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On Rough Relations

by

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Summary. In this article we define a concept of a rough relation, based on the idea of the rough set defined by the author in a previous paper. Some elementary properties of rough relations are given. The presented approach may be considered as an alternative to "fuzzy" philosophy.

1. Introduction. In [1] we have introduced the concept of the rough set, which can be regarded as an alternative to fuzzy sets (see [3]), however there are some essential differences between these two concepts. This note contains an extension of the ideas given in [2].

The basic idea underlying the concept of a rough relation is connected with the fact that in some cases we might be unable to say for sure whether some objects, states processes, etc., are in a certain relationship or not. This may be caused by our limited accuracy of observation, measurement or description of some phenomena, processes, states, etc.

In our approach the limitation of our knowledge about the real world is expressed by an indiscernibility relation, which is a basic tool in our considerations.

The concept of the rough relation seems to be of some value for pattern recognition, measurement theory, identification of object, etc. Some of the possible applications will be presented in a separate paper.

2. Rough relations

2.1. *An approximation space.* An approximation space is an ordered pair $A = (U, R)$, where $U \neq \emptyset$ is a set, called the universe and $R \subseteq U \times U$ is a binary relation called the indiscernibility relation. In what follows we assume that R is an equivalence relation. If $(x, y) \in R$, we say that x and y are indiscernible with respect to R in A.

The equivalence classes of the indiscernibility relation R are called R-elementary sets in A. A finite union of R-elementary sets will be called an R-definable set in A.

Let $A_1 = (U_1, R_1), \ldots, A_n = (U_n, R_n)$ be a family of approximation spaces, and let

$$A^n = (U^n, R)$$

where $U^n = U_1 \times U_2 \times \ldots \times U_n$, $R = R_1 \times R_2 \times \ldots \times R_n$ defined thus

$$\big((u_1, u_2, \ldots, u_n), (r_1, r_2, \ldots, r_n)\big) \in R$$

$$\text{iff } (u_j, r_j) \in R_j, \text{ for each } j = 1, 2, \ldots, n.$$

Obviously, R is also an equivalence relation and A^n is an approximation space, called the product of A_i. The equivalence classes of the relation R are called R-elementary relations in A^n and a finite union of R-elementary relations is called on R-definable relation in A^n.

Let $A^n = (U^n, R)$ be a product of approximation space. For any relation $Q \subseteq U^n$, we define two relations $\underline{R}Q$ and $\bar{R}Q$ called the lower and upper R-approximation of Q in A^n, respectively, and defined thus:

$$\underline{R}Q = \{(u_1, u_2, \ldots, u_n) \in U^n \colon [(u_1, u_2, \ldots, u_n)]_R \subseteq Q\}$$

$$\bar{R}Q = \{(u_1, u_2, \ldots, u_n) \in U^n \colon [(u_1, u_2, \ldots, u_n)]_R \cap Q \neq \emptyset\},$$

where $[(u_1, u_2, \ldots, u_n)]_R$ denotes the equivalence class of the relation R containing tuple $(u_1, u_2, \ldots, u_n)$.

The product space of $A_1, A_2, \ldots, A_n$ will be also denoted by $A^n = A_1 \times A_2 \times \ldots \times A_n$.

If each R-elementary set contains one element only, the product space is called selective.

Let $A^2 = A \times A$, where $A = (U, R)$, and let $Q \subseteq U \times U$. One can easily verify the following properties:

1) If Q is an identity relation and A is not selective, then neither $\underline{R}Q$ nor $\bar{R}Q$ is an identity relation.

2) If Q is a reflexive relation, so is $\bar{R}Q$, but not necessarily $\underline{R}Q$.

3) If Q is a symmetric relation, so are $\underline{R}Q$ and $\bar{R}Q$.

4) If Q is an antisymmetric relation, so is $\underline{R}Q$, but not necessarily $\bar{R}Q$.

5) If Q is a nonsymmetric relation, so is $\underline{R}Q$, but not necessarily $\bar{R}Q$.

6) If Q is a transitive relation then in general, neither $\underline{R}Q$ nor $\bar{R}Q$ are transitive.

7) If Q is an equivalence relation then in general, neither $\underline{R}Q$ nor $\bar{R}Q$ are equivalence relations.

8) If Q is an ordering relation and Q is not R-definable, then, in general, neither $\underline{R}Q$ nor $\bar{R}Q$ are ordering relations.

9) $\underline{R}(Q^{-1}) = (\underline{R}Q)^{-1}$ and $\bar{R}(Q^{-1}) = (\bar{R}Q)^{-1}$.

2.2. *Roughly selective, symmetric, antisymmetric and transitive relations.* Let $A^2 = A \times A$ be an approximation space, where $A = (U, R)$ and let $Q \subseteq U \times U$ be a binary relation in U. We shall employ the following definitions.

A relation Q is roughly reflexive (r-reflexive) in A^2 if $[(x, x)]_R \subseteq Q$ for each $x, y \in U$. The $[(x, y)]_R$ denotes an equivalence class of the relation R containing pair (x, y).

A relation Q is roughly symmetric (r-symmetric) in A^2 if $[(x, y)]_R \subseteq Q$, implies $[(y, x)]_R \subseteq Q$ for each $x, y \in U$.

A relation Q is roughly antisymmetric (r-antisymmetric) in A^2 if $[(x, y)]_R \subseteq Q$ implies $[(y, x)]_R \not\subseteq Q$, for each $x, y \in U$.

A relation Q is roughly nonsymmetric (r-nonsymmetric) in A^2 if $[(x, y)]_R \subseteq Q$ and $[(y, x)]_R \subseteq Q$ implies $[(x, y)]_R = [(y, x)]_R$, for each $x, y \in U$.

A relation Q is roughly transitive (r-transitive) in A^2 if $[(x, y)]_R \subseteq Q$ and $[(y, z)]_R \subseteq Q$ implies $[(x, z)]_R \subseteq Q$ for each $x, y, z \in U$.

A relation Q is called a rough equivalence (r-equivalence) relation in A^2 if Q is r-reflexive, r-symmetric and r-transitive in A^2.

A relation Q is called a rough ordering (r-ordering) relation in A^2 if Q is r-reflexive, r-transitive and r-nonsymmetric in A^2.

Obviously, if Q is r-reflexive in A^2, then Q is also reflexive; if Q is r-symmetric in A^2, then Q may be not symmetric, and if Q is symmetric then Q is also r-symmetric for any approximation space.

If Q is r-symmetric in A^2, $[(x, y)]_R \subseteq Q$, $(x, x') \in R$ and $(y, y') \in R$, then $(y', x') \in Q$.

If Q is r-antisymmetric in A^2, $[(x, y)]_R \subseteq Q$, $[(x, x')] \in R$ and $(y, y') \in R$ then $(y', x') \notin Q$.

If Q is r-transitive relation in A^2, $[(x, y)]_R \subseteq Q$, $[(y, z)]_R \subseteq Q$, $(x, x') \in R$, $(y, y') \in R$ and $(z, z') \in R$, then $(x', z') \in Q$.

Let $A = (U, R)$ and $B = (U, S)$ be two approximation spaces. If $S \subset R$, we say that B is finer that A; if $S \supset R$ we say that B is coarser than A.

The following properties are obvious: if Q is r-equivalence (r-ordering) relation in A^2 and B is finer than A, then Q is also r-equivalence (r-ordering) relation in B^2; if B is coarser than A, then Q may be not r-equivalence (r-ordering) relation in B^2.

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З. Павляк, **О приближенных отношениях**

В настоящей статье нами определяется концепция приближенного отношения на основе понятия приближенного множества, изложенного автором в предыдущей работе.

Приводятся некоторые элементарные свойства приближенных отношений. Предлагемый подход может рассматриваться как альтернатива к философии "расплывчатости".