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On Superreducts

by

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Summary. If $S = (U, A, V, f)$ is an information system, then any set of attributes $X \subseteq A$ defines an equivalence $EQ_S(X) = \{(u_1, u_2) \in U \times U;\ f(u_1, a) = f(u_2, a)$ for any $a \in X\}$ on the set U of objects. A superreduct of a set $X \subseteq A$ is a maximal subset Y of X such that the system of equivalences defined by all subsets of Y coincides with the system of equivalences defined by all subsets of X. Superreducts are studied in a more abstract setting and an algorithm for finding superreducts is presented.

1. Introduction. Any set of attributes of an information system defines an equivalence on the set of its objects, i.e. a classification of objects. Sometimes the same classification may be obtained on the basis of a smaller set of attributes which is more advantageous and more economical. These economical aspects lead to the notion of reduct of a set of attributes. However, there exists another aspect of this problem: the given set of attributes should be replaced by its—as small as possible — subset in such a way that any classification defined by a subset of the first set can be defined by a subset of the second one. This problem was investigated in the seminar text [3]. Later on, we succeeded in finding a more abstract basis for some investigations of information systems (cf. [7]). The aim of this paper is to incorporate also the matter of [3] in the abstract theory. Semilattice theory appears as a suitable general framework for our investigations.

2. Finite semilattices (see [8], Sect. 13, 14, 19). Let $(S, \vee)$ be a finite semilattice with an identity element (i.e. with an element $o \in S$ such that $o \vee x = x = x \vee o$ for any $x \in S$). We set $x \leqslant y$ if an only if $x \vee y = y$ where $x, y \in S$ are arbitrary. It is well known that the relation $\leqslant$ is an ordering on S such that $x \vee y = \sup\{x, y\}$. Furthermore, any subset B of S has a supremum sup B in S with respect to the ordering $\leqslant$. It will be denoted also by $\bigvee\{b;\ b \in B\}$ or by $b_1 \vee \ldots \vee b_n$ in case

$B = \{b_1, \ldots, b_n\}$. (Sometimes, the operation of the semilattice will be denoted by $\cup$; then we shall write $\bigcup$ for $\bigvee$). The identity element o in S is its least element and, therefore, $o = \sup \emptyset$. It follows that the ordered set $(S, \vee)$ is a (complete) lattice. For any $x \in S$, we put $A(x) = \{t \in S;\ t < x\}$ where $t < x$ means $t \leqslant x$, $t \neq x$.

An element $a \in S$ is said to be (totally) *irreducible* in $(S, \vee)$ if $B \subseteq S$, $a = \sup B$ imply $a \in B$. We denote by $\mathrm{Irr}(S, \vee)$ the set of all irreducible elements in $(S, \vee)$. Since $o = \sup \emptyset$, we have $o \notin \mathrm{Irr}(S, \vee)$.

A set $T \subseteq S$ is said to *generate* $(S, \vee)$ if for any $a \in S$ there exists $B(a) \subseteq T$ such that $a = \sup B(a)$.

Irreducible elements will play an important role in the sequel. The following results will be useful.

THEOREM 1. *If $(S, \vee)$ is a finite semilattice with an identity element, then $Irr(S,\vee)$ is the least subset of S generating $(S,\vee)$.*

Proof. For any $x \in S$, we denote by $V(x)$ the following property: x is a supremum of a subset of $\mathrm{Irr}(S, \vee)$. Then $V(o)$ holds.

Let $x \in S$, $x \neq o$ be arbitrary and suppose that $V(t)$ holds for any $t \in A(x)$. If $x \in \mathrm{Irr}(S, \vee)$, then $x = \sup\{x\}$ and $V(x)$ holds. If $x \notin \mathrm{Irr}(S, \vee)$, there exists $B(x) \subseteq S$ such that $\sup B(x) = x$, $x \notin B(x)$. By hypothesis, for any $t \in B(x)$, there exists $I(t) \subseteq \mathrm{Irr}(S, \vee)$ such that $t = \sup I(t)$. If putting $I(x) = \bigcup\{I(t);\ t \in B(x)\}$, then $I(x) \subseteq \mathrm{Irr}(S, \vee)$ and, clearly,

$$\sup I(x) = \sup\{\sup I(t);\ t \in B(x)\} = \sup\{t;\ t \in B(x)\} = x$$

which is $V(x)$. Since S is finite, $V(x)$ holds for any $x \in S$ which implies that $\mathrm{Irr}(S, \vee)$ generates $(S,\vee)$.

If $T \subseteq S$ generates $(S, \vee)$ and $a \in \mathrm{Irr}(S, \vee)$, then $a = \sup B(a)$ or some $B(a) \subseteq T$. The irreducibility of a implies that $a \in B(a)$ and, thus, $a \in T$. We have proved that $\mathrm{Irr}(S, \vee) \subseteq T$. □

We may recognize irreducible elements using the following.

THEOREM 2. *If $(S,\vee)$ is a finite semilattice with an identity element, $B \subseteq A$ is a set generating $(S, \vee)$, and $x \in S$ is an arbitrary element, then the following assertions are equivalent:*

(*i*) $x \in Irr(S, \vee)$.

(*ii*) $x \neq \bigvee\{t;\ t \in B \cap A(x)\}$.

Proof. If $x = \bigvee\{t;\ t \in B \cap A(x)\}$, then $x \notin \mathrm{Irr}(S, \vee)$ because $x \notin A(x)$. Hence, (*i*) implies (*ii*). If $x \notin \mathrm{Irr}(S, \vee)$, there exists $B(x) \subseteq S$ such that $x \notin B(x)$ and that $x = \bigvee\{t;\ t \in B(x)\}$. Furthermore, for any $t \in B(x)$ there exists $B(t) \subseteq B$ such that $t = \bigvee\{s;\ s \in B(t)\}$. It follows that $x = \bigvee\{s;\ s \in \bigcup\{B(t);\ t \in B(x)\}\}$. Clearly, $\bigcup\{B(t);\ t \in B(x)\} \subseteq B \cap A(x)$ and, therefore, $x = \bigvee\{t;\ t \in B \cap A(x)\}$. Hence, (*ii*) implies (*i*). □

We prove that sets generating semilattices are preserved under surjective homomorphisms.

THEOREM 3. *Let h be a surjective homomorphism of a semilattice* $(S, \cup)$ *onto a semilattice* $(S', \vee)$. *If* $B \subseteq S$ *generates* $(S, \cup)$, *then* $h[B]$ *generates* $(S', \vee)$ *where* $h[B]; = \{h(s);\ s \in B\}$.

Indeed, if $x' \in S'$ is arbitrary, there exists $x \in S$ with $h(x) = x'$. There exists $B(x) \subseteq B$ such that $x = \bigcup\{t;\ t \in B(x)\}$. It follows that

$$x' = h(x) = \bigvee\{h(t);\ t \in B(x)\} = \{t';\ t' \in h[B(x)]\}$$

where $h[B(x)] \subseteq h[B]$. □

Clearly, this is a special case of an analogous theorem holding for universal algebras. As a consequence of Theorems 1 and 3, we obtain

COROLLARY 1. *Let* $(S, \cup)$ $(S', \vee)$ *be finite semilattices with identity elements and h be a surjective homomorphism of* $(S, \cup)$ *onto* $(S', \vee)$. *Then* $Irr(S', \vee) \subseteq h[Irr(S, \cup)]$.

3. Quotient semilattices. Let A be a finite nonempty set, $\boldsymbol{B}(A)$ the system of all its subsets, $\cup$ the operation of union. Then $(\boldsymbol{B}(A), \cup)$ is a finite semilattice with the identity element $\emptyset$. If $X \in \boldsymbol{B}(A)$ is arbitrary, then $(\boldsymbol{B}(X), \cup)$ is a subsemilattice of $(\boldsymbol{B}(A), \cup)$ having the identity element $\emptyset$.

LEMMA 1. *If A is a finite nonempty set and* $X \in \boldsymbol{B}(A)$ *is arbitrary, then* $Irr(\boldsymbol{B}(X), \cup) = \{\{x\};\ x \in X\}$.

Our abstract model for the study of information systems is formed of a finite semilattice $(\boldsymbol{B}(A), \cup)$ and a congruence on $(\boldsymbol{B}(A), \cup)$ (see [7]). We now present some results about this structure that will be needed in studying superreducts.

Suppose that a congruence K on $(\boldsymbol{B}(A), \cup)$ is given. We are interested in the restriction of K to a subset X of A. More exactly:

LEMMA 2. *Let A be a finite nonempty set, K be a congruence on the semilattice* $(\boldsymbol{B}(A), \cup)$, *X be an arbitrary element in* $\boldsymbol{B}(A)$. *Put* $K_X = K \cap (\boldsymbol{B}(X) \times \boldsymbol{B}(X))$. *Then* K_X *is a congruence on the semilattice* $(\boldsymbol{B}(X), \cup)$.

Thus, we may define a quotient structure $(\boldsymbol{B}(X), \cup)/K_X$. Clearly, it is a finite semilattice whose carrier is $\boldsymbol{B}(X)/K_X$, and whose operation is defined "by means of representatives"; it will be denoted by $\vee$. The block of K_X containing $\emptyset$ is the identity element in $(\boldsymbol{B}(X), \cup)/K_X$. For any $Y \in \boldsymbol{B}(X)$, we denote by nat $K_X(Y)$ the block of K_X containing the set Y. Then nat K_X is a surjective homomorphism of $(\boldsymbol{B}(X), \cup)$ onto $(\boldsymbol{B}(X), \cup)/K_X = (\boldsymbol{B}(X)/K_X, \vee)$.

In what follows, the hypotheses of Lemma 2 will frequently appear. For the sake of brevity, we introduce the following condition:

(*H*) A is a finite nonempty set, K a congruence on the semilattice $(\boldsymbol{B}(A), \cup)$, X is an arbitrary element in $\boldsymbol{B}(A)$.

Regarding the fact the nat K_X is a surjective homomorphism, the equality $Y = \bigcup\{\{y\};\ y \in Y\}$, Corollary 1 and Lemma 1, we obtain:

LEMMA 3. *If* $(\boldsymbol{H})$ *holds, then*

(*i*) *nat* $K_X(Y) = \bigvee\{nat\ K_X(\{y\});\ y \in Y\}$ *or any* $Y \subseteq X$,

(*ii*) *Irr* $(\boldsymbol{B}(X)/K_X, \vee) \subseteq \{nat\ K_X(\{x\});\ x \in X\}$.

If $(\boldsymbol{H})$ holds, we are interested in recognizing irreducible elements among all elements of the form nat $K_X(\{x\})$ where $x \in X$. To this aim, we put

$$C(Y) = \bigcup\{Z;\ Z \in \text{nat}\ K_X(Y)\} \text{ for any } Y \in \boldsymbol{B}(X).$$

Clearly, $C(Y)$ is the greatest element Z with $(Y,\ Z) \in K_X$. Some properties of the operator C are needed.

LEMMA 4. *If* $(\boldsymbol{H})$ *holds, and* $Y \in \boldsymbol{B}(X)$, $Z \in \boldsymbol{B}(X)$ *are arbitrary, then*

(*i*) *the conditions* $C(Y) = C(Z)$ *and nat* $K_X(Y) = nat\ K_X(Z)$ *are equivalent;*

(*ii*) *the conditions* $C(Y) \cup C(Z) = C(Z)$ *and nat* $K_X(Y) \vee nat\ K_X(Z) = nat\ K_X(Z)$ *are equivalent.*

P r o o f. By definition of C, the equality $C(Y) = C(Z)$ is equivalent with $(Y,\ Z) \in K_X$ which means nat $K_X(Y) =$ nat $K_X(Z)$ and (*i*) holds.

Since $C(Y) \in$ nat $K_X(Y)$, $C(Z) \in$ nat $K_X(Z)$ hold, the equality $C(Y) \cup C(Z) = C(Z)$ implies nat $K_X(Y) \vee$ nat $K_X(Z) =$ nat $K_x(Z)$. On the other hand, the last equation entails that $C(Y) \cup C(Z) \in$ nat $K_X(Z)$ and, hence, $C(Y) \cup C(Z) \subseteq C(Z)$ by definition of C which implies that $C(Y) \cup C(Z) = C(Z)$. Thus (*ii*) holds. □

LEMMA 5. *Let* $(\boldsymbol{H})$ *hold. Then for any* $x \in X$ *the following assertions are equivalent*:

(*i*) *nat* $K_X(\{x\}) \in Irr(\boldsymbol{B}(X)/K_X, \vee)$.

(*ii*) $\bigcup\{C(\{y\});\ y \in X,\ C(\{y\}) \subseteq C(\{x\}),\ C(\{y\}) \neq C(\{x\})\} \notin nat\ K_X(\{x\})$.

P r o o f. In Theorem 2, we take $(\boldsymbol{B}(X)/K_X, \vee)$ for $(S, \vee)$ and $\{\text{nat}\ K_X(\{x\});\ x \in X\}$ for B which is possible by Lemma 3 and Theorem 1. We obtain that (*i*) equivalent with

$$(iii) \qquad \bigvee\{\text{nat}\ K_X(\{y\});\ y \in X,\ \text{nat}\ K_X(\{y\}) \leqslant \text{nat}\ K_X(\{x\}),\ \text{nat}\ K_X(\{y\}) \neq \neq \text{nat}\ K_X(\{x\})\} \neq \text{nat}\ K_X(\{x\}).$$

By Lemma 4, (*iii*) may be written as

$$(iv)\ \bigvee\{\text{nat}\ K_X(\{y\});\ y \in X,\ C(\{y\}) \subseteq C(\{x\}),\ C(\{y\}) \neq C(\{x\})\} \neq \text{nat}\ K_X(\{x\}).$$

Since $C(\{y\}) \in$ nat $K_X(\{y\})$, we have

$$(v) \qquad \bigcup\{C(\{y\});\ y \in X,\ C(\{y\}) \subseteq C(\{x\}),\ C(\{y\}) \neq C(\{x\})\} \in \in \bigvee\{\text{nat}\ K_X(\{y\});\ y \in X,\ C(\{y\}) \subseteq C(\{x\}),\ C(\{y\}) \neq C(\{x\})\},$$

which implies that (*iv*) is equivalent with (*ii*). □

4. Reducts and superreducts. Let $(\boldsymbol{H})$ hold. A set $Y \in \boldsymbol{B}(A)$ is said to be a *K-reduct* of X it has the following properties:

(A) $Y \subseteq X$.

(B) $(X, Y) \in K_X$.

(C) Y is minimal with respect to inclusion among all elements meeting conditions (A) and (B).

A set $Y \in \boldsymbol{B}(A)$ is said to be a *K-superreduct* of X if it has the following properties:

(A) $Y \subseteq X$.

(B') For any $X' \subseteq X$ there exists $Y' \subseteq Y$ such that $(Y', X') \in K_X$.

(C') Y is minimal with respect to inclusion among all elements meeting conditions (A) and (B').

THEOREM 4. *If $(\boldsymbol{H})$ holds, then X has at least one K-superreduct.*

P r o o f. Let $\boldsymbol{P}$ be the system of sets $Y \in \boldsymbol{B}(A)$ meeting conditions (A), (B'). Then, clearly, $X \in \boldsymbol{P}$ and, therefore, $\boldsymbol{P} \neq \emptyset$. The finiteness of A implies that $\boldsymbol{P}$ has at least one element that is minimal with respect to inclusion, i.e. a K-superreduct of X exists. □

Remark 1. Similarly the existence of a K-reduct may be proved. □

The following examples may clear up the relationship between reducts and superreducts. In the first two examples we suppose $A = \{b, c, d\}$, $B = \{b\}$, $C = \{c\}$, $D = \{d\}$, $E = \{b, c\}$, $F = \{b, d\}$, $G = \{c, d\}$, $O = \emptyset$.

Example 1. Let K have the following blocks: $\{O\}, \{B\}, \{C\}, \{D\}, \{E, F, G, A\}$. Clearly, K is a congruence on the semilattice $(\boldsymbol{B}(A), \cup)$. Furthermore, E, F, G are K-reducts of A and $K_A = K$.

Let $Y \in \boldsymbol{B}(A)$ be a K-superreduct of A. Since $B \subseteq A$, there exists $B' \subseteq Y$ such that $(B', B) \in K$. It follows that $\{b\} = B = B' \subseteq Y$ and, therefore, $b \in Y$. Similarly we prove that $c \in Y$, $d \in Y$, i.e. $Y = \{b, c, d\} = A$. Consequently A is the only K-superreduct of A. □

Observation 1. There exists a finite nonempty set A, a congruence K on $(\boldsymbol{B}(A), \cup)$, and a set $X \in \boldsymbol{B}(A)$ such that the set of its K-reducts and the set of its K-superreducts are mutually disjoint. □

Observation 2. If $(\boldsymbol{H})$ is satisfied, then any K-superreduct of X includes a K-reduct of X. Indeed, if Y is a K-superreduct of X and Z is a K-reduct of Y, then Z is a K-reduct of X. □

Example 2. Let K have the following blocks: $\{O\}, \{B\}, \{D\}, \{C, E, F, G, A\}$. Clearly, K is a congruence on $(\boldsymbol{B}(A), \cup)$ and $K_A = K$. We prove that F is a K-superreduct of A by indicating, for any $T \subseteq A$, the set $S \subseteq F$ with $(T, S) \in K$.

T	O	B	C	D	E	F	G	A
S	O	B	F	D	F	F	F	F

Let Y be one of the sets C, E. If Y were a K-superreduct of A, then there would be $S \subseteq Y$ such that $(S, D) \in K$. This would imply $d \in D = S \subseteq Y$ which is a contradiction. Thus, C, E are not K-superreducts of A. Similarly, G is not K-superreduct of A. It follows that F is the only K-superreduct of A. On the other hand, C is a K-reduct of A. □

Observation 3. There exists a finite nonempty set A, a congruence K on $(\boldsymbol{B}(A), \cup)$, a set $X \in \boldsymbol{B}(A)$, and a K-reduct of X that is included in no K-superreduct of X. □

Hence, any K-superreduct includes a K-reduct but a K-reduct need not be included in a K-superreduct.

Example 3. Put $A = \{b, c\}$, $B = \{b\}$, $C = \{c\}$, $0 = \emptyset$. Let K have the following blocks: $\{0\}$, $\{C\}$, $\{B, A\}$. Clearly, K is a congruence on $(\boldsymbol{B}(A), \cup)$ and $K_A = K$. We prove that A is the only K-superreduct of A. Indeed, B is not a K-superreduct of A because $C \subseteq A$ and $S \subseteq B$, $(S, C) \in K$ would imply $c \in C = S \subseteq B$ which is a contradiction. On the other hand, B is the only K-reduct of A. □

Observation 4. There exists a finite nonempty set A, a congruence K on $(\boldsymbol{B}(A), \cup)$, a set $X \in \boldsymbol{B}(A)$, and a K-superreduct Y of X that does not coincide with the union of K-reducts of X included in Y. □

5. Algorithm for superreducts. We have the following natural

PROBLEM 1. *If $(\boldsymbol{H})$ holds, find all K-superreducts of X.*

In order to enable the formulation of results, we present the following definition:

Let $(\boldsymbol{H})$ be satisfied. A set $Y \in \boldsymbol{B}(A)$ is said to be *K-suitable* for X if it has the following properties:

(A) $Y \subseteq X$.

(b) $\mathrm{Irr}((\boldsymbol{B}(X), \cup)/K_X) = \{\mathrm{nat}\ K_X(\{y\});\ y \in Y\}$.

(c) $\mathrm{nat}\ K_X(\{x\}) \neq \mathrm{nat}\ K_X(\{y\})$ for any $x \in Y$, $y \in Y$ with $x \neq y$.

LEMMA 6. *If $(\boldsymbol{H})$ holds and $Y \in \boldsymbol{B}(X)$, then the following conditions are equivalent:*

(B') For any $X' \subseteq X$ there exists $Y' \subseteq Y$ such that $(Y', X') \in K_X$.

(b') $\mathrm{Irr}((\boldsymbol{B}(X), \cup)/K_X) \subseteq \{\mathrm{nat}\ K_X(\{y\});\ y \in Y\}$.

Proof. If (B') holds, then

$$\mathrm{nat}\ K_X(X') = \mathrm{nat}\ K_X(Y') = \bigvee\{\mathrm{nat}\ K_X(\{t\});\ t \in Y'\}$$

by Lemma 3. Hence, $\{\text{nat } K_X(\{y\});\ y \in Y\}$ generates $(\boldsymbol{B}(X), \cup)/K_X$ which implies (b') by Theorem 1.

If (b') holds, then for any $X' \subseteq X$ there exists $Y' \subseteq Y$ such that $\text{nat } K_X(X') = \bigvee\{\text{nat } K_X(\{t\});\ t \in Y'\}$ by Theorem 1. By Lemma 3, we obtain $Y' \in \text{nat } K_X(X')$ which is (B'). □

THEOREM 5. *If* $(\boldsymbol{H})$ *holds, then the system of all K-suitable sets for X coincides with the system of all K-superreducts of X.*

Proof. By Lemma 6, it is sufficient to prove that a set Y is minimal with respect to inclusion in the system of all sets meeting (A), (b') if and only if it meets (A), (b), (c). Indeed, if a set is minimal in the system of sets meeting (A), (b'), then, clearly, it meets (b) and (c). On the other hand, if Y meets (A), (b), (c) and $Y' \subseteq Y$ meets (A), (b'), then

$$\{\text{nat } K_X(\{y\});\ y \in Y\} = \text{Irr}((\boldsymbol{B}(X), \cup)/K_X) \subseteq \{\text{nat } K_X(\{t\});\ t \in Y'\}.$$

Hence, for any $y \in Y$, there exists $t \in Y'$ such that $\text{nat } K_X(\{y\}) = \text{nat } K_X(\{t\})$. Since $Y' \subseteq Y$, we obtain $t = y$ by (c). Thus, $Y \subseteq Y'$ and we have $Y = Y'$ and, hence, Y is minimal in the system of sets meeting (A), (b'). □

By Lemma 4, we obtain:

LEMMA 7. *Let* $(\boldsymbol{H})$ *be satisfied and let* $Y \subseteq X$ *be a set meeting condition* (c). *Put* $x \leqslant y$ *if and only if* $C(\{x\}) \subseteq C(\{y\})$ *for any* $x \in Y$ *and* $y \in Y$. *Then the relation* $\leqslant$ *is an ordering on* Y.

Using Lemma 5 and the definition of $\leqslant$, we obtain:

THEOREM 6. *If* $(\boldsymbol{H})$ *holds and* $Y \subseteq X$ *is an arbitrary set, then the following conditions are equivalent:*

(i) Y *is a K-suitable set for X.*

(ii) Y *meets conditions* (A), (c) *and* (b''), *where* (b'') *is the following condition:*

$$\bigcup\{C(\{t\});\ t \in X,\ t \leqslant y,\ t \neq y\} \notin \textit{nat } K_X(\{y\}) \textit{ for any } y \in Y.$$

Using Theorems 5 and 6, we obtain:

ALGORITHM 1. (for finding K-superreducts):

(1) A is a finite nonempty set given by the list of its elements;
K is a congruence on the semilattice $(\boldsymbol{B}(A), \cup)$ given by the list of elements of its blocks;
$X \in \boldsymbol{B}(A)$ is a set given by the list of its elements.

(2) If $X = \emptyset$, then $\emptyset$ is the only K-superreduct of X. Otherwise go to (3).

(3) For any block of K form its intersection with X. The set of nonempty intersections coincides with the set of blocks of K_X.

(4) For any $x \in X$ construct the block nat $K_X(\{x\})$ of K_X containing $\{x\}$; furthermore, construct the set $C(\{x\})$ to be the union of all elements in nat $K_X(\{x\})$.

(5) Construct $R = \{(x, y) \in X \times X; \text{nat } K_X(\{x\}) = \text{nat } K_X(\{y\})\}$. Let Z be a set having exactly one element in common with any block of R. (There can be several possibilities for the choice of Z).

(6) For any $x, y \in Z$ put $x \leqslant y$ if and only if $C(\{x\}) \subseteq C(\{y\})$.

(7) For any $x \in Z$ test whether $\bigcup\{C(\{y\}); y \leqslant x, y \neq x\} \in \text{nat } K_X(\{x\})$ or not. Form the set Y of all elements in Z that do not meet this condition.

Then Y is a K-superreduct of X corresponding to the set Z chosen in (5).

All K-superreducts of X correspond to the sets Z obtained in (5) by all possible choices. □

6. Applications to information systems. Let U, A, V be finite nonempty sets and f a mapping of the set $U \times A$ into V. Then the ordered quadruple $S = (U, A, V, f)$ is said to be an *information system* (cf. [1, 2, 4–6]). Elements in U are interpreted to be *objects*, elements in A are called *attributes*, elements in V are said to be *values of attributes*. The condition $f(u, a) = v$ means that the attribute a has the value v for the object u.

For any set $X \subseteq A$, we put

$$EQ_S(X) = \{(u, u') \in U \times U;\ f(u, a) = f(u', a) \quad \text{for any } a \in X\}.$$

Clearly, $EQ_S(X)$ is an equivalence on the set U, i.e. a classification of objects of S. It will be called the *classification* of objects *defined by means of the set* X of attributes. The following is easy to see:

LEMMA 8. *If $S = (U, A, V, f)$ is an information system and $X \subseteq A$, $Y \subseteq A$ hold, then $EQ_S(X \cup Y) = EQ_S(X) \cap EQ_S(Y)$.*

Furthermore, we define for an information system $S = (U, A, V, f)$:

$$K^S = \{(X, Y) \in \boldsymbol{B}(A) \times \boldsymbol{B}(A);\ EQ_S(X) = EQ_S(Y)\}.$$

As a consequence of Lemma 8, we obtain

THEOREM 7. *If $S = (U, A, V, f)$ is an information system, then K^S is a congruence on the semilattice $(\boldsymbol{B}(A), \cup)$.*

Indeed, if $(X, Y) \in K^S$ and $Z \in \boldsymbol{B}(A)$, we have

$$EQ_S(X \cup Z) = EQ_S(X) \cap EQ_S(Z) = EQ_S(Y) \cap EQ_S(Z) = EQ_S(Y \cup Z)$$

which means $(X \cup Z, Y \cup Z) \in K^S$. □

Theorem 7 enables to apply our general results to the semilattice $(\boldsymbol{B}(A), \cup)$ provided with the congruence K^S. Hence the meaning of K_X^S and nat K_X^S is defined in accordance with Section 3; furthermore, C and $\leqslant$ are defined in accordance with Sections 3 and 5 starting with K_X^S.

An information system $S = (U, A, V, f)$ may be expressed by a table. We put $U = \{u_1, \ldots, u_m\}$, $A = \{a_1, \ldots, a_n\}$, where $m \geqslant 1$, $n \geqslant 1$ and $u_i \neq u_j$, $a_h \neq a_k$ for any i, j, h, k, with $1 \leqslant i < j \leqslant m$, $1 \leqslant h < k \leqslant n$. Then we define $b_{ij} = f(u_i, a_j)$ for any i, j with $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$. Then the matrix of type (m, n) formed of all elements b_{ij} expresses the information system. If we add the entries formed of elements in U and elements in A in their corresponding orders, we obtain the table of the information system S.

Let $S = (U, A, V, f)$ be an information system and $X \in \boldsymbol{B}(A)$ is a set of attributes. By definition, a set $Y \in \boldsymbol{B}(A)$ is a K^S-superreduct of X if and only if the system of classifications of objects defined by all subsets of Y coincides with the system of classifications defined by all subsets of X and if Y is a minimal subset of X with this property. Thus, the following problem is reasonable:

Problem 2. *If $S = (U, A, V, f)$ is an information system and $X \in \boldsymbol{B}(A)$ is a set of attributes, find all K^S-superreducts of X.*

Before formulating solution of this problem, we give some useful results.

Lemma 9. *Let $S = (U, A, V, f)$ be an information system $X \in \boldsymbol{B}(A)$ an arbitrary set, $Z \subseteq X$ a set having exactly one element in common with any block of K_X^S. Then for any $a \in Z$ and any $a' \in Z$ the following conditions are equivalent:*

(i) For any $u \in U$, $u' \in U$, $u \neq u'$, the condition $f(u, a') = f(u', a')$ implies $f(u, a) = f(u', a)$.

(ii) $C(\{a\}) \subseteq C(\{a'\})$ (where $C(Y)$ is the greatest element Y' such that $(Y, Y') \in K_X^S$ for any $Y \in \boldsymbol{B}(X)$).

Proof. Regarding Lemma 8, we see that any two consecutive conditions in the following sequence are equivalent.

(1) $EQ_S(\{a'\}) \subseteq EQ_S(\{a\})$.

(2) $EQ_S(C(\{a'\})) \subseteq EQ_S(C(\{a\}))$.

(3) $EQ_S(C(\{a\}) \cup C(\{a'\})) = EQ_S(C(\{a'\}))$.

(4) $(C(\{a\}) \cup C(\{a'\}), C(\{a'\})) \in K_X^S$.

(5) $C(\{a\}) \cup C(\{a'\}) \subseteq C(\{a'\})$.

Clearly, (*i*) is equivalent with (1), and (*ii*) is equivalent with (5). □

LEMMA 10. *Let $S = (U, A, V, f)$ be an information system, $X \in B(A)$ an arbitrary set, $Z \subseteq X$ a set having exactly one element in common with any block of K_X^S. Then, for any $x \in Z$, the following conditions are equivalent:*

(*i*) $\bigcup\{C(\{y\});\ y \in Z,\ y \leqslant x,\ y \neq x\} \notin nat\ K_X^S(\{x\})$.

(*ii*) *There exist $u, u' \in U$ such that $u \neq u'$, $f(u,x) \neq f(u',x)$ while $f(u,y) = f(u',y)$ for any $y \in Z$ with $y \leqslant x$, $y \neq x$.*

Proof. By Lemmas 7 and 9 the condition $x, y \in Z$, $y \leqslant x$ means $EQ_S(\{x\}) \subseteq EQ_S(\{y\})$. Hence (*ii*) may be expressed in the form $EQ_S(\{x\}) \neq \bigcap\{EQ_S(\{y\});\ y \in Z,\ y \leqslant x,\ y \neq x\}$. The last set equals

$$\bigcap\{EQ_S(C(\{y\}));\ y \in Z,\ y \leqslant x,\ y \neq x\} = EQ_S(\bigcup\{C(\{y\});\ y \in Z,\ y \leqslant x,\ y \neq x\})$$

by Lemma 8 which means that $\bigcup(\{C(\{y\});\ y \in Z, y \leqslant x, y \neq x\}, \{x\}) \notin K_X^S$. The last condition may be expressed as (*i*). □

If we adapt Algorithm 1 using Lemmas 9 and 10, we obtain:

ALGORITHM 2 (for finding K^S-superreducts):

(1) An information system $S = (U, A, V, f)$ is given by its table; $X \in B(A)$ is a set given by the list of its elements.

(2) If $X = \emptyset$, then $\emptyset$ is its only K^S-superreduct. Otherwise go to (3).

(3) In the column labelled by $a \in X$, replace all occurrences of the symbol appearing in the first row by the integer 1. Suppose that we have replaced some symbols of this column by the integers $1, \ldots, i$ where $i \geqslant 1$. Then passing through this column from the top to the bottom, find the first symbol not replaced by an integer and replace its all occurrences by the integer $i+1$. In this way, replace all elements of this column by integers. The resulting column will be called the *column corresponding to a*.

Construct columns corresponding to all elements in X.

(4) For any $a \in X$, $a' \in X$ put $(a, a') \in R$ if and only if their corresponding columns are equal.

(5) Choose exactly one element in any block of R and denote by Z the set of all chosen elements. (There can be several possibilities for the choice of Z).

(6) Put $T = (U, Z, V, g)$ where g is the restriction of f to the set $U \times Z$, $U = \{u_1, \ldots, u_m\}$, $Z = \{a_1, \ldots, a_n\}$; suppose $u_i \neq u_j$, $a_h \neq a_k$ for any i, j, h, k with $1 \leqslant i < j \leqslant m$, $1 \leqslant h < k \leqslant n$. Let T be given by a table with the elements b_{ij}. For a_k, a_l with $1 \leqslant k \leqslant n$, $1 \leqslant l \leqslant n$ put $a_k \leqslant a_l$ if and only if for any i, j with $1 \leqslant i < j \leqslant m$ the condition $b_{il} = b_{jl}$ implies $b_{ik} = b_{jk}$.

(7) For any $k \in \{1, \ldots, n\}$ construct $A_k = \{l;\ a_l \leqslant a_k,\ a_l \neq a_k\}$.

(8) Put $Y = \{a_k \in Z$; there exist i, j such that $1 \leqslant i < j \leqslant m$, $b_{ik} \neq b_{jk}$, $b_{il} = b_{jl}$ for any $l \in A_k\}$.

Then Y is a K^S- superreduct of X corresponding to the set Z chosen in (5).

All K^S-superreducts of X correspond to the sets Z obtained in (5) by all possible choices. □

Example 4. Let S be an information system given by the following table:

	a_1	a_2	a_3	a_4	a_5	a_6	a_7
u_1	v_1	v_1	v_2	v_2	v_3	v_1	v_2
u_2	v_1	v_1	v_2	v_2	v_3	v_2	v_1
u_3	v_1	v_2	v_1	v_2	v_2	v_2	v_2
u_4	v_1	v_2	v_1	v_1	v_1	v_1	v_1
u_5	v_1	v_2	v_1	v_1	v_1	v_1	v_2

where $S = (U, A, V, f)$, $U = \{u_1, \dots, u_5\}$, $A = \{a_1, \dots, a_7\}$, $V = \{v_1, v_2, v_3\}$. Suppose that $X = \{a_1, a_2, a_3, a_4, a_5\}$. By (3) of Algorithm 2, we obtain:

	a_1	a_2	a_3	a_4	a_5
u_1	1	1	1	1	1
u_2	1	1	1	1	1
u_3	1	2	2	1	2
u_4	1	2	2	2	3
u_5	1	2	2	2	3

By (4), (5) of Algorithm 2, we obtain, e.g., $Z = \{a_1, a_2, a_4, a_5\}$. Put $a'_1 = a_1$, $a'_2 = a_2$, $a'_3 = a_4$, $a'_4 = a_5$. By (6) of Algorithm 2, the ordering $\leqslant$ on Z is given by the following table:

$\leqslant$	a'_1	a'_2	a'_3	a'_4
a'_1	1	1	1	1
a'_2	0	1	0	1
a'_3	0	0	1	1
a'_4	0	0	0	1

It follows that $A_1 = \emptyset$, $A_2 = \{1\} = A_3$, $A_4 = \{1, 2, 3\}$ by (7) of Algorithm 2. By (8), we obtain $Y = \{a'_2, a'_3\} = \{a_2, a_4\}$ which is a K^S-superreduct of X corresponding to the set Z. Another possibility for the K^S-superreduct of X is $\{a_3, a_4\}$.

This may be interpreted as follows. Elements $u_1, \dots, u_5$ are persons, $a_1, \dots, a_7$ are body attributes, e.g. a_1 = *body force*, a_2 = *body weight*, a_3 = *sprint speed*, a_4 = *run speed*, a_5 = *reaction speed*, a_6 = *gymnastic abilities*, a_7 = *adaptability*. Furthermore, v_1, v_2, v_3 may be interpreted as *great*, *middle*, *little*, respectively. Then the attribute set X and its subsets define various classifications of persons with respect to their body abilities. The set Y and its subsets define the same classifications, though its cardinality is smaller than the cardinality of X.

For example, we may understand the set U as a set of young members of an athletic club. Subsets of X represent tests of body abilities that should enable the specialization of any new member. It follows that only test represented by $\emptyset$, $\{a_2\}$, $\{a_4\}$, $\{a_2, a_4\}$ are needed; since $EQ_S(\emptyset) = U \times U$ and $EQ_S(\{a_2, a_4\}) = EQ_S(\{a_2\}) \cap EQ_S(\{a_4\})$, only tests represented by $\{a_2\}$ and by $\{a_4\}$, are relevant. If the set U is representative enough, then the experience obtained with testing this set may be used for any set U' of persons in the future, i.e. only tests represented by $\{a_2\}$ and $\{a_4\}$ are sufficient for classifying U'. This situation would be more convincing if the set U had a larger number of elements; only such a set can be considered to be representative. We preferred presenting a transparent information system with a small number of objects and attributes. Algorithm 2 enables to process large information systems by the same methods.

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REFERENCES

[1] W. Marek, Z. Pawlak, *Mathematical foundations of information storage and retrieval I–III.* CC PAS Reports, **135–137,** Warszawa, 1973.

[2] W. Marek, Z. Pawlak, *Information storage and retrieval system — mathematical foundations,* CC PAS Reports, **149**, Warszawa, 1974. Also in Theoretical Computer Science, **1**, (1976), 331–354.

[3] M. Novotný, *Reducing the number of attributes in an information system,* Seminar on General Algebra and Ordered Sets, Brno, 1980.

[4] Z. Pawlak, *Mathematical foundations of information retrieval,* CC PAS Reports, **101,** *Warszawa, 1973.*

[5] Z. Pawlak, *Information system,* ICS PAS Reports, **338**, Warszawa, 1978.

[6] Z. Pawlak, *Systemy informacyjne — podstawy teoretyczne,* WNT, Warszawa 1983.

[7] M. Novotný, Z. Pawlak, *Independence of attributes,* Bull. Pol. Ac.: Math., **36** (1988) 459–465

[8] G. Szász, *Introduction to lattice theory,* Budapest, 1963.