more closely Example 3, referring to conflict analysis. In this case we do not distinguish condition and decision attributes in the information table. Each row of the table is a discription of the attitude of an agent to issues debated in the dispute. In the conflict analysis primarily we are interested in finding the relationship between agents taking part in the dispute, and investigate what can be done in order to improve the relationship between agents, or in other words how the conflict can be resolved. To this end new concepts must be introduced. Some of them will be discussed in what follows.

Conflict, Alliance and Neutrality

In order to express relations between agents we define three basic binary relations on the universe: *conflict*, *neutrality* and *alliance*. To this end we need the following auxiliary function:

$$\phi_a(x, y) = \begin{cases} 1, & \text{if } a(x)a(y) = 1 \text{ or } x = y, \\ 0, & \text{if } a(x)a(y) = 0 \text{ and } x \ne y, \\ -1, & \text{if } a(x)a(y) = -1. \end{cases}$$

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This means that, if $\phi_a(x,y) = 1$, agents x and y have the same opinion about issue a (are *allied* on a); if $\phi_a(x,y) = 0$ means that at least one agent x or y has neutral apprach to issue a (is *neutral* on a), and if $\phi_a(x,y) = -1$, means that both agents have different opinions about issue a (are in *conflict* on a).

In what follows we will need three basic relations R_a^+, R_a^0 and R_a^-, called *alliance*, *neutrality* and *conflict* relations respectively, and defined as follows:

$$R_a^+(x,y) \text{ iff } \phi_a(x,y) = 1,$$

$$R_a^0(x,y) \text{ iff } \phi_a(x,y) = 0,$$

$$R_a^-(x,y) \text{ iff } \phi_a(x,y) = -1.$$

It is easily seen that the alliance relation has the following properties:

(i) $R_a^+(x,x)$,

(ii) $R_a^+(x,y)$ implies $R_a^+(y,x)$,

(iii) $R_a^+(x,y)$ and $R_a^+(y,z)$ implies $R_a^+(x,z)$,

i.e., R_a^+ os an *equivalence* relation for every a. Each equivalence classe of alliance relation will be called *coalition* on a. Let us note that the condition (iii) can be expressed as "friend of my friend is my friend".

For the conflict relation we have the following properties:

(iv) non $R_a^-(x,x)$,

(v) $R_a^-(x,y)$ implies $R_a^-(y,x)$,

(vi) $R_a^-(x,y)$ and $R_a^-(y,z)$ implies $R_a^+(x,z)$,

(vii) $R_a^-(x,y)$ and $R_a^+(y,z)$ implies $R_a^-(x,z)$.

Conditions (vi) and (vii) refers to well know sayings "enemy of may enemy is my friend" and "friend of my enemy is my enemy".

For the neutrality relation we have:

(viii) non $R_a^0(x,x)$,

(ix) $R_a^0(x,y) = R_a^0(y,x)$.

Let us observe that in the conflict and neutrality relations there are no coalitions.

Obviously $R_a^+ \cup R_a^0 \cup R_a^- = U^2$ and all the three relation are pairwise disjoint, i.e., every pair of objects (x,y) belongd to exactly one of the above defined relations (is in conflict, is allied or is neutral).

For example in the Middle East situation Egypt, Palestinians an Syria are allied on issue a (autonomous Palestinian state on the West Bank and Gaza), Jordan and Israel are neutral to this issue whereas, Israel and Egypt, Israel and Palestinian, and Israel and Syria are in conflict about this issue.

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Degree of Conflict

Let and $a \in A$. If there exits a pair (x, y) such that $R_a^-(x, y)$ will say that the attribute a is *conflicting* (agents), otherwise the attribute is *conflictless*. The following property is obvious.

If a is an conflicting attribute, then the relation R_a^+ has exactly two equivalence classes X_a^+ and X_a^-, where $X_a^+ = \{x \in U : a(x) = +\}$, $X_a^- = \{x \in U : a(x) = -\}$, $X_a^+ \cup X_a^- \cup X_a^0 = U$, and $X_a^0 = \{x \in U : a(x) = 0\}$. Moreover $R_a^-(x, y)$ iff $x \in X_a^+$ and $y \in X^-$ for every $x, y \in U$.

The above proposition says that if a is conflicting attribute, then all agents are divided into two coalitions (bloks) X_a^+ and X_a^-, all members of two different coalitions are in conflict, and the remaining (if any) agents are neutral to the issue a.

This can be easily illustrated by graph as shown in Figure 1.
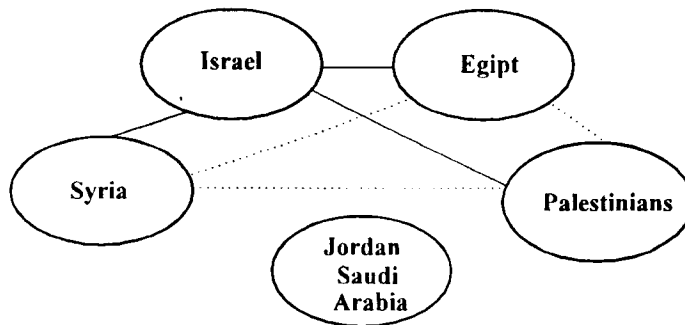
Fig. 1

Vertices of the graph are labelled by agents, whereas branches of the graph are representing relation between agents. Solid lines are denoting conflicts, dotted line – alliance, and neutrality, for simplicity, is not shown explicity on the graph. The proposition says that the graph shown in Fig. 1 can be presented as shown in Fig. 2.
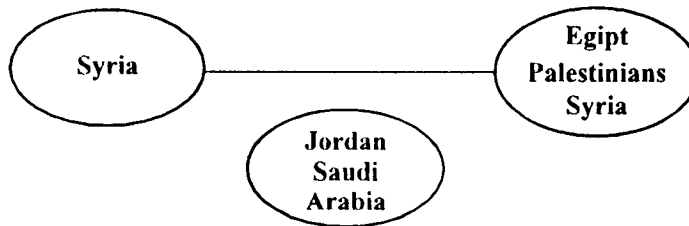
Fig. 2

The *degree* of *conflicts* between agents about a issue a can be easily expressed numerically, as follows:

$$Con(a) = \frac{|X_a^+| . |X_a^-|}{E(n/2) . (n - E(n/2))}.$$

The number $|X_a^+| . |X_a^-|$ is equal to the number of conflicts generated by the issue a (i.e., pairs of agents being in conflict because of issue a), whereas $E(n/2) . (n - E(n/2))$ is the number of maximal conflicts possible between n agents and $E(n/2)$ denotes the whole part of the division of n by 2. Of course $0 \leq Con(a) \leq 1$.

The coefficient $Con(a)$ can be easily extended to the whole set of attributes as follows:

$$Con(A) = \frac{\sum_{a \in A} Con(a)}{|A|}.$$

Evidently $0 \leq Con(A) \leq 1$, and $Con(A)$ can be viewed as an overall measure of conflicts in a conflict situation and will be referred to as *conflictness* of A, or *tension* of a conflict situation.

The following proposition is true

$$Con(A) \geq Con(A - Max(A)),$$

where $Max(A)$ is a most conflicting attribute in A. smallskip

From the proposition it follows that removing from the debate the most conflicting attribute (issue) reduces tension. This can be used as a guidance for a negotiation process. For example, for the Middle East situation considered previously we have

$$Con(a) = 1/3,$$
$$Con(b) = 2/3,$$
$$Con(c) = 4/9,$$
$$Con(d) = 1/3,$$
$$Con(e) = 2/3.$$

The most conflicting attributes are b and e, and $Con(A) \cong 0.49$. If we remove the attribute b we get $Con(\{a, c, d, e, \}) = 0.45$, but removing the attribute a we obtain $Con(\{b, c, d, e\}) \cong 0.53$.

Distance between Agents

The relations $R_a^+(x, y)$, $R_a^0(x, y)$ and $R_a^-(x, y)$ can be seen as a description of views on a issue a between agents x and y. We will also need an evaluation of views between x and y with respect to the whole set of attributes A. To this end we define a function $\rho_A(x, y)$ or in short $\rho(x, y)$ defined as shown below:

$$\rho(x, y) = \frac{\sum_{a \in A} \phi_a^*(x, y)}{|A|},$$

where

$$\phi_a^*(x, y) = \frac{1 - \phi_a(x, y)}{2}.$$

Obviously $0 \leq \rho(x, y) \leq 1$. If $\rho(x, y) \neq 0$) we will say that x and y are in conflict on A in a *degree* $\rho(x, y)$, and of course if $\rho(x, y) = 0$, x and y are in coalition on A. In particular, if $\rho(x, y) = 0.5$ x and y are neutral on A. Thus neutrality in this case is considered as a form of a (weak) conflict.

The following properties are obvious

1) $\rho(x, x) = 0$,
2) $\rho(x, y) = \rho(y, x)$,
3) $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$,

thus the $\rho(x, y)$ is a distance function.

For example for the considered Middle East situation we have the following distances matrix between agents.

Discernibility of Agents

Differences in views of agents concerning specifiec issues can be also expressed not only in a quantitative way (numerically), as in the previous section, but also in a qualitative way. To this end we will use discernibility matrix discussed previously.

The discernibility matrix for conflict presented in Table 3 is given below:

	1	2	3	4	5	6
1						
2	a,b,c,d,e					
3	a,b,c,d,e	b,e				
4	a,b,c,d,e	a,b,d	a,d,e			
5	a,b,c,d,e	b ᐧ	$e,$	a,d		
6	a,c,d	a,b,e,d	a,b,d,e	b,e	a,b,d,e	

Table 19

Each entry of the discernibility matrix shows all issues for which the corresponding agents have different opinions. The difference between Table 18 and Table 19 is that in the first table we have numerical evaluation of differences between agents e.g., $\rho(2,3) = 0.2$, whereas in the second one differences are expresssed literally e.g.,$\delta(2,3) = \{b, e\}$.

Beside problems discussed above we might be also interested in analysis of dependencies between issues, computing reducts etc.

For example, it is easy to see that reduction of attributes (issues) as defined in this section not necessarily reduces the tension in a conflict situation, i.e., it is not general true that if B is a reduct of A, then $Con(B) < Con(A)$. The aim of finding reducts consists in preserving overall structure of the conflict situation, i.e., after reducing attributes general relations (conflict, coalition and neutrality) between agents remain intact.

12   Explanation and specification

Let us consider first the problem of explanation of data. Suppose we are given data obtained from observation of certain processes, phenomena etc. Our task is to find out, on the basis of observed data, rules governing the perceived process. That means that we have to find all (or some) decision rules which describe relationships between data. For example, let us consider medical data given in Example 1 about patients suffering from flu. It follows from this example that the decision rules (Muscle-pain, no) implies (Temperature-high), (Headache, no) implies (Muscle-pain, yes), (Temperature-high) implies (Flu, yes) are valid in the data. Decision rules are sometimes interpreted as descriptions of "cause-effect" relations in data, however, we will refrain from a philosophical discussion of this problem.

For easy of notation let us replace Table 1, by Table 20

	a	b	c	d
1	0	1	1	1
2	1	0	1	1
3	1	1	2	1
4	0	1	0	0
5	1	0	1	0
6	0	1	2	1

Table 20

where attributes Headache, Muscle-pain, Temperature and Flu are replace by letters a, b, c and d respectively, and similarly values of attributes are replaced by numbers 0,1 and 2 in an obvious way.

More precisely, by explanation we will understand here finding all simple minimal decision rules from data. By a simple decision rule we mean the rule having only one decision attribute, and a decision rule is minimal if all its condition attribute-value pairs are reduced with respect to the decision attribute. To this end we can employ discernibility function discussed in previous sections, computed for every non conflicting object.

Let us start with the decison attribute a, for which discernibility matrix is shown in Table 21.

	1	2	3	4	5	6
1						
2	b					
3	c	–				
4	–	b,c,d	c,d			
5	b,d	–	–	b,c		
6	–	b,c	–	–	b,c,d	

Table 21

The corresponding decision table is inconsistent and the conflicting objects in this table are 3 and 6. Hence we have the following discernibiliy functions:

$$f_a^1(\{b,c,d\}) = bc,$$
$$f_a^2(\{b,c,d\}) = b,$$
$$f_a^4(\{b,c,d\}) = c + bd,$$
$$f_a^5(\{b,c,d\}) = b + cd.$$

These discernibility functions give a-reducts of condition attributes $\{b,c,d\}$- for every object. Fore xample $f_a^1(\{b,c,d\}) = bc$ means that the attribute-value pairs b_1, c_1 detrmine uniquelly attribute-value pair a_0, i.e., that the decision rule

$$b_1 c_1 \rightarrow a_0$$

is valid. Proceeding similarly we get from the remaining discernibility functions the following decision rules:

$$b_0 \rightarrow a_1,$$
$$c_0 \rightarrow a_0,$$
$$b_1 d_0 \rightarrow a_0,$$
$$c_1 d_0 \rightarrow a_1.$$

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In the same way we get for the attribut b the discernibility functions

$$f_b^1(\{a,c,d\}) = a,$$
$$f_b^2(\{a,c,d\}) = ac,$$
$$f_b^3(\{a,c,d\}) = c,$$
$$f_b^4(\{a,c,d\}) = ac,$$
$$f_b^5(\{a,c,d\}) = ac + ad + cd,$$
$$f_b^6(\{a,c,d\}) = a + c,$$

ance the following decision rules are valid:

$$a_0 \rightarrow b_1,$$
$$a_1 c_1 \rightarrow b_0,$$
$$c_2 \rightarrow b_1,$$
$$c_0 \rightarrow b_1,$$
$$a_1 d_0 \rightarrow b_0,$$
$$c_1 d_0 \rightarrow b_0.$$

For the attribute c we have

$$f_c^2(\{a,b,d\}) = b,$$
$$f_c^3(\{a,b,d\}) = ab,$$
$$f_c^4(\{a,b,d\}) = ab + ad,$$
$$f_c^5(\{a,b,d\}) = b + ad,$$

which yielas decision rules

$$b_0 \rightarrow c_1,$$
$$a_1 b_1 \rightarrow c_2,$$
$$a_0 d_0 \rightarrow c_0,$$
$$b_1 d_0 \rightarrow c_0,$$
$$a_1 d_0 \rightarrow c_1.$$

For the attribute d we get

$$f_d^1(\{a,b,c\}) = ac + bc,$$
$$f_d^3(\{a,b,c\}) = c + ab,$$
$$f_d^4(\{a,b,c\}) = c,$$
$$f_d^6(\{a,b,c\}) = c,$$

which gives:

$$a_0 c_1 \rightarrow d_1,$$
$$b_1 c_1 \rightarrow d_1,$$
$$c_2 \rightarrow d_1,$$
$$c_0 \rightarrow d_0.$$

The above decision rules can be presented in a more compact form

$$b_1 c_1 \lor b_1 d_0 \lor c_0 \;\rightarrow\; a_0,$$
$$b_0 \lor c_1 d_0 \;\rightarrow\; a_1,$$
$$a_1 c_1 \lor a_1 d_0 \lor c_1 d_0 \;\rightarrow\; b_0,$$
$$a_0 \lor c_0 \lor c_2 \;\rightarrow\; b_1,$$
$$a_0 d_0 \lor b_1 d_0 \;\rightarrow\; c_0,$$
$$b_0 \lor a_1 d_0 \;\rightarrow\; c_1,$$
$$a_1 b_1 \;\rightarrow\; c_2,$$
$$c_0 \;\rightarrow\; d_0,$$
$$a_0 c_1 \lor b_1 c_1 \lor c_2 \;\rightarrow\; d_1.$$

Relations between all decision rules can be depicted as a *connection graph*. Vertices of the graph represent attribute-value pairs (drawn in a similar way as multistate devices, e.g., flip-flops, in the case of binary circuits), whereas branches of the graph show connections (implications) between corresponding attribute-value pairs. The connection graphs for decision attributes a, b, c and d are given in Fig. 3, 4, 5 and 6 respectively, and Fig. 7 contains all connections (implications) which occur in the whole decision table. Thus Table 20 can be seen as a concise denotation of all decision rules defined by the table.
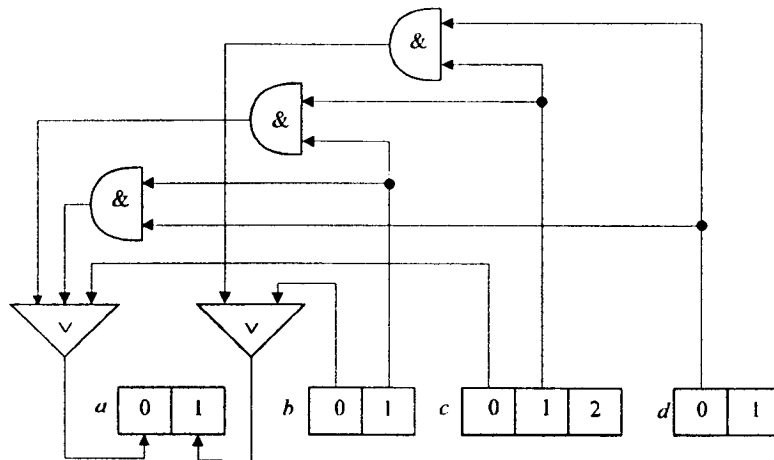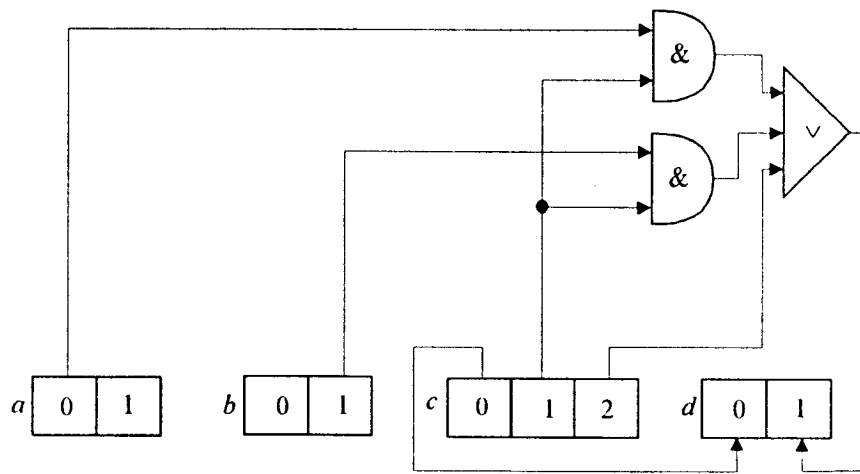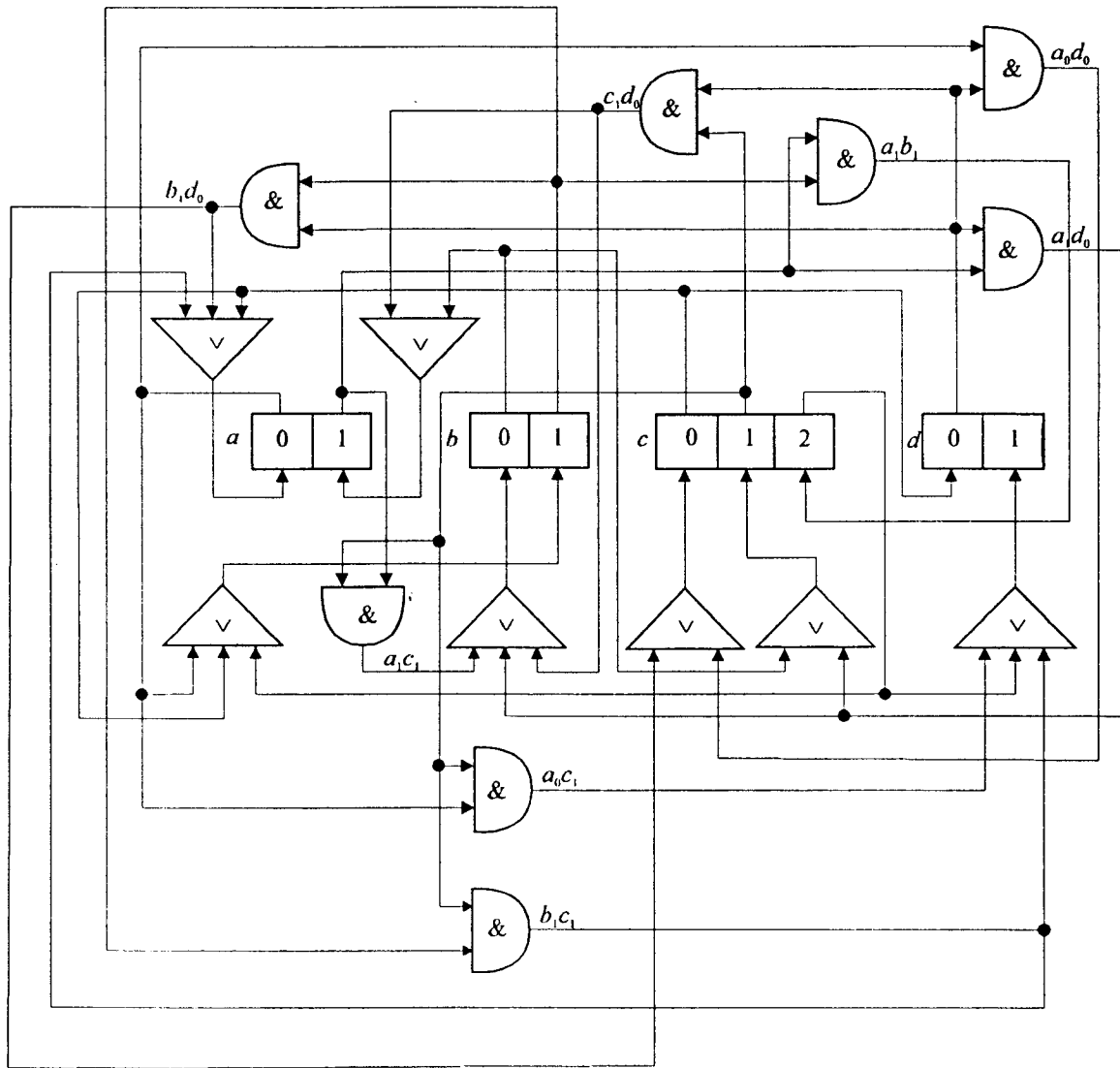

Fig. 3
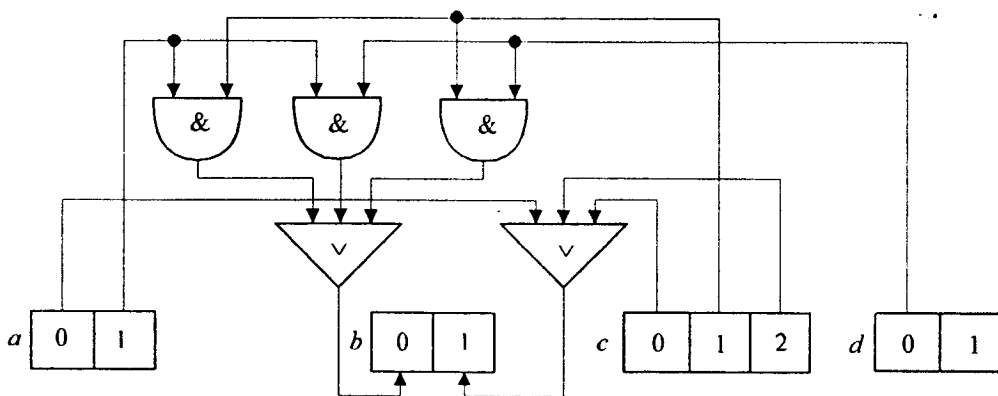
Fig. 4
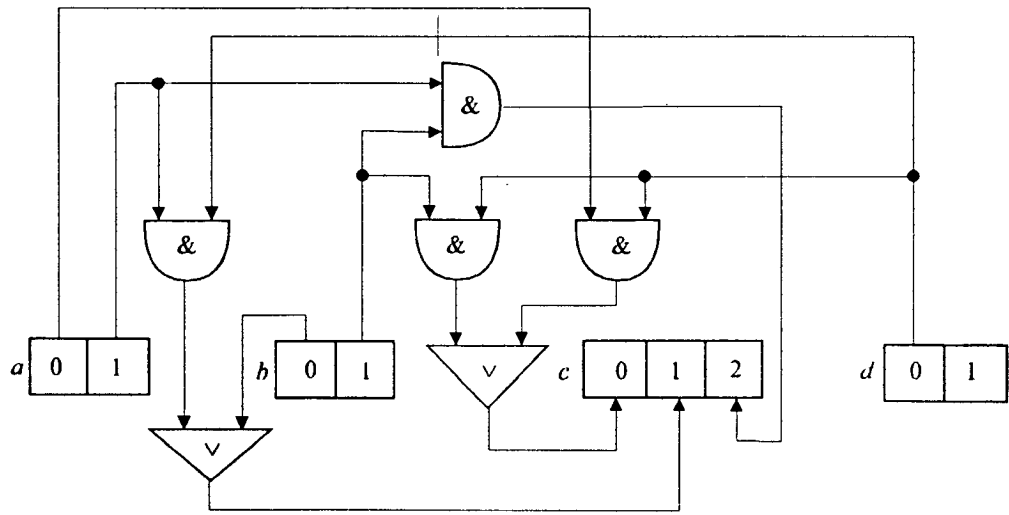
Fig. 5

Fig. 6

Fig. 7

Table 20 considered in this section can be also understood not as a result of observations but, as a specification of requirements for a special discrete systems behavior (e.g., controller, program etc.). In this case the diagram presented in Fig. 7 is to be understood not as a visual aid of "cause-effect" relations occurring in the observed data, but as a schemata of an algorithm implemented in hardware or software, and performing the task specified be the table. This approach can be seen as alternative to the well know Petri Nets notation of concurrent systems behavior.

We will illustrate this idea by more intuitive example of distributed traffic signals control. Let us consider a very simple intersection (T-intersection) shown in Fig. 8.

⊗ - TRAFFIC SIGNAL
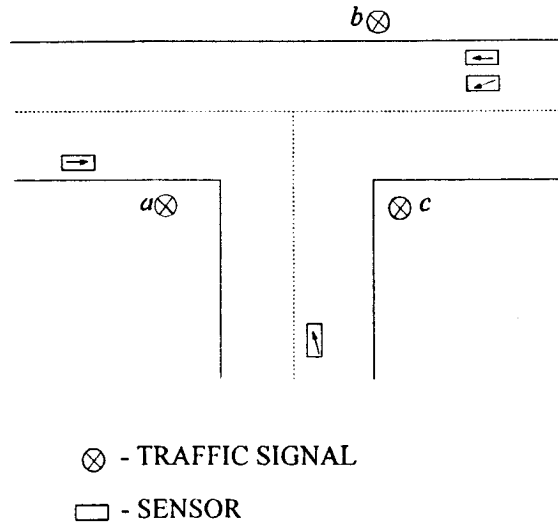
☐ - SENSOR

Fig. 8

We would like to design a distributed control algorithm which will supervise the traffic on the basis of local conditions. We assume that the conditions are determined by sensors placed in lanes and indicating the desired turn of a car approaching the intersection. For the sake of simplicity we omit many important factors, needed in a real-life control, e.g., traffic intensity, the busiest directions, length of the green period in each direction etc. Distributed control means that the control cycle is not fixed but it depends on the situation on the intersection. If the traffic flow is small, fixed traffic signals changes are not optimal for a smooth traffic flow. Therefore we assume that in the main directions (a-b) the lights stay green until some cars approach the intersection and want to turn left. We could implement the control algorithm by changing the possible states of signals periodically, independently of the traffic flow. For the sake of illustration we assume that the signal changes are not changed periodically, independently of the traffic intensity, but are govern by states of sensors. The specification of the control admissible situations is given in Table 22 below

	a	b	c
1	1	1	0
2	0	2	0
3	0	0	2

Table 22

where

0 – red

1 – green

2 – green arrow (left turn)

Using methods shown previously we get from the table the following control rules

$$b_0 \vee b_2 \rightarrow a_0$$

$$b_1 \rightarrow a_1$$

$$c_2 \rightarrow b_0$$

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$$a_1 \rightarrow b_1$$
$$a_0 c_0 \rightarrow b_2$$
$$b_1 \vee b_2 \rightarrow c_0$$
$$b_0 \rightarrow c_2$$

The connection graph for the set of control rules is shown in Fig. 9
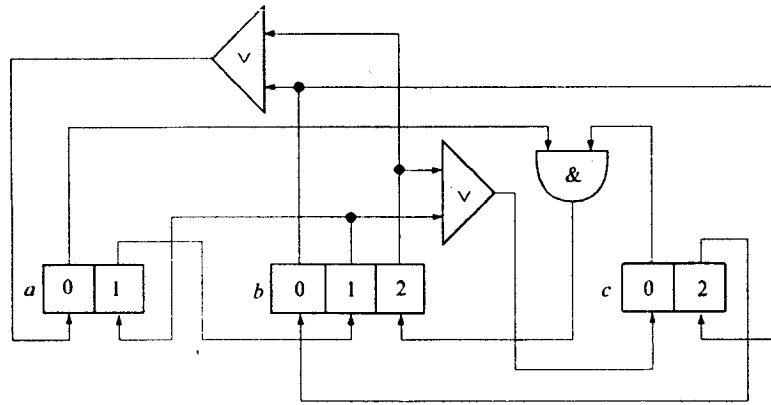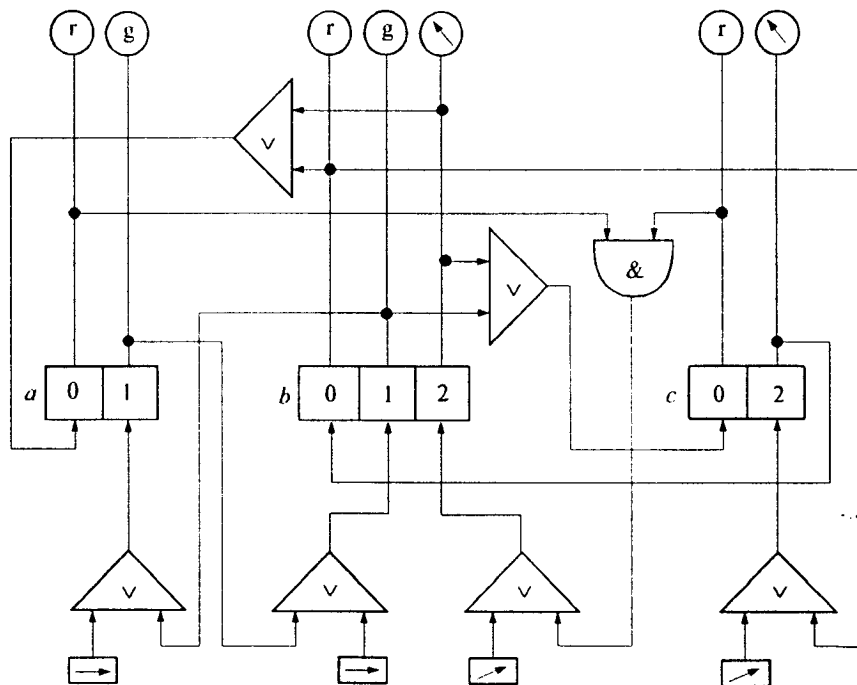

Fig. 9

and the corresponding switching circuits (controller) is depicted in Fig. 10.

☐ - SENSOR

○ - SIGNAL

Fig. 10

It can be observed that some states of sensors may cause conflicts. For example, suppose that simultaneously two cars are approaching the intersection, as shown in Fig. 11,
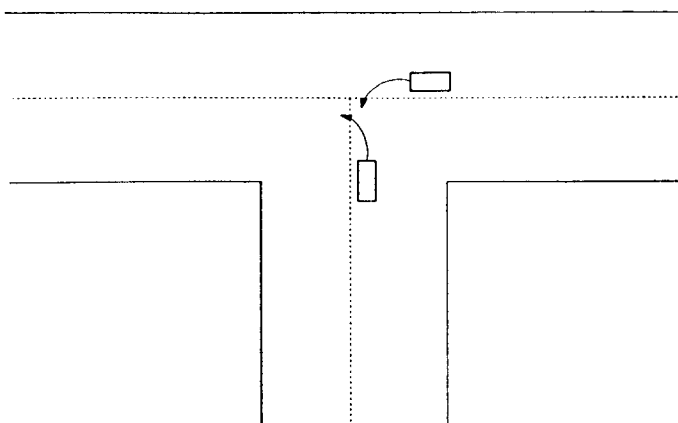

Fig. 11

i.e., both want to turn left. This causes conflict, since $c2 \to b0$ and $b2 \to c0$ (states $c2$ and $b2$ do not occur simultaneously in the specification table) Hence special means to resolve this kind of conflicts must be incorporated in the method, but we will not discuss this issue here.

References

[1] Bazan, J., Skowron, A., and Synak, P., (1995), "Discovery of decision rules from experimental data", in: T.Y. Lin and A.M. Wildberger (eds.), *The Third International Workshop on Rough Sets and Soft Computing Proceedings* (*RSSC'94*), San Jose State University, San Jose, California, USA, November 10-12, 276-279.

[2] Bazan, J., Skowron, A., and Synak, P., (1994), "Dynamic reducts as a tool for extracting laws from decision tables", *Proc. of the Symp. on Methodologies for Intelligent Systems*, Charlotte, NC, October 16-19, 1994, Lecture Notes in Arificial Intelligence Vol. 869, Springer Verlag, 346-355.

[3] Bazan, J., Skowron, A., and Synak P., (1994), "Discovery of decision rules from experimental data", in: T.Y. Lin (ed.), *The Third International Workshop on Rough Sets and Soft Computing Proceedings* (*RSSC'94*), San Jose State University, San Jose, California, USA, November 10-12, 526-535.

[4] Beaubouef, T., and Petry, F.E., (1993), "A rough set model for relational databases", in: W. Ziarko (ed.), *Rough Sets, Fuzzy Sets and Knowledge Discovery. Proceedings of the International Workshop on Rough Sets and Knowledge Discovery* (*RSKD'93*), Banff, Alberta, Canada, October 12-15, Springer-Verlag, Berlin, 100-107.

[5] Beaubouef, T., and Petry, F.E., (1995), "Rough querying of crisp data in relational databases", in: T.Y. Lin and A.M. Wildberger (eds.), (1995), *The Third International Workshop on Rough Sets and Soft Computing Proceedings* (*RSSC'94*), San Jose State University, San Jose, California, USA, November 10-12, 85-88.

[6] Beaubouef, T., Petry, F.E., and Buckles, B.P., (1995), "Extension of the relational Database and its algebra with rough set techniques", *Computational Intelligence: An International Journal Vol. 11*, 233-245.

[7] Brindle, D., (1994), "Speaker-independent speech recognition by rough sets analysis", in: T.Y. Lin (ed.), *The Third International Workshop on Rough Sets and Soft Computing Proceedings (RSSC'94)*, San Jose State University, San Jose, California, USA, November 10-12, 376-383.

[8] Czyżewski , A., (1995) , "Speaker-Independent Recognition of Digits - Experiments with Neural Networks, fuzzy logic and rough sets", *Journal of the Intelligent Automation and Soft Computing* (to appear).

[9] Czyżewski, A., and Kaczmarek, A., (1993), "Multilayer knowledge based system for speaker-independent recognition of isolated words", in: W. Ziarko (ed.), *Rough Sets, Fuzzy Sets and Knowledge Discovery. Proceedings of the International Workshop on Rough Sets and Knowledge Discovery (RSKD'93)*, Banff, Alberta, Canada, October 12-15, Springer-Verlag, Berlin, 387-394.

[10] Czyżewski, A., and Kaczmarek, A., (1995), "Speaker-independent recognition of isolated words using rough sets", in: P.P. Wang (ed.), *Second Annual Joint Conference on Information Sciences PROCEEDINGS*, September 28 - October 1, Wrightsville Beach, North Carolina, USA, 397-400.

[11] Czyżewski A., and Kaczmarek, A., (1995), "Speech recognition systems based on rough sets and neural networks", in: T.Y. Lin and A.M. Wildberger (eds.), *The Third International Workshop on Rough Sets and Soft Computing Proceedings (RSSC'94)*, San Jose State University, San Jose, California, USA, November 10-12, 97-100.

[12] Czogala, E., Mrozek, A. and Pawlak, Z. (1995). "The Idea of Rough-Fuzzy Controller". *International Journal of Fuzzy Sets and Systems*, 72, 61-63.

[13] Dubois, D. and Prade, H. (1992),"Putting Rough Sets and Fuzzy Sets Together", in: R. Slowinski (ed.), *Intelligent Decision Support - Handbook of Advances and Applications of the Rough Set Theory*, , Kluwer Academic Publishers, Dordrecht, Boston, London, 203-232.

[14] Frege, G. (1903). Grundgesetze der Arithmetik, 2, in: *Selections from the Philosophical Writings of Gotlob Frege*, Blackweil, Oxford, 1970.

[15] Golan, R., and Edwards, D., (1993), "Temporal rules discovery using datalogic/R+ with stock market data", in: W. Ziarko *Rough Sets, Fuzzy Sets and Knowledge Discovery. Proceedings of the International Workshop on Rough Sets and Knowledge Discovery (RSKD'93)*, Banff, Alberta, Canada, October 12-15, Springer-Verlag, Berlin, 74-81.

[16] Grzymala-Busse, J.W., (1988), "Knowledge acquisition under uncertainty - A rough set approach", *Journal of Intelligent & Robotic Systems*, 1/1, 3-16.

[17] Grzymala-Busse J.W., (1991), *Managing Uncertainty in Expert Systems.* Kluwer Academic Publishers, Dordrecht, Boston, London.

[18] Grzymala-Busse, J.W., (1992), "LERS-a system for learning from examples based on rough sets", in: R. Slowiński (ed.), *Intelligent Decision Support. Handbook of Applications and Advances of the Rough Set Theory*, Kluwer Academic Publishers, Dordrecht, 3–18.

[19] Grzymala-Busse, J. (1995). "Rough Sets", *Advances in Imaging and Electrons Physics*, 94, to appear.

[20] Grzymala-Busse, J.W., Sedelow, S.Y., and Sedelow, W.A. Jr., (1995), "Machine learning & knowledge acquisition, rough sets, and the English semantic code". *Proc. of the Workshop on Rough Sets and Database Mining, 23rd Annual ACM Computer Science Conference CSC'95*, Nashville, TN, March 2, 91–109.

[21] Grzymala-Busse, Stefanowski, J., and Ziarko W., (1995), "Rough sets: facts versus misconceptions", *ICS Research Report 61/95*.

[22] Grzymala-Busse, J.W., and Than, S. (1993), "Data compression in machine learning applied to natural language". *Behavior Research Methods, Instruments, & Computers* 25, 318–321.

[23] Grzymala-Busse, and Woolerly, L., (1994), "Improving prediction of preterm birth using a new classification scheme and rule induction", *Proc. of the 18-th Annual Symposium on Computer Applications in Medical Care (SCAMC)*, Washington D.C. November 5–9, 730–734.

[24] Gunn, J.D., and Grzymala-Busse, J.W., (1994), "Global temperature stability by rule induction: an interdisciplinary bridge", *Human Ecology*, 22, 59–81.

[25] Hadjimichael, M., and Wasilewska, A. (1992), "Rough sets-based study of voter preference in 1988 USA presidential election", in: R. Slowiński (ed.), *Intelligent Decision Support. Handbook of Applications and Advances of the Rough Set Theory*, Kluwer Academic Publishers, Dordrecht, 137–152.

[26] Haines, S., (1995), "The practical application of rough sets to semantics and simulation", in: T.Y. Lin and A.M. Wildberger (eds.), *The Third International Workshop on Rough Sets and Soft Computing Proceedings (RSSC'94)*, San Jose State University, San Jose, California, USA, November 10–12, 77–80.

[27] Haines, S., Longshaw, T., and Magee, G., (1994), "The practical application of rough sets to sementics and simulation", in: T.Y. Lin (ed.), *The Third International Workshop on Rough Sets and Soft Computing Proceedings (RSSC'94)*, San Jose State University, San Jose, California, USA, November 10–12, 392–398.

[28] Hu, X., Cercone, N., and Han, J., (1993), "An attribute-oriented rough set approach for knowledge discovery in databases", in: W. Ziarko (ed.), *Rough Sets, Fuzzy Sets and Knowledge Discovery. Proceedings of the International Workshop on Rough Sets and Knowledge Discovery (RSKD'93)*, Banff, Alberta, Canada, October 12–15, Springer-Verlag, Berlin. 90–99.

[29] Jackson, A.G., Ohmer, M., and Al-Kamhawi, H., (1994), "Rough sets analysis of chalcopyrite semiconductor band gap data", in: T.Y. Lin (ed.), The Third International Workshop on Rough Sets and Soft Computing Proceedings (RSSC'94), San Jose State University, San Jose, California, USA, November 10-12, 408-417.

[30] Jackson, A.G., LeClair, S.R., Ohmer, M.C., Ziarko. W., and Al-Kamhwi, H., "Rough sets applied to material data", Acta Metallurgica et Materialia (to appear).

[31] Jelonek, J., Krawiec, K., Slowiński, R., Stefanowski, J., and Szymas, J., (1993), "Neural networks and rough sets - comparison and combination for classification of histological pictures", in: W. Ziarko (ed.), Rough Sets, Fuzzy Sets and Knowledge Discovery. Proceedings of the International Workshop on Rough Sets and Knowledge Discovery (RSKD'93), Banff, Alberta, Canada, October 12-15, Springer-Verlag, Berlin, 426-433.

[32] Jelonek, J., Krawiec, K., Slowiński, R., and Szymas, J., (1994), "Rough set reduction of features for picture-based reasoning", in: T.Y. Lin (ed.), The Third International Workshop on Rough Sets and Soft Computing Proceedings (RSSC'94), San Jose State University, San Jose, California, USA, November 10-12, 418-425.

[33] Kobayashi, S., Yokomori, T., (1995), "Approximately learning regular languages with respect to reversible languages: A rough set based analysis", in: P.P. Wang (ed.), Second Annual Joint Conference on Information Sciences PROCEEDINGS, September 28 – October 1, Wrightsville Beach, North Carolina, USA, 91-94.

[34] Kostek B., (1995), "Rough classification as a tool for acoustical analysis", in: T.Y. Lin and A.M. Wildberger (eds.), The Third International Workshop on Rough Sets and Soft Computing Proceedings (RSSC'94), San Jose State University, San Jose, California, USA, November 10-12, 81-84.

[35] Kostek B., (1995), "Computer based recognition of musical phrases using the rough set apprach", in: P.P. Wang (ed.), Second Annual Joint Conference on Information Sciences PROCEEDINGS, September 28 – October 1, Wrightsville Beach, North Carolina, USA, 401-404.

[36] Kostek, B., (1995), "Statistical versus artificial intelligence based processing of subjective test results", 98th Convention of the Audio Engineering Society, Paris, February 25-28, Preprint No. 4018.

[37] Kostek B., (1995), "Rough set and fuzzy set methods applied to acoustical analyses", Journal of the Intelligent Automation and Soft Computing (to appear).

[38] Krusińska E., Slowiński R., and Stefanowski J., (1992), "Discriminant versus rough set approach to vague data analysis", Applied Stochastic Models and Data Analysis, 8, 43-56.

[39] Krysiński, J., (1990), "Rough set approach to the analysis of structure - activity relationship of quaternary imidazolium compounds", Arzneimittel Forschung / Drug Research, 40/II, 795-799.

[40] Krysiński, J., (1992), "Grob Mengen Theorie in der Analyse der Struktur Wirkungs Beziehungen von quartaren Pyridiniumverbindungen", *Pharmazie*, 46/12, 878–881.

[41] Krysiński, J., (1992), "Analysis of structure - activity relationships of quaternary ammonium compounds", in: R. Słowiński (ed.), *Intelligent Decision Support. Handbook of Applications and Advances of the Rough Set Theory*, Kluwer Academic Publishers, Dordrecht, 119–136.

[42] Krysiński, J., (1995), "Application of the rough sets theory to the analysis of structure-activity-relationships of antimicrobial pyridinium compounds", *Die Pharmazie*, 50, 593–597.

[43] Kryszkiewicz, M., and Rybiński, H., (1993), "Finding Reducts in Composed Information Systems", in: W. Ziarko (ed.), *Rough Sets, Fuzzy Sets and Knowledge Discovery. Proceedings of the International Workshop on Rough Sets and Knowledge Discovery (RSKD'93)*, Banff, Alberta, Canada, October 12–15, Springer-Verlag, Berlin, 261–273.

[44] Lenarcik, A., and Piasta, Z., (1993), "Rough classfiers", in: W. Ziarko (ed.), *Rough Sets, Fuzzy Sets and Knowledge Discovery. Proceedings of the International Workshop on Rough Sets and Knowledge Discovery (RSKD'93)*, Banff, Alberta, Canada, October 12–15, Springer-Verlag, Berlin, 298–316.

[45] Lin, T.Y., (ed.), (1994), *The Third International Workshop on Rough Sets and Soft Computing Proceedings (RSSC'94)*, San Jose State University, San Jose, California, USA, November 10–12.

[46] Lin, T.Y., (1995), "Topological view of stability in rough-fuzzy controllers", in: P.P. Wang (ed.), *Second Annual Joint Conference on Information Sciences PROCEEDINGS*, September 28 - October 1, Wrightsville Beach, North Carolina, USA.

[47] Lin, T.Y., and Wildberger A.M., (eds.), (1995), *The Third International Workshop on Rough Sets and Soft Computing Proceedings (RSSC'94)*, San Jose State University, San Jose, California, USA, November 10–12.

[48] Lau, S.S.Y., (1993), "Image segmentation based on the indiscernibility relation", in: W. Ziarko (ed.), *Rough Sets, Fuzzy Sets and Knowledge Discovery. Proceedings of the International Workshop on Rough Sets and Knowledge Discovery (RSKD'93)*, Banff, Alberta, Canada, October 12–15, Springer-Verlag, Berlin, 395–403.

[49] Mrózek, A., (1989), "Rough sets and dependency analysis among attributes in computer implementations of expert's inference models", *International Journal of Man-Machine Studies*, 30, 457–471.

[50] Mrózek, A., (1992), "Rough sets in computer implementation of rule-based control of industrial processes", in: R. Słowiński (ed.), *Intelligent Decision Support. Handbook of Applications and Advances of the Rough Set Theory*. Kluwer Academic Publishers, Dordrecht, 19–31.

[51] Moradi, H., Grzymala-Busse, J., and Roberts, J., (1995), "Entropy of English text: Experiments with humans nad machine learning system based on rough sets", in: P.P. Wang (ed.), *Second Annual Joint Conference on Information Sciences PRO-CEEDINGS*, September 28 - October 1, Wrightsville Beach, North Carolina, USA, 87-88.

[52] Munakata, T., (1995), "Rough control: Basic ideas and applications", in: Wang, P.P., (ed.), *Second Annual Joint Conference on Information Sciences PROCEEDINGS*, September 28 - October 1, Wrightsville Beach, North Carolina, USA, 340-343.

[53] Nowicki, R., Slowiński, R., and Stefanowski, J.,(1992), "Rough sets analysis of diagnostic capacity of vibroacoustic symptoms", *Journal of Computers and Mathematics with Applications*, 24/2, 109-123.

[54] Nowicki, R., Slowiński, R., and Stefanowski, J., (1992), "Evaluation of vibroacoustic diagnostic symptoms by means of the rough sets theory", *Journal of Computers in Industry*, 20, 141-152.

[55] Nowicki, R., Slowiński, R., and Stefanowski, J., (1992), "Analysis of diagnostic symptoms in vibroacoustic diagnostics by means of the rough set theory", in: R. Slowiński (ed.), *Intelligent Decision Support. Handbook of Applications and Advances of the Rough Set Theory*, Kluwer Academic Publishers, Dordrecht, 33-48.

[56] Pawlak Z., (1982), "Rough sets". *International Journal of Computer and Information Sciences*, 11, 341-356.

[57] Pawlak Z., (1991), *Rough Sets - Theoretical Aspects of Reasoning about Data*. Kluwer Academic Publishers, Dordrecht, Boston, London.

[58] Pawlak Z., Grzymala-Busse J. W., Slowiński R., and Ziarko, W., (1995),"Rough sets", *Communication of the ACM*, 38, 88-95.

[59] Pawlak Z., and Skowron A., (1994), "Rough membership functions", in: R.R Yaeger, M. Fedrizzi and J. Kacprzyk (eds.), *Advances in the Dempster Shafer Theory of Evidence*, John Wiley & Sons, Inc., New York, Chichester, Brisbane, Toronto, Singapore, 251-271.

[60] Pawlak Z., and Slowiński R., (1994), "Rough set approach to multi-attribute decision analysis, Invited Review", *European Journal of Operational Research*, 72, 443-459.

[61] Pawlak, Z., Wong, S.K.M. and Ziarko, W. (1988). "Rough Sets: Probabilistic versus Deterministic Approach", *International Journal of Man Machine Studies*, 29, 81-85.

[62] Piasta, Z., (1995), "A comparative analysis of classifiers' performance by using a simulation study", in: T.Y. Lin and A.M. Wildberger (eds.), *The Third International Workshop on Rough Sets and Soft Computing Proceedings (RSSC'94)*, San Jose State University, San Jose, California, USA, November 10-12, 65-68.

[63] Peterson, G.I., (1993), "Rough classification of pneumonia patients using a clinical database", in: W. Ziarko (ed.), *Rough Sets, Fuzzy Sets and Knowledge Discovery. Proceedings of the International Workshop on Rough Sets and Knowledge Discovery (RSKD'93)*, Banff, Alberta, Canada, October 12–15, Springer-Verlag, Berlin, 412–419.

[64] Plonka, L., and Mrózek, A., (1995), "Rule-based stabilization of the inverted pendulum", *Computational Intelligence: An International Journal*, 11, 348–356.

[65] Raś, Z., Kudelska, A., and Chilumula, N., (1995), "Can we simplify international physical performance test profile using rough sets approach?", in: P.P. Wang (ed.), *Second Annual Joint Conference on Information Sciences PROCEEDINGS*, September 28 – October 1, Wrightsville Beach, North Carolina, USA, 393–396.

[66] Shenoi, S., (1995), "Rough sets in fuzzy databases", in: P.P. Wang (ed.), *Second Annual Joint Conference on Information Sciences PROCEEDINGS*, September 28 – October 1, Wrightsville Beach, North Carolina, USA, 263–264.

[67] Skowron, A., (1995), "Extracting laws from decision tables", *Computational Intelligence*, 11/2, 371–388.

[68] Skowron, A., (1995), "Synthesis of adaptive decision systems from experimantal data", in: A. Aamadt and J. Komorowski (eds.), *Proc. of the Fifth Scandinavian Conference on Artificial Intelligence SCAI-95*, Amsterdam: IOS Press , 220–238.

[69] Skowron A., and Grzymala-Busse, J.W., (1994), "From rough set theory to evidence theory", in: R.R Yaeger, M. Fedrizzi and J. Kacprzyk (eds.), *Advances in the Dempster Shafer Theory of Evidence*, John Wiley & Sons, Inc., New York, Chichester, Brisbane, Toronto, Singapore, 193–236.

[70] Skowron A.,and Rauszer, C., (1992), "The discernibility matrices and functions in information systems", in: R. Slowiński (ed.), *Intelligent Decision Support. Handbook of Applications and Advances of the Rough Set Theory*, Kluwer Academic Publishers, Dordrecht, 311–362.

[71] Skowron, A., and Stepaniuk, J., (1994), "Decision rules based on discernibility matrices and decision", in: T.Y. Lin (ed.), *The Third International Workshop on Rough Sets and Soft Computing Proceedings (RSSC'94)*, San Jose State University, San Jose, California, USA, November 10–12, 602–609.

[72] Slowiński, R., (ed.), (1992), *Intelligent Decision Support. Handbook of Applications and Advances of the Rough Set Theory*, Kluwer Academic Publishers, Dordrecht.

[73] Slowiński, R., (1993), "Rough set learning of preferential attitude in multi-criteria decision making", in: J. Komorowski and Z.W. Raś (eds.), *Methodologies for Intelligent Systems. Lecture Notes in Artificial Intelligence Vol. 689*, Springer-Verlag, Berlin, 642–651.

[74] Slowiński, R. (1995). "Rough Set Approach to Decision Analysis", *AI Expert*, 10, 18–25.

[75] Slowiński, R., and Stefanowski, J., (1992), "RoughDAS and RoughClass' software implementations of the rough sets approach", in: R. Slowiński (ed.), *Intelligent Decision Support. Handbook of Applications and Advances of the Rough Set Theory*, Kluwer Academic Publishers, Dordrecht, 445–456.

[76] Slowiński R., and Stefanowski J., (1994), "Rough classification with valued closeness relation", in: E.Diday, Y.Lechevallier, M.Schrader, P.Bertrand and B.Burtschy (eds.), *New Approaches in Classification and Data Analysis*, Springer-Verlag, Berlin, 482–489.

[77] Slowiński, R., and Zopounidis, C., (1993), "Applications of the rough set approach to evaluation of bankruptcy risk", Working Paper 93-08, Decision Support System Laboratory, Technical University of Crete, Chania, June.

[78] Slowiński, R., and Zopounidis, C., (1994), "Rough set sorting of firms according to bankruptcy risk", in: M. Paruccini (ed.), *Applying Multiple Criteria Aid for Decision to Environmental Management*, Kluwer, Dordrecht, Netherlands, 339–357.

[79] Slowiński, K., (1992), "Rough classification of HSV patients", in: R. Slowiński (ed.), *Intelligent Decision Support. Handbook of Applications and Advances of the Rough Set Theory*, Kluwer Academic Publishers, Dordrecht, 77–93.

[80] Slowiński, K., Slowiński, R., and Stefanowski, J., (1988), "Rough sets approach to analysis of data from peritoneal lavage in acute pancreatitis", *Medical Informatics*, 13/3, 143–159.

[81] Slowiński, K., and Sharif, E.S., (1993), "Rough sets approach to analysis of data of diatnostic peritoneal lavage applied for multiple injuries patients", in: W. Ziarko (ed.), *Rough Sets, Fuzzy Sets and Knowledge Discovery. Proceedings of the International Workshop on Rough Sets and Knowledge Discovery (RSKD'93)*, Banff, Alberta, Canada, October 12-15, Springer-Verlag, Berlin, 420–425.

[82] Slowiński, K., Stefanowski, J., Antczak, A. and Kwias, Z., (1995), "Rough sets approach to the verification of indications for treatment of urinary stones by extracorporeal shock wave lithotripsy (ESWL)", in: T.Y. Lin and A.M. Wildberger (eds.), *The Third International Workshop on Rough Sets and Soft Computing Proceedings (RSSC'94)*, San Jose State University, San Jose, California, USA, November 10-12, 93–96.

[83] Stefanowski, J., and Vanderpooten, D., (1993), "A general two-stage approach to inducing rules from examples", in: W. Ziarko (ed.), *Rough Sets, Fuzzy Sets and Knowledge Discovery. Proceedings of the International Workshop on Rough Sets and Knowledge Discovery (RSKD'93)*, Banff, Alberta, Canada, October 12-15, Springer-Verlag, Berlin, 317–325.

[84] Szladow, A. J., and Ziarko, W., (1992), "Knowledge-based process control using rough sets", in: R. Slowiński (ed.), *Intelligent Decision Support. Handbook of Applications and Advances of the Rough Set Theory*, Kluwer Academic Publishers, Dordrecht, 49–60.

[85] Szladow, A., (1993), "Datalogic/R: Mining the knowledge in databases", *PC AI* , 7/1, 40–41.

[86] Szladow, A., and Ziarko W., (1993), "Rough sets: Working with imperfect data", *AI Expert*, 7, 36–41.

[87] Tanaka, H., Ishibuchi, H., and Shigenaga, T., (1992) "Fuzzy inference system based on rough sets and its application to medical diagnostic", in: R. Slowiński (ed.), *Intelligent Decision Support. Handbook of Applications and Advances of the Rough Set Theory*, Kluwer Academic Publishers, Dordrecht, 111–117.

[88] Teghem, J., and Charlet J.-M., (1992), "Use of 'rough sets' method to draw pre-monitory factors for eathquakes by emphasing gas geochemistry: The case of a low seismic activiti context, in Belgium", in: R. Slowiński (ed.), *Intelligent Decision Support. Handbook of Applications and Advances of the Rough Set Theory*, Kluwer Academic Publishers, Dordrecht, 165–180.

[89] Tsumoto, S., and Tanaka, H., (1994), "Characterization of structure of decision trees based on rough sets and greedoid theory", in: T.Y. Lin (ed.), *The Third International Workshop on Rough Sets and Soft Computing Proceedings (RSSC'94)*, San Jose State University, San Jose, California, USA, November 10-12, 450–461.

[90] Tsumoto, S., and Tanaka, H., (1993), "PRIMEROSE: "Probabilistic rule induction method based on rough set theory", in: W. Ziarko (ed.), *Rough Sets, Fuzzy Sets and Knowledge Discovery. Proceedings of the International Workshop on Rough Sets and Knowledge Discovery (RSKD'93)*, Banff, Alberta, Canada, October 12-15, Springer-Verlag, Berlin, 274–281.

[91] Tsumoto, S., and Tanaka, H., (1995), "PRIMEROSE: "Probabilistic rule induction method based on rough set and resampling methods", *Computational Intelligence: An International Journal*, 11/2, 389–405.

[92] Tsumoto, S., and Tanaka, H., (1995), "Induction of expert system rules from databases based pm rough set theory and resampling methods", in: P.P Wang (ed.), *Second Annual Joint Conference on Information Sciences PROCEEDINGS*, September 28 - October 1, Wrightsville Beach, North Carolina, USA.

[93] Wang, P.P., (ed.), (1995), *Second Annual Joint Conference on Information Sciences PROCEEDINGS*, September 28 - October 1, Wrightsville Beach, North Carolina, USA.

[94] Woolery, L.K. and Grzymala-Busse, J. (1994). "Machine learning for an expert system to predict preterm birth risk", *Journal of the American Medical Informatics Association*, 1, 439–446.

[95] Wróblewski, J., (1995), "Finding minimal reducts using genetic algorithm (extended version)", in: P.P. Wang (ed.), *Second Annual Joint Conference on Information Sciences PROCEEDINGS*, September 28 - October 1, Wrightsville Beach, North Carolina, USA, 186–189.

[96] Wu, B.H., (1995), "An intelligent tutoring system using a rough set approach", in: P.P. Wang (ed.), *Second Annual Joint Conference on Information Sciences PRO-CEEDINGS*, September 28 – October 1, Wrightsville Beach, North Carolina, USA, 409-412.

[97] Ziarko, W., (1989), "Data analysis and case-based expert system development tool 'Rough'", in: *Proc. Case-Based Reasoning Workshop*, Morgan Kaufmann, Los Altos, CA, 356-361.

[98] Ziarko, W., (1991), "The discovery, analysis and representation of data dependencies in databases", in: G. Piatetsky-Shapiro, and W. J. Frawley (eds.),*Knowledge Discovery in Databases*, AAAI Press/MIT Press, 177-195.

[99] Ziarko, W., (1992), "Acquisition of control algorithms from operation data", in: R. Slowiński (ed.), *Intelligent Decision Support. Handbook of Applications and Advances of the Rough Set Theory*, Kluwer Academic Publishers, Dordrecht, 61-75.

[100] Ziarko, W., (1993), "Analysis of uncertain information in the framework of variable precision rough sets", *Foundations of Computing and Decision Sciences* 18, 3/4 381-396.

[101] Ziarko, W., (ed.), (1993), *Rough Sets, Fuzzy Sets and Knowledge Discovery. Proceedings of the International Workshop on Rough Sets and Knowledge Discovery (RSKD'93)*, Banff, Alberta, Canada, October 12-15, Springer-Verlag, Berlin.

[102] Ziarko, W. (1993). "Variable Precision Rough Set Model", *Journal of Computer and System Sciences*, 40, 39-59.

[103] Ziarko, W., Golan, R., and Edwards, D., (1993), "An application of DATALOGIC/R knowledge discovery tool to identify strong predictive rules in stock market data" in: *Proc. AAAI Workshop on Knowledge Discovery in Databases*, Washington, DC, 89-101

[104] Ziarko, W., and Katzberg, J., (1989), "Control algorithms acquisition, analysis and reduction: machine learning approach", in: *Knowledge-Based Systems Diagnosis, Supervision and Control*, Plenum Press, Oxford, 167-178.