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THEORETICAL COMPUTER SCIENCE

On Rough Equalities

by

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Summary. Rough equalities appear in a natural way if studying information systems. In the article [5], the so called rough top equalities and rough bottom equalities are completely characterized. Complete characterizations of rough equalities are given in the present article.

1. Introduction. Rough equalities appear in a natural way in connection with information systems.

An information system (cf. [3] p. 16) is an ordered quadruple $\langle X, A, V, \varrho\rangle$ where X, A, V are finite nonempty sets and ϱ is a mapping of $X \times A$ into V. The elements in X are called objects, the elements in A attributes, the elements in V values of attributes. The mapping ϱ assigns to any object x in X and to any attribute a in A the value $\varrho(x, a)$ in V that the attribute a assumes for the object x. We set $R = \{(x, y) \in X \times X; \varrho(x, a) = \varrho(y, a)$ for any $a \in A\}$. Then R is an equivalence on X, its blocks are said to be elementary sets. A union of elementary sets is said to be a definable set. Not any subset of X is definable in a general case, but any subset can be approximated by definable sets. For any $Z \subseteq X$, we set

$$(\mathbf{u}R)(Z) = \bigcup \{Q; Q \in X/R, Q \cap Z \neq \emptyset\}, \ (\mathbf{l}R)(Z) = \bigcup \{Q; Q \in X/R, Q \subseteq Z\};$$

the definable sets $(\mathbf{u}R)(Z)$, $(\mathbf{l}R)(Z)$ are said to be the upper and lower approximation of Z, respectively.

Two subsets Z, T of X are said to be roughly equal if $(\mathbf{u}R)(Z) = (\mathbf{u}R)(T)$, $(\mathbf{l}R)(Z) = (\mathbf{l}R)(T)$.

1. EXAMPLE. Let X be the set of all pupils in a school, let $A = \{class\}$ where the values of the attribute *class* are: $1a, 1b, 1c, \ldots, 5a, 5b, 5c$. For any pupil x in X, the function $\varrho(x, class)$ denotes the class attended by the pupil x. Then $\langle X, A, V, \varrho\rangle$ is an information system where V denotes the

set of all values of the attribute *class*. Clearly, R is the equivalence on X whose blocks are exactly the classes of the school.

Suppose that an infectious disease appeared in the school and that Z is the set of all pupils suffering from this disease at a certain moment. The sick pupils and their class mates are supposed to need vaccination against the disease. Hence $(\mathbf{u}R)(Z)$ is the set of all pupils that have been vaccinated. Furthermore, the classes where all pupils are sick, are closed. Thus, $(\mathbf{l}R)(Z)$ is the set of all pupils for which the compulsory education has been stopped temporarily.

If we know the set of all pupils that have been vaccinated and the set of all pupils with stopped education, we cannot deduce the set of all sick pupils. All sets roughly equal to the set of all sick pupils produce the same set of vaccinated pupils and the same set of pupils with stopped education.

In what follows, we shall investigate rough equalities in a more abstract way.

2. Equivalences and subsets of finite lattices. Let $(L, \vee, \wedge)$ be a finite nonempty lattice. Then it has a least element O and a greatest element 1. A subset C of L is said to be *completely closed* in $(L, \vee, \wedge)$ if it has the following properties.

(i) $O \in C$, $1 \in C$;

(ii) $x \vee y \in C$, $x \wedge y \in C$ for any $x \in C$ and $y \in C$.

1. LEMMA. *Let $(L, \vee, \wedge)$ be a finite nonempty lattice with a least element O and a greatest element 1, let C be a subset of L. Then the following conditions are equivalent.*

(a) *C is completely closed.*

(b) *For any $x \in L$ the set $\{t \in C;\ t \geqslant x\}$ has a least element and the set $\{t \in C;\ t \leqslant x\}$ has a greatest element.*

Proof. If (a) holds, then $1 \in \{t \in C; t \geqslant x\}$ for any $x \in L$. By (ii), there exists $\inf_C \{t \in C; t \geqslant x\}$ and $\inf_C \{t \in C; t \geqslant x\} \geqslant x$ which implies that $\inf_C \{t \in C; t \geqslant x\}$ is the least element of the set $\{t \in C; t \geqslant x\}$. Similarly, the set $\{t \in C; t \leqslant x\}$ has a greatest element and (b) holds.

If (b) holds, the set $\{t \in C; t \geqslant 1\} = \{1\}$ has at least element 1 and, thus, $1 \in C$. Similarly, $O \in C$. Furthermore, for any $x, y \in C$, there exists a greatest element z in the set $\{t \in C;\ t \leqslant x \vee y\}$. Since x and y are in the last set, we obtain $x \leqslant z$, $y \leqslant z$ and, thus, $x \vee y \leqslant z$. On the other hand, z is also in the last set which implies that $z \leqslant x \vee y$. Thus, $x \vee y = z \in C$. Similarly, $x \wedge y \in C$, and (a) holds. □

Let $(L, \vee, \wedge)$ be a finite nonempty lattice, C its completely closed subset, and $x \in L$ an arbitrary element. We put

$$(\mathbf{i}C)(x) = \inf_C \{t \in C; t \geqslant x\}, (\mathbf{o}C)(x) = \sup_C \{t \in C; t \leqslant x\}.$$

For any $x \in L$, $y \in L$, we set $(x, y) \in \mathbf{M}(C)$ if and only if $(\mathbf{i}C)(x) = (\mathbf{i}C)(y)$, $(\mathbf{o}C)(x) = (\mathbf{o}C)(y)$. Clearly, $\mathbf{M}(C)$ is an equivalence on L.

Let $(L, \vee, \wedge)$ be a finite nonempty lattice, M an equivalence on L, C a completely closed subset of $(L, \vee, \wedge)$. The equivalence M is said to be *induced* by C if and only if $M = \mathbf{M}(C)$.

For an arbitrary equivalence M on L, we put $\mathbf{S}(M) = \{x \in L; \{x\} \in L/M\}$.

3. Characterization of rough equality. For any set U, we denote by $\mathbf{B}(U)$ the family of all subsets of U, by $Co\, X$ the set $U - X$, for any $X \in \mathbf{B}(U)$.

Let U be a finite nonempty set, R an equivalence on U. For any $X \in \mathbf{B}(U)$, we put

$$(\mathbf{u}R)(X) = \bigcup \{Q; Q \in U/R, Q \cap X \neq \emptyset\}, (\mathbf{l}R)(X) = \bigcup \{Q; Q \in U/R, Q \subseteq X\}.$$

We shall need some properties of the operators $(\mathbf{u}R)$, $(\mathbf{l}R)$. They can be found in [5] 3.1–3.4 where they are proved from some results of [2].

1. Lemma. $(\mathbf{u}R)$ *is a closure operator, i.e.,* $(\mathbf{u}R)$ *is extensive, monotone, and idempotent.* □

2. Lemma. $(\mathbf{l}R)((\mathbf{u}R)(X)) = (\mathbf{u}R)(X)$, $(\mathbf{u}R)((\mathbf{l}R)(X)) = (\mathbf{l}R)(X)$ *for any* $X \in \mathbf{B}(U)$. □

We set $\mathbf{F}(R) = \{(\mathbf{u}R)(X); X \in \mathbf{B}(U)\}$.

3. Lemma. $(\mathbf{F}(R), \cup, \cap, Co, \emptyset, U)$ *is a Boolean algebra.* □

A subset C of $\mathbf{B}(U)$ is said to be *closed* in the Boolean algebra $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$ if $(C, \cup, \cap, Co, \emptyset, U)$ is a subalgebra of $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$.

4. Lemma. *If C is a subset closed in the Boolean algebra* $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$, *then there exists an equivalence R on U such that* $\mathbf{F}(R) = C$. □

For any $X \in \mathbf{B}(U)$, $Y \in \mathbf{B}(U)$, we put $(X, Y) \in \mathbf{N}(R)$ if and only if $(\mathbf{u}R)(X) = (\mathbf{u}R)(Y)$, $(\mathbf{l}R)(X) = (\mathbf{l}R)(Y)$. The equivalence $\mathbf{N}(R)$ on $\mathbf{B}(U)$ is said to be the *rough equality corresponding to* R. An equivalence N on $\mathbf{B}(U)$ is called a *rough equality* if there exists an equivalence R on U such that $N = \mathbf{N}(R)$.

5. Lemma. $\mathbf{S}(\mathbf{N}(R)) = \mathbf{F}(R)$.

Proof. For any $X \in \mathbf{F}(R)$, we have $(\mathbf{u}R)(X) = X$ and, therefore, $(\mathbf{l}R)(X) = (\mathbf{l}R)((\mathbf{u}R)(X)) = (\mathbf{u}R)(X) = X$ by 2. Thus, $(X, Y) \in \mathbf{N}(R)$ implies that $X = (\mathbf{l}R)(X) = (\mathbf{l}R)(Y) \subseteq (\mathbf{u}R)(Y) = (\mathbf{u}R)(X) = X$ and, hence, $X = Y$ which entails $X \in \mathbf{S}(\mathbf{N}(R))$. We have proved $\mathbf{F}(R) \subseteq \mathbf{S}(\mathbf{N}(R))$.

On the other hand, $X \notin \mathbf{F}(R)$ means $X \neq (\mathbf{u}R)(X)$ which implies the existence of $Q \in U/R$ and $x \in X \cap Q$, $y \in Q - X$. We put $Y = (X - Q) \cup ((Q \cap X) -$

$-\{x\}) \cup \{y\}$. Then $Y \neq X$ and $(\mathbf{u}R)(Y) = (\mathbf{u}R)(X)$, $(\mathbf{l}R)(Y) = (\mathbf{l}R)(X)$. Thus $X \notin \mathbf{S}(\mathbf{N}(R))$. We have proved that $\mathbf{S}(\mathbf{N}(R)) \subseteq \mathbf{F}(R)$. □

REMARK. Clearly, a closed subset of the algebra $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$ is a completely closed subset of the lattice $(\mathbf{B}(U), \cup, \cap)$.

By 3 and 5, $\mathbf{S}(\mathbf{N}(R))$ is a completely closed subset of the lattice $(\mathbf{B}(U), \cup, \cap)$. Thus, the operators $(\mathbf{iS}(\mathbf{N}(R)))$, $(\mathbf{oS}(\mathbf{N}(R)))$ can be defined.

6. LEMMA. $(\mathbf{iS}(\mathbf{N}(R))) = (\mathbf{u}R)$, $(\mathbf{oS}(\mathbf{N}(R))) = (\mathbf{l}R)$.

Proof. For any $X \in \mathbf{B}(U)$, we obtain by definition that $(\mathbf{iS}(\mathbf{N}(R)))(X)$ is the least element in $\mathbf{S}(\mathbf{N}(R))$ including X. By 5, $(\mathbf{iS}(\mathbf{N}(R)))(X)$ is the least element in $\mathbf{F}(R)$ including X, i.e., $(\mathbf{iS}(\mathbf{N}(R)))(X) = (\mathbf{u}R)(X)$. Thus, the first equality is proved and the second can be proved similarly. □

7. CHARACTERIZATION THEOREM FOR ROUGH EQUALITIES. *Let U be a finite nonempty set, N an equivalence on $\mathbf{B}(U)$. Then the following assertions are equivalent.*

(α) *N is a rough equality.*

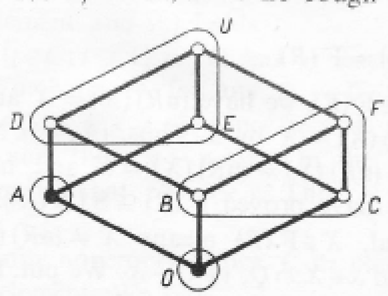
(β) *$\mathbf{S}(N)$ is a closed subset of the algebra $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$ and N is induced by $\mathbf{S}(N)$.*

Proof. (1) If (α) holds, there exists an equivalence R on U such that $N = \mathbf{N}(R)$. By 5 and 3, $\mathbf{S}(N) = \mathbf{F}(R)$ is a closed subset of the algebra $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$. The equivalence $\mathbf{N}(R)$ is induced by the set $\mathbf{S}(\mathbf{N}(R))$ by 6. Thus, (β) holds.

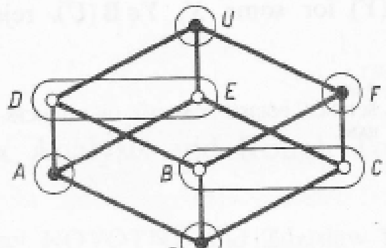
(2) If (β) holds, there exists an equivalence R on U such that $\mathbf{S}(N) = \mathbf{F}(R) = \mathbf{S}(\mathbf{N}(R))$ by 4 and 5. By Remark, the set $\mathbf{S}(\mathbf{N}(R))$ is completely closed in the lattice $(\mathbf{B}(U), \cup, \cap)$. By 6, we have $(\mathbf{iS}(N)) = (\mathbf{u}R)$, $(\mathbf{oS}(N)) = (\mathbf{l}R)$, and, therefore, $\mathbf{M}(\mathbf{S}(N)) = \mathbf{N}(R)$. Since N is induced by $\mathbf{S}(N)$, we obtain $N = \mathbf{M}(\mathbf{S}(N))$ whence $N = \mathbf{N}(R)$ and (α) holds. □

4. Examples. In all examples, we suppose $U = \{a, b, c\}$, $O = \emptyset$, $A = \{a\}$, $B = \{b\}$, $C = \{c\}$, $D = \{a, b\}$, $E = \{a, c\}$, $F = \{b, c\}$.

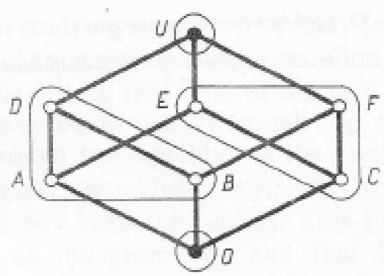
1. EXAMPLE. Let N be an equivalence on $\mathbf{B}(U)$ whose blocks are $\{O\}$, $\{A\}$, $\{B, C, F\}$, $\{D, E, U\}$. Then $\mathbf{S}(N) = \{O, A\}$ which is not closed in the algebra $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$. Thus, N is no rough equality.



2. Example. Let N have the blocks $\{O\}$, $\{A\}$, $\{B, C\}$, $\{D, E\}$, $\{F\}$, $\{U\}$. Then $\mathbf{S}(N) = \{O, A, F, U\}$ is closed in $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$. Furthermore, $(\mathbf{iS}(N))(D) = U = (\mathbf{iS}(N))(E)$, $(\mathbf{oS}(N))(D) = A = (\mathbf{oS}(N))(E)$, $(\mathbf{iS}(N))(B) = F = (\mathbf{iS}(N))(C)$, $(\mathbf{oS}(N))(B) = O = (\mathbf{oS}(N))(C)$, which implies that $N = \mathbf{M}(\mathbf{S}(N))$ and, therefore, N is a rough equality.



3. Example. Let N have the blocks $\{O\}$, $\{A, B, D\}$, $\{C, E, F\}$, $\{U\}$. Then $\mathbf{S}(N) = \{O, U\}$ which is closed in $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$. We have $(\mathbf{iS}(N))(D) = U = (\mathbf{iS}(N))(E)$, $(\mathbf{oS}(N))(D) = O = (\mathbf{oS}(N))(E)$, but $(D, E) \notin N$. Thus, N is not induced by $\mathbf{S}(N)$ and N is no rough equality.



5. Algorithm recognizing rough equalities.

Data: Let a finite nonempty set U be given. Suppose that the Boolean algebra $(\mathbf{B}(U), \cup, \cap, Co, \emptyset, U)$ is given by tables for binary operations $\cup, \cap$, and for the unary operation Co. Let an equivalence N on $\mathbf{B}(U)$ be given in such a way that the elements of any block of N are enumerated.

Preprocessing of data: Construct the set $\mathbf{S}(N) = \{x; \{x\} \in \mathbf{B}(U)/N\}$.

Algorithm:

(1) If either $\emptyset \notin \mathbf{S}(N)$ or $U \notin \mathbf{S}(N)$, reject N; otherwise go to (2).

(2) If $Co\, X \notin \mathbf{S}(N)$ for some $X \in \mathbf{S}(N)$, reject N; otherwise go to (3).

(3) If either $X \cup Y \notin \mathbf{S}(N)$ or $X \cap Y \notin \mathbf{S}(N)$ for some $X, Y \in \mathbf{S}(N)$, reject N; otherwise go to (4).

(4) For any $X \in \mathbf{B}(U)$ construct $(\mathbf{iS}(N))(X) = \bigcap \{T; T \in \mathbf{S}(N), X \subseteq T\}$, $(\mathbf{oS}(N))(X) = \bigcup \{T; T \in \mathbf{S}(N), T \subseteq X\}$. Go to (5).

(5) If either $(\mathbf{iS}(N))(X) = (\mathbf{iS}(N))(Y)$, $(\mathbf{oS}(N))(X) = (\mathbf{oS}(N))(Y)$, $(X, Y) \notin N$ or $(X, Y) \in N$, $(\mathbf{iS}(N))(X) \neq (\mathbf{iS}(N))(Y)$ or $(X, Y) \in N$, $(\mathbf{iS}(N))(X) = (\mathbf{iS}(N))(Y)$, $(\mathbf{oS}(N))(X) \neq (\mathbf{oS}(N))(Y)$ for some $X, Y \in \mathbf{B}(U)$, reject N; otherwise N is a rough equality.

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М. Новотны, З. Павляк, **О приближенных равенствах**

В настоящей работе излагается теорема о характеризации приближенных равенств.